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SOME RESULTS IN TYPES OF EXTENSIONS OF MV-ALGEBRAS

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Abstract. We introduce the notions of zero divisor and extension, contraction of ideals in MV-algebras and several interesting types of extensions of MV-algebras. In particular, we show what kinds of extensions MV-algebras will lead in a homeomorphism of the spectral topology and inverse topology on minimal prime ideals. Finally, we investigate the relations among types of extensions of MV-algebras.

1. Introduction and preliminaries

Chang introduced MV-algebras to provide algebraic semantics for Łukasiewicz in finite-valued propositional logic [**3**]. Also, Busneag and Piciu introduced $\wedge$-closed systems of an MV-algebra and they introduced the notion of MV-algebra of fractions and proved constructively the existence of a maximal MV-algebra of quotients [**2**].

Eslami introduced prime spectrum of a BL-algebra and investigated some properties of them [**5**]. Forouzesh et al. introduced the spectral topology and quasi-spectral topology of prime A-ideals in MV-modules and proved some properties of them. They showed that the set of all prime A-ideals in an MV-module is Hausdorff and disconnected [**6**]. Forouzesh et al. introduced the inverse topology on $\mathrm{Min}(A)$ and proved that it is compact, Hausdorff space, T_0-space and T_1-space [**8**].

In this paper, the set of all zero divisors of an MV-algebra A, is denoted by Z_A and the relation between zero divisors and the chain MV-algebra A is studied. In addition, we show that if A is a chain, then only idempotent elements of A are 0 and 1.

Also, we prove that if A is an MV-algebra, $P_S\colon A \to A[S]$ is a homomorphism MV-algebra, $I \in \mathrm{Id}(A)$ and $\{0\} \in \mathrm{Spec}(A)$, then $I^e = \{\lambda \in A[S] : \lambda = \frac{x}{S},\, x \in I\}$. Also, we define min-extension of MV-algebras and prove that if $B \hookrightarrow A$ *is a* min-extension of MV-algebras, then $\psi\colon \mathrm{Min}(A) \to \mathrm{Min}(B)$ *by* $\psi(P) = P \cap B$ *is continuous with respect to both the spectral topology and the inverse topology.*

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We define good extension,f-extension, f^*-extension , quasi good extension, quasi f-extension and quasi f^*-extension of MV-algebras and prove that good extension of MV-algebras has property violations. Furthermore, we show that A is an f-extension of MV-algebra B if and only if map of $\psi\colon \operatorname{Min}(A) \to \operatorname{Min}(B)$ by $\psi(P) = P \cap B$ with respect to the spectral topology is a homeomorphism.

Also, we show that A is a quasi f^*-extension of an MV-algebra B if and only if map $\psi\colon \operatorname{Min}(A) \to \operatorname{Min}(B)$ by $\psi(P) = P \cap B$ with respect to the inverse topology is a homeomorphism.

We recollect some definitions and results which will be used in the following:

Definition 1.1. [**3**] An MV-algebra is a structure $(A, \oplus, *, 0)$ where $\oplus$ is a binary operation, $*$, is a unary operation, and 0 is a constant such that the following axioms are satisfied for any $a, b \in A$:

(MV1) $(A, \oplus, 0)$ is an Abelian monoid,
(MV2) $(a^*)^* = a$,
(MV3) $0^* \oplus a = 0^*$,
(MV4) $(a^* \oplus b)^* \oplus b = (b^* \oplus a)^* \oplus a$.

Note that we have $1 = 0^*$ and the auxiliary operation $\odot$ which are as follows:

$$x \odot y = (x^* \oplus y^*)^*.$$

We recall that the natural order determines a bounded distributive lattice structure such that

$$x \vee y = x \oplus (x^* \odot y) = y \oplus (x \odot y^*) \quad \text{and} \quad x \wedge y = x \odot (x^* \oplus y) = y \odot (y^* \oplus x).$$

We recall that an element $a \in A$ *is complemented if there is an element* $b \in A$ *such that* $a \vee b = 1$ *and* $a \wedge b = 0$.

Definition 1.2. [**3**] A subalgebra of an MV-algebra A is a subset B of A containing the zero element of A, that it is closed under the operations of A and is equipped with the restriction to B of these operations.

Lemma 1.1. [**4**] *In each MV-algebra, the following relations hold for all $x, y, z \in A$:*

1. $x \leqslant y$ *if and only if* $y^* \leqslant x^*$,
2. *If* $x \leqslant y$, *then* $x \oplus z \leqslant y \oplus z$ *and* $x \odot z \leqslant y \odot z, x \wedge z \leqslant y \wedge z$,
3. $x \leqslant y$ *if and only if* $x^* \oplus y = 1$ *if and only if* $x \odot y^* = 0$,
4. $x, y \leqslant x \oplus y$ *and* $x \odot y \leqslant x, y$, $x \leqslant nx = x \oplus x \oplus \cdots \oplus x$ *and* $x^n = x \odot x \odot \cdots \odot x \leqslant x$,
5. $x \oplus x^* = 1$ *and* $x \odot x^* = 0$,
6. *If* $x \leqslant y$ *and* $z \leqslant t$, *then* $x \oplus z \leqslant y \oplus t$,
7. $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$,
8. $x \wedge (y \oplus z) \leqslant (x \wedge y) \oplus (x \wedge z), x \wedge (x_1 \oplus \cdots \oplus x_n) \leqslant (x \wedge x_1) \oplus \cdots \oplus (x \wedge x_n)$, *for all* $x_1, \ldots, x_n \in A$; *in particular* $(mx) \wedge (ny) \leqslant mn(x \wedge y)$ *for every* $m, n \geqslant 0$.
9. *If* $e \in B(A)$, *then* $e \wedge e^* = 0$, $e \vee e^* = 1$, *where* $B(A)$ *is the set of all complemented elements of* $L(A)$ *such that* $L(A)$ *is distributive lattice with* 0 *and* 1 *on* A.

Also for any two elements $x, y \in A$, $x \leqslant y$ if and only if x and y satisfy condition (3) in the above lemma.

Definition 1.3. [**4**] An ideal of an MV-algebra A is a nonempty subset I of A satisfying the following conditions:

(I1) If $x \in I$, $y \in A$ and $y \leqslant x$, then $y \in I$,
(I2) If $x, y \in I$, then $x \oplus y \in I$.

We denote by $\mathrm{Id}(A)$ the set of all ideals of an MV-algebra A.

Definition 1.4. [**4**] Let I be an ideal of an MV-algebra A. Then I is a proper ideal of A, *if* $I \neq A$.

- [**4**] A proper ideal I of an MV-algebra A is called prime ideal if for all $x, y \in A$, $x \wedge y \in I$, then $x \in I$ or $y \in I$. We denote by $\mathrm{Spec}(A)$ the set of all prime ideals of an MV-algebra A.
- [**4**] An ideal I of an MV-algebra A is called a minimal prime ideal of A:
  1) $I \in \mathrm{Spec}(A)$;
  2) If there exists $Q \in \mathrm{Spec}(A)$ such that $Q \subseteq I$, then $I = Q$.

We denote by $\mathrm{Min}(A)$ the set of all prime minimal ideals of an MV-algebra A.

Definition 1.5. [**4**] Let X and Y be two MV-algebras. A function $f\colon X \to Y$ is called homomorphism of MV-algebras if and only if

$$(1)\ f(0) = 0, \quad (2)\ f(x \oplus y) = f(x) \oplus f(y), \quad (3)\ f(x^*) = (f(x))^*.$$

Theorem 1.1. [**4**] *Let A and B be MV-algebras, $f\colon A \to B$ be a homomorphism of MV-algebras and $P \in \mathrm{Spec}(B)$. Then $f^{-1}(P) \in \mathrm{Spec}(A)$. Also, the intersection of prime ideals of an MV-algebra is a prime ideal.*

Definition 1.6. [**10**] A nonempty subset S of an MV-algebra is called $\wedge$-closed system in A, if $1 \in S$ and $x, y \in S$ implies $x \wedge y \in S$.

We denote by $S(A)$ the set of all $\wedge$-closed systems of A (clearly 1, $A \in S(A)$). For $S \in S(A)$ in the MV-algebra A, we consider the relation θ_S defined by $(x, y) \in \theta_S$ if and only if there exists $e \in S \cap B(A)$ such that $x \wedge e = y \wedge e$.

Lemma 1.2. [**10**] *θ_S is a congruence on A.*

For $x \in A$, we denote by $\frac{x}{S}$ the equivalence class of x relative to θ_S and by

$$A[S] = \frac{A}{\theta_S}.$$

By $P_S\colon A \to A[S]$, we denote the canonical map defined by $P_S(x) = \frac{x}{S}$, for every $x \in A$. Clearly, $0 = \frac{0}{S}$ and $1 = \frac{1}{S}$ are in $A[S]$ and for every $x, y \in A$, we have

$$\frac{x}{S} \oplus \frac{y}{S} = \frac{(x \oplus y)}{S}, \qquad \Big(\frac{x}{S}\Big)^* = \frac{x^*}{S}.$$

So, P_S is an onto morphism of MV-algebras [**10**].

Remark 1.1. [**10**] Since for every $s \in S \cap B(A)$, $s \wedge s = s \wedge 1$, we conclude that $\frac{s}{S} = \frac{1}{S}$. Hence $P_S(S \cap B(A)) = 1$.

THEOREM 1.2. [**10**] *Let S be a $\wedge$-closed system of an MV-algebra A and $I \in \mathrm{Id}(A)$ such that $I \cap S = \emptyset$. Then there exists a prime ideal P of A such that $I \subseteq P$ and $P \cap S = \emptyset$.*

DEFINITION 1.7. [**10**] Let X be a nonempty subset of MV-algebra A. Then $\mathrm{Ann}_A(X) = \{a \in A : a \wedge x = 0,\ \forall x \in X\}$ is called the annihilator of X.

DEFINITION 1.8. [**10**] An MV-algebra A is called chain if for all $a, b \in A$, a and b are comparable elements of A.

REMARK 1.2. [**10**] Let A be an MV-algebra. Then $a \wedge a^* = 0$, for all $a \in A$ if and only if A is Boolean algebra. It is proved $a \odot a = a$, for all $a \in A$ in an MV-algebra.

NOTE: Let A be an MV-algebra and I be an ideal of A. We define

$$v_A(I) = \{P \in \mathrm{Spec}(A) : I \subseteq P\} \qquad u_A(I) = \{P \in \mathrm{Spec}(A) : I \nsubseteq P\}$$

$$V_A(I) = \mathrm{Min}(A) \cap v_A(I) \qquad U_A(I) = \mathrm{Min}(A) \cap u_A(I)$$

$$V_A(a) = \{P \in \mathrm{Min}(A) : a \in P\} \qquad U_A(a) = \{P \in \mathrm{Min}(A) : a \notin P\}$$

LEMMA 1.3. *[**5**] Let $\tau(A) = \{v(I) : I \in \mathrm{Id}(A)\}$ and $X = \mathrm{Spec}(A)$. Then $\tau(A)$ satisfies the axioms for closed set in a topological space. Hence $\tau_A = \{u_A(I) : I \in \mathrm{Id}(A)\}$ is a topology on $\mathrm{Spec}(A)$, which is called spectral topology of A.*

LEMMA 1.4. [**1**] *Let A be a nonempty MV-algebra. Then $\beta = \{u_A(a) : a \in A\}$ is a base for a topology on $\mathrm{Spec}(A)$.*

LEMMA 1.5. [**8**] *Let A be a nonempty MV-algebra. The collection $\beta = \{V_A(I) : I \in \mathrm{Id}(A)\}$ is a base for a topology on $\mathrm{Min}(A)$.*

REMARK 1.3. [**8**] The induced topology of base

$$\beta = \{V_A(I) : I \text{ is finitely generated ideal of } A\}$$

is called the inverse topology. When equipped with the inverse topology on $\mathrm{Min}(A)$, we shall write $\mathrm{Min}^{-1}(A)$.

NOTATION. $U_A(a)$ is called a basic open set with respect to spectral topology on $\mathrm{Min}(A)$ and $V_A(I)$ is a basic open set with respect to the inverse topology on $\mathrm{Min}(A)$. Obviously, $\{V_A(a) : a \in A\}$ is a subbasis for inverse topology.

THEOREM 1.3. [**8**] *Let A be an MV-algebra and $P \in \mathrm{Spec}(A)$. Then $P \in \mathrm{Min}(A)$ if and only if for each $x \in P$, there exists $r \in A - P$ such that $x \wedge r = 0$.*

THEOREM 1.4. [**8**] *Let A be an MV-algebra, $P \in \mathrm{Min}(A)$ and I is finitely generated ideal. Then $I \subseteq P$ if and only if $\mathrm{Ann}_A(I) \nsubseteq P$.*

LEMMA 1.6. [**8**] *Let A be an MV-algebra. If we have $0 \neq x \in A$, then there exists $P \in \mathrm{Min}(A)$ such that $x \notin P$.*

DEFINITION 1.9. [**7**] Let A be an MV-algebra and $I \in \mathrm{Id}(A)$ be a proper ideal of A. Radical of an ideal I is intersection of all maximal ideals of A contain I, it is denoted by $\sqrt{I}$. We proved that radical of I is as follows:

$$\sqrt{I} = \{a \in A : a \odot na \in I, \text{ for all } n \in \mathbb{N}\}.$$

DEFINITION 1.10. [**10**] Let $X \subseteq A$. The ideal of A generated by X will be denoted by $(X]$. We have

$$(X] = \{a \in A \mid a \leqslant x_1 \oplus x_2 \oplus \cdots \oplus x_n, \text{ for some } n \in \mathbb{N} \text{ and } x_i \in X,\ 1 \leqslant i \leqslant n\}$$

In particular, $(a] = \{x \in A \mid x \leqslant na, \text{ for some } n \in \mathbb{N}\}$.

We denote by $(a_1, a_2, \ldots, a_n]$, the ideal of A generated by $X = \{a_1, a_2, \ldots, a_n\}$.

REMARK 1.4. [**2**] If P is a prime ideal of an MV-algebra A, then $S = A - P$ is an $\wedge$-closed system. We denote $A[S]$ by A_P. The set $PA_P = \{\frac{x}{S} : x \in P\}$ is a unique maximal ideal of A_P. In other words, A_P is a local MV-algebra. The process of passing from A to A_P is called localization at P.

REMARK 1.5. Let I and J be finitely generated ideals of an MV-algebra A. Obviously $I \vee J = (I \cup J]$ is a finitely generated ideal.

2. Zero divisors of a subset of an MV-algebra

DEFINITION 2.1. Let X be a nonempty subset of A. The set of all zero-divisors of X is denoted by $Z_X(A)$ and is defined as follows:

$$Z_X(A) = \{a \in A : \exists 0 \neq x \in X \text{ such that } x \wedge a = 0\}.$$

Zero element of an MV-algebra is a zero-divisor, which is called trivial zero-divisor. We denote by Z_A the set of all zero-divisors of an MV-algebra A.

NOTATION. It can be easily shown that $\mathrm{Ann}_A(X) \subseteq Z_X(A)$.

EXAMPLE 2.1. Let $A = \{0, a, b, c, d, 1\}$, where $0 < a$, $c < d < 1$ and $0 < a < b < 1$. Define $\odot$, $\oplus$ and $*$ as follows:

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	a	0	0	a
b	0	a	b	0	a	b
c	0	0	0	c	c	c
d	0	0	a	c	c	d
1	0	a	b	c	d	1

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	b	b	d	1	1
b	b	b	b	1	1	1
c	c	d	1	c	d	1
d	d	1	1	d	1	1
1	1	1	1	1	1	1

$*$	0	a	b	c	d	1
	1	d	c	b	a	0

Then $(A, \oplus, \odot, *, 0, 1)$ is an MV-algebra [**10**]. Obviously, $Z_A = \{0, a, b, c\}$ and $\mathrm{Ann}_A(A) = \{0\}$.

NOTATION. Let A be an MV-algebra. A is without non-trivial zero divisor if and only if A is a chain. We can easily show that an MV-algebra A is chain if and only if for all $x, y \in A$, $x \wedge y = 0$, implies $x = 0$ or $y = 0$. Hence we have:

THEOREM 2.1. *Let A be an MV-algebra. Then $I_0 = \{0\}$ is prime ideal if and only if A is a chain.*

We recall that an element a of A is called idempotent if $a^2 = a \odot a$, then we can prove that if a and b are idempotent elements, then $a \odot b = a \wedge b$ [**10**].

THEOREM 2.2. *Let A be a chain. Then only idempotent elements of A are 0 and 1.*

PROOF. Obviously 0 and 1 are idempotent elements of A. Let $0 \neq e \in A$ be idempotent. We know that $e \wedge e^* = e \odot e^* = 0$, since A is a chain, so $e^* = 0$. It follows that $e^{**} = 0^* = 1$. Hence $e = 1$. $\square$

REMARK 2.1. Let A be an MV-algebra. If $a, b \in Z_A$, then $a \oplus b$ and a^* may not be zero-divisors. In Example 2.1, we have $a \oplus c = d$ and $a^* = d$ such that a and c are zero-divisors but d is not zero-divisor of A and $a^* = d$.

THEOREM 2.3. *Let L and K be nonempty subsets of A. Then the following statements hold:*

(1) $0 \in Z_L(A)$;
(2) *If* $L \subseteq K$, *then* $Z_L(A) \subseteq Z_K(A)$;
(3) $Z_{L\cap K}(A) \subseteq Z_L(A) \cap Z_K(A)$.
(4) *If* $\{0\}$ *is a prime ideal and* $L, K \in \mathrm{Id}(A)$, *then* $Z_{L\cap K}(A) = Z_L(A) \cap Z_K(A)$.
(5) *If* $L = \{1\}$, *then* $Z_L(A) = \{0\}$,
(6) $Z_{L\cup K}(A) = Z_L(A) \cup Z_K(A)$.

PROOF. (1) Since for all $x \in L, 0 \wedge x = 0$, we get $0 \in Z_L(A)$

(2) Let $x \in Z_L(A)$. Then there exists $0 \neq l \in L$ such that $x \wedge l = 0$, since $L \subseteq K$, we get $l \in K$. Thus $Z_L(A) \subseteq Z_K(A)$.

(3) We have

$$\begin{aligned} a \in Z_{L\cap K}(A) &\Rightarrow \exists 0 \neq x \in L \cap K \text{ such that } a \wedge x = 0 \\ &\Rightarrow 0 \neq x \in L \text{ such that } a \wedge x = 0 \text{ and } x \in K \text{ such that } a \wedge x = 0 \\ &\Rightarrow a \in Z_L(A) \cap Z_K(A) \end{aligned}$$

(4) By (3), $Z_{L\cap K}(A) \subseteq Z_L(A) \cap Z_K(A)$. Since $K, L \in \mathrm{Id}(A)$, we get $L \cap K \in \mathrm{Id}(A)$ [**4**]. Now we have

$$\begin{aligned} a \in Z_L(A) \cap Z_K(A) &\Rightarrow a \in Z_L(A) \text{ and } a \in Z_K(A) \\ &\Rightarrow \exists 0 \neq x \in L, a \wedge x = 0 \text{ and } \exists 0 \neq y \in L, a \wedge y = 0 \\ &\Rightarrow (a \wedge x) \wedge (a \wedge y) = a \wedge (x \wedge y) = 0. \end{aligned}$$

Since $x \wedge y \leqslant x \in L$, and $x \wedge y \leqslant y \in K$, we obtain $x \wedge y \in L \cap K$ and since $\{0\}$ is a prime ideal, so $0 \neq x \wedge y$. We get $a \in Z_{L\cap K}(A)$. Thus $Z_L(A) \cap Z_K(A) \subseteq Z_{L\cap K}(A)$. Therefore $Z_L(A) \cap Z_K(A) = Z_{L\cap K}(A)$.

(5) It is clear.

(6) We have

$$\begin{aligned} a \in Z_{L\cup K}(A) &\Rightarrow \exists 0 \neq x \in L \cup K \text{ such that } a \wedge x = 0 \\ &\Rightarrow 0 \neq x \in L \text{ or } 0 \neq x \in K \text{ such that } a \wedge x = 0 \\ &\Rightarrow a \in Z_L(A) \cup Z_K(A). \end{aligned}$$

So $Z_{L\cup K}(A) \subseteq Z_L(A) \cup Z_K(A)$.

$$\begin{aligned} a \in Z_L(A) \cup Z_K(A) &\Rightarrow a \in Z_L(A) \text{ or } a \in Z_K(A) \\ &\Rightarrow \exists 0 \neq x \in L, a \wedge x = 0 \text{ or } \exists 0 \neq y \in K, a \wedge y = 0 \\ &\Rightarrow \exists 0 \neq x \in L \subseteq L \cup K, a \wedge x = 0 \text{ or} \\ &\quad\ \exists 0 \neq y \in K \subseteq L \cup K, a \wedge y = 0 \\ &\Rightarrow a \in Z_{L\cup K}(A). \end{aligned}$$

Thus $Z_L(A) \cup Z_K(A) \subseteq Z_{L\cup K}(A)$. Therefore $Z_L(A) \cup Z_K(A) = Z_{L\cup K}(A)$. □

In the following example, we show that equality in (4) is not true in general.

Example 2.2. In Example 2.1, let $L = \{b, d\}$, $K = \{1, a, d\}$. Then $L \cap K = \{d\}$. Now we have $Z_L = Z_K = \{0, c\}$, but $Z_{L\cap K} = \{0\}$.

3. Extension and contraction ideals of MV-algebras

The following example shows that the MV-homomorphic image of an ideal is not necessarily an ideal.

Example 3.1. Let $A = \{0, a, b, 1\}$, where $0 < a$, $b < 1$. Define $\odot$, $\oplus$ and $*$ as follows:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\oplus$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

$*$	0	a	b	1
	1	b	a	0

Then $(A, \oplus, \odot, *, 0, 1)$ is an MV-algebra [**10**]. Consider MV-homomorphism $f\colon A \to A$ such that $f(0) = 0$, $f(a) = 1$, $f(b) = 0$ and $f(1) = 1$. It is clear $I = \{0, a\}$ is an ideal of A, while $f(I) = \{0, 1\}$ is not an ideal of A.

Theorem 3.1. *Let A, B be MV-algebras and $f\colon A \to B$ be a homomorphism of MV-algebras. If we have $I \in \mathrm{Id}(A)$, then*

$$(f(I)] = \{b \in B : b \leqslant f(a_1) \oplus f(a_2) \oplus \cdots \oplus f(a_n) \text{ such that } a_1, a_2, \ldots, a_n \in I\}$$

is an ideal of B.

Proof. Obviously, $0 \in (f(I)]$. Let $x, y \in (f(I)]$. Then there exist $a_i, b_j \in I$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$, such that $x \leqslant f(a_1) \oplus f(a_2) \oplus \cdots \oplus f(a_n)$ and $y \leqslant f(b_1) \oplus f(b_2) \oplus \cdots \oplus f(b_m)$, we get $x \oplus y \leqslant f(a_1) \oplus f(a_2) \oplus \cdots \oplus f(a_n) \oplus f(b_1) \oplus f(b_2) \oplus \cdots \oplus f(b_m)$, so $x \oplus y \in (f(I)]$. Let $x \in (f(I)]$ and $y \leqslant x$. So there exists $a_i \in I$, $1 \leqslant i \leqslant n$ such that $x \leqslant f(a_1) \oplus f(a_2) \oplus \cdots \oplus f(a_n)$, thus $y \leqslant x \leqslant f(a_1) \oplus f(a_2) \oplus \cdots \oplus f(a_n)$, we obtain $y \in (f(I)]$. Therefore $(f(I)] \in \mathrm{Id}(B)$. □

Remark 3.1. Let $f\colon A \to B$ be a homomorphism of MV-algebras.

(1) Let $I \in \mathrm{Id}(A)$ and $J \in \mathrm{Id}(B)$. We denoted by I^e ideal generated by $f(I)$ and J^c ideal $f^{-1}(J)$, which I^e is called, extension ideal of I and I^c is called contraction ideal of I.

(2) Let $I_1, I_2 \in \mathrm{Id}(A)$ and $J_1, J_2 \in \mathrm{Id}(B)$. Obviuosly, if we have $I_1 \subseteq I_2$ and $J_1 \subseteq J_2$, then $I_1^e \subseteq I_2^e$ and $J_1^c \subseteq J_2^c$.

THEOREM 3.2. *Let $f\colon A \to B$ be a homomorphism of MV-algebras, $I_1, I_2 \in \mathrm{Id}(A)$ and $J_1, J_2 \in \mathrm{Id}(B)$. Then*

(1) $I_1 \subseteq I_1^{ec}$;	(5) $(I_1 \vee I_2)^e = I_1^e \vee I_2^e$;
(2) $J_1^{ce} \subseteq J_1$;	(6) $(J_1 \cap J_2)^c = J_1^c \cap J_2^c$;
(3) $I_1^e = I_1^{ece}$;	(7) $\sqrt{J_1^c} = (\sqrt{J_1})^c$.
(4) $J_1^{cec} = J_1^c$	

PROOF. (1) We have $x \in I_1 \Rightarrow f(x) \in f(I_1) \subseteq I_1^e \Rightarrow f(x) \in I_1^e \Rightarrow x \in I_1^{ec}$.

(2) We have $f(f^{-1}(J_1)) \subseteq J_1 \Rightarrow (f(f^{-1}(J_1))] \subseteq (J_1] \Rightarrow J_1^{ce} \subseteq J_1$.

(3) By (1), we have $I_1 \subseteq I_1^{ec}$, now by Remark 3.1(2), we get $I_1^e \subseteq I_1^{ece}$. By (2), we have $I_1^{ece} \subseteq I_1^e$. Therefore $I_1^e = I_1^{ece}$.

(4) By (2), we have $J_1^{ce} \subseteq J_1$, so by Remark 3.1(2), $J_1^{cec} \subseteq J_1^c$. By (1) on J_1^c, we have $J_1^c \subseteq J_1^{cec}$. Therefore $J_1^{cec} = J_1^c$.

(5) First we show that $f(I_1 \vee I_2) = f(I_1) \vee f(I_2)$. Obviously, $f(I_1) \vee f(I_2) \subseteq f(I_1 \vee I_2)$. We have

$$\begin{aligned} b \in f(I_1 \vee I_2) &\Rightarrow b = f(a) \text{ such that } a \in I_1 \vee I_2 \\ &\Rightarrow a \leqslant x \oplus y \text{ such that } x \in I_1, y \in I_2 \text{ and } b = f(a) \leqslant f(x) \oplus f(y) \\ &\Rightarrow b \in f(I_1) \vee f(I_2) \Rightarrow f(I_1 \vee I_2) \subseteq f(I_1) \vee f(I_2). \end{aligned}$$

Therefore $f(I_1 \vee I_2) = f(I_1) \vee f(I_2)$. Now we have

$$\begin{aligned} f(I_1 \vee I_2) = f(I_1) \vee f(I_2) \subseteq I_1^e \vee I_2^e &\Rightarrow (f(I_1 \vee I_2)] \subseteq I_1^e \vee I_2^e \\ &\Rightarrow (I_1 \vee I_2)^e \subseteq I_1^e \vee I_2^e. \end{aligned}$$

Also, we have

$$\begin{aligned} I_1 \subseteq I_1 \vee I_2, I_2 \subseteq I_1 \vee I_2 &\Rightarrow f(I_1) \subseteq f(I_1 \vee I_2), f(I_2) \subseteq f(I_1 \vee I_2) \\ &\Rightarrow I_1^e \subseteq (I_1 \vee I_2)^e, I_2^e \subseteq (I_1 \vee I_2)^e \\ &\Rightarrow I_1^e \vee I_2^e \subseteq (I_1 \vee I_2)^e. \end{aligned}$$

Therefore $I_1^e \vee I_2^e = (I_1 \vee I_2)^e$.

(6) We know that $J_1 \cap J_2 \subseteq J_1$ and $J_1 \cap J_2 \subseteq J_2$. Also, by Remark 3.1(2), $(J_1 \cap J_2)^c \subseteq J_1^c$ and $(J_1 \cap J_2)^c \subseteq J_2^c$, hence $(J_1 \cap J_2)^c \subseteq J_1^c \cap J_2^c$. Let

$$\begin{aligned} x \in J_1^c \cap J_2^c &\Rightarrow x \in J_1^c \text{ and } x \in J_2^c \Rightarrow f(x) \in J_1 \text{ and } f(x) \in J_2 \\ &\Rightarrow f(x) \in J_1 \cap J_2 \Rightarrow x \in (J_1 \cap J_2)^c \Rightarrow J_1^c \cap J_2^c \subseteq (J_1 \cap J_2)^c. \end{aligned}$$

Therefore $J_1^c \cap J_2^c = (J_1 \cap J_2)^c$.

(7) By Definitions 1.5 and 1.9 , we have

$$\begin{aligned} x \in (\sqrt{J_1})^c &\Leftrightarrow f(x) \in (\sqrt{J_1}) \Leftrightarrow f(x) \odot nf(x) \in J_1, \ \ \forall n \in \mathbb{N} \\ &\Leftrightarrow f(x \odot nx) \in J_1 \Leftrightarrow x \odot nx \in f^{-1}(J_1) = J_1^c, \ \ \forall n \in \mathbb{N} \\ &\Leftrightarrow x \in \sqrt{J_1^c} \end{aligned}$$ □

THEOREM 3.3. *Let A be an MV-algebra, $P_S\colon A \to A[S]$ be homomorphism MV-algebra, $I \in \mathrm{Id}(A)$ and $\{0\} \in \mathrm{Spec}(A)$. Then $I^e = \big\{\lambda \in A[S] : \lambda = \frac{x}{S},\ x \in I\big\}$.*

PROOF. Let $\lambda \in I^e$. Then we have $\lambda = \frac{x}{S}$ such that $\lambda \leqslant P_S(a_1) \oplus P_S(a_2) \oplus \cdots \oplus P_S(a_n)$ such that $a_i \in I$, $1 \leqslant i \leqslant n, x \in A$. We define $b := a_1 \oplus a_2 \oplus \dots a_n$ and show that $x \in I$.

$$\begin{aligned} &\frac{x}{S} \leqslant P_S(a_1) \oplus P_S(a_2) \oplus \cdots \oplus P_S(a_n) \\ &\quad \Rightarrow \frac{x}{S} \leqslant \frac{a_1}{S} \oplus \frac{a_2}{S} \oplus \dots \frac{a_n}{S} \Rightarrow \frac{x}{S} \leqslant \frac{a_1 \oplus a_2 \oplus \dots a_n}{S} \Rightarrow \frac{x}{S} \leqslant \frac{b}{S} \\ &\quad \Rightarrow \frac{x}{S} \odot \Big(\frac{b}{S}\Big)^* = \frac{0}{S} \Rightarrow \exists e \in B(A) \cap S \text{ such that } (x \odot b^*) \wedge e = 0 \wedge e = 0 \\ &\quad \Rightarrow x \odot b^* = 0 \text{ (Since } \{0\} \text{ is a prime ideal of } A) \\ &\quad \Rightarrow x \leqslant b \Rightarrow x \in I \end{aligned}$$ □

LEMMA 3.1. *Let A be an MV-algebra and $P_S\colon A \to A[S]$ be homomorphism MV-algebra and $Q \in \mathrm{Id}(A)$, $\{0\} \in \mathrm{Spec}(A)$ such that $B(A) \cap S \cap Q \neq \emptyset$. Then $Q^e = A[S]$.*

PROOF. Suppose that $x \in B(A) \cap S \cap Q$. By Theorem 3.3, we obtain $\frac{x}{S} \in Q^e$. Since $x \wedge x = 1 \wedge x$, we get $\frac{1}{S} = \frac{x}{S}$. Hence $\frac{1}{S} \in Q^e$. Thus $Q^e = A[S]$. □

THEOREM 3.4. *Let A be an MV-algebra and $P_S\colon A \to A[S]$ be homomorphism MV-algebra, $\rho \in \mathrm{Spec}(A[S])$, and $0 \in \mathrm{Spec}(A)$. Then*

(1) $\rho = \rho^{ce}$. (2) $B(A) \cap S \cap \rho^c = \emptyset$.

PROOF. (1) Let $x \in \rho$. Then $x = \frac{a}{S}$ such that $a \in A$. We get $P_S(a) = \frac{a}{S} = x \in \rho$, then $a \in \rho^c$. It follows from Theorem 3.3 that $x \in \rho^{ce}$. Then we obtain $\rho \subseteq \rho^{ce}$. By Theorem 3.2(2), we have $\rho^{ce} \subseteq \rho$. Therefore $\rho = \rho^{ce}$

(2) Let $B(A) \cap S \cap \rho^c \neq \emptyset$. Since $\rho \in \mathrm{Spec}(A[S])$, by Theorem 1.1, we imply $\rho^c \in \mathrm{Spec}(A)$. Now by (1) and Lemma 3.1, we have $\rho^{ce} = \rho = A[S]$, which is a contradiction. □

THEOREM 3.5. *Let A be an MV-algebra. $P_S\colon A \to A[S]$ is homomorphism MV-algebra such that $Q \in \mathrm{Id}(A)$, $\{0\} \in \mathrm{Spec}(A)$ and $B(A) \cap S \cap Q = \emptyset$. Then*

(1) $Q^{ec} = Q$, (2) $Q^e \in \mathrm{Spec}(A[S])$.

PROOF. (1) By Theorem 3.2(1), we get $Q \subseteq Q^{ec}$. Let $a \in Q^{ec}$. Then $P_S(a) = \frac{a}{S} \in Q^e$, by Theorem 3.3, we get $a \in Q$. So $Q^{ec} \subseteq Q$. Therefore $Q^{ec} = Q$.

(2) We must show that $Q^e \subsetneqq A[S]$. Let $Q^e = A[S]$. Then $Q^{ec} = (A[S])^c$ by (1), we have $Q = Q^{ec} = (A[S])^c = A$, which is a contradiction. Let $\frac{a}{S}, \frac{b}{S} \in A[S]$ such that $\frac{a}{S} \wedge \frac{b}{S} \in Q^e$. It follows from Theorem 3.3 that

$$\frac{a}{S} \wedge \frac{b}{S} = \frac{a \wedge b}{S} \in Q^e \Rightarrow a \wedge b \in Q \Rightarrow a \in Q \text{ or } b \in Q \Rightarrow \frac{a}{S} \in Q^e \text{ or } \frac{b}{S} \in Q^e$$ □

4. Extensions of MV-algebras

Definition 4.1. Let B be subalgebra of MV-algebra A. Then A is an extension of B, and $B \hookrightarrow A$, is called an inclusion.

Theorem 4.1. *Let $B \hookrightarrow A$ be an inclusion of MV-algebras. Then for each $P \in \mathrm{Spec}(B)$, there exists $Q \in \mathrm{Min}(A)$ such that $Q \cap B \subseteq P$. Furthermore, if $P \in \mathrm{Min}(B)$, then there exists $Q \in \mathrm{Min}(A)$ such that $P = Q \cap B$.*

Proof. Seting $D = \{Q \in \mathrm{Spec}(A) : Q \cap B \subseteq P\}$. By Zorn's lemma, it is sufficient to show that $D \neq \emptyset$. Let

$$\psi\colon B \to B_P,\ b \mapsto \frac{b}{S}; \quad \varphi\colon A \to A_P,\ a \mapsto \frac{a}{S}; \quad \phi\colon B_P \to A_P,\ b \mapsto \frac{b}{S}.$$

Let T be a maximal ideal of A_P. Put $Q = \varphi^{-1}(T)$, which by Theorem 1.1, Q is a prime ideal of A. But $\phi^{-1}(T)$ is a prime ideal of B_P, so $\phi^{-1}(T) \subseteq PB_P$. Now let $a \in Q \cap B$. We get

$$\begin{aligned}
a \in Q \Rightarrow \varphi(b) = \frac{b}{S} &\Rightarrow \frac{b}{S} \in \phi^{-1}(T) \subseteq PB_P \Rightarrow \exists t \in P \text{ such that } \frac{b}{S} = \frac{t}{S} \\
&\Rightarrow \exists e \in B(A) \cap S \text{ such that } b \wedge e = t \wedge e \leqslant t \in P \\
&\Rightarrow b \wedge e \in P \text{ (Since } P \text{ is a prime ideal of } A \text{ and } e \notin P) \\
&\Rightarrow b \in P.
\end{aligned}$$

So $Q \cap B \subseteq P$, it follows that $D \neq \emptyset$. We define $\leqslant$ on D by

$$\forall Q_1, Q_2 \in D; \quad Q_1 \leqslant Q_2 \Leftrightarrow Q_2 \subseteq Q_1.$$

Obviously, $\leqslant$ is a partial order relation on D. Let $\{Q_i\}_{i \in I}$ be a chain of elements of D. It follows from Theorem 1.1 that

$$\bigcap_{i \in I} Q_i \in \mathrm{Spec}(A) \ \text{ and } \ \left(\bigcap_{i \in I} Q_i\right) \cap B \subseteq Q_i \cap B \subseteq P.$$

$\bigcap_{i \in I} Q_i$ is an upper bounded of chain in D, so by Zorn's lemma, D has a maximal element F. Now, we show that $F \in \mathrm{Min}(A)$. Let $E \in \mathrm{Spec}(A)$ such that $E \subseteq F$. It follows that $F \leqslant E$, so $E \cap A \subseteq F \cap B \subseteq P$, hence $F = E$. Thus $F \in \mathrm{Min}(A)$. □

Definition 4.2. An inclusion $B \hookrightarrow A$ of MV-algebras is called min-extension, if for all $P \in \mathrm{Min}(A)$, $P \cap B \in \mathrm{Min}(B)$. When $B \hookrightarrow A$ is min-extension, we let $\psi\colon \mathrm{Min}(A) \to \mathrm{Min}(B)$ be the map defined by $\psi(P) = P \cap B$.

Definition 4.3. Let $S = \{a \in A : a \notin Z_A\}$. Obviously S is a $\wedge$-closed system of A. $A[S]$ is called classic MV-algebra and we denote it by $q(A)$.

Example 4.1. If $\{0\}$ is a prime ideal of an MV-algebra A, then extension $A \hookrightarrow q(A)$ is a min-extension. Let $\rho \in \mathrm{Min}(q(A))$. By Theorem 1.1, $\rho \cap A \in \mathrm{Spec}(A)$ and by Theorem 3.4(1), we have $\rho^{ce} = \rho$. If $\rho \cap A \notin \mathrm{Min}(A)$, then there exists $Q \in \mathrm{Spec}(A)$ such that $Q \subsetneqq \rho \cap A$, so $Q^e \subsetneqq \rho^{ce} = \rho$. But $Q \in \mathrm{Spec}(A)$, we consider two cases.

Case 1. If $Q \cap S \cap B(A) = \emptyset$, then by Theorem 3.5(2), $Q^e \in \mathrm{Spec}(q(A))$. On the other hand $Q^e \subsetneqq \rho$, which is a contradiction, (since $\rho \in \mathrm{Min}(q(A))$).

Case 2. If $Q \cap S \cap B(A) \neq \emptyset$, then by Lemma 3.1, we get $Q^e = q(A)$. So $q(A) \subsetneqq \rho$, which is a contradiction. Therefore $\rho \cap A \in \mathrm{Min}(A)$.

Theorem 4.2. *The inclusion of MV-algebras $B \hookrightarrow A$ is a min-extension if and only if whenever $P \in \mathrm{Min}(A)$ and $b \in P \cap B$, then there exists $a \in B - P$ such that $a \wedge b = 0$.*

Proof. Let $B \hookrightarrow A$ be a min-extension, $P \in \mathrm{Min}(A)$ and $b \in P \cap B$. Then $P \cap B \in \mathrm{Min}(B)$. By Theorem 1.4, $\mathrm{Ann}_B(b) \nsubseteq P \cap B$, we get that there exists $a \in \mathrm{Ann}_A(b)$ such that $a \notin P \cap B$. Hence there exists $a \in B - P$ such that $a \wedge b = 0$.

Conversely, let $P \in \mathrm{Min}(A)$ and $P \cap B \notin \mathrm{Min}(B)$. By Theorem 1.3, there exists $b \in P \cap B$ such that for all $a \in B - P$, $a \wedge b \neq 0$, which is a contradiction. □

Theorem 4.3. *Let $B \hookrightarrow A$ be a min-extension of MV-algebra. Then $\psi\colon \mathrm{Min}(A) \to \mathrm{Min}(B)$ by $\psi(P) = P \cap B$ is continuous with respect to both the spectral topology and the inverse topology.*

Proof. Let I be an ideal of B and $b \in B$. Then

$$\psi^{-1}(V_B(I)) = \{P \in \mathrm{Min}(A) : I \subseteq P\} = V_A(I),$$
$$\psi^{-1}(U_B(a)) = \{P \in \mathrm{Min}(A) : b \notin P\} = U_A(b).$$

We get that map ψ is continuous with respect to both the spectral and the inverse topologies. □

Definition 4.4. (1) An inclusion $B \hookrightarrow A$ is a good extension, if for each $a \in A$, then there exists $b \in B$ such that $\mathrm{Ann}_A(a) = \mathrm{Ann}_A(b)$.

(2) An inclusion $B \hookrightarrow A$ is an f-extension, if for each $P \in \mathrm{Min}(A)$ and each $a \in A - P$, there exists $b \in B - P$ such that $\mathrm{Ann}_A(a) \subseteq \mathrm{Ann}_A(b)$.

(3) An inclusion $B \hookrightarrow A$ is f^*-extension, if for each $P \in \mathrm{Min}(A)$ and each $a \in P$ then there exists $b \in B \cap P$ such that $\mathrm{Ann}_A(b) \subseteq \mathrm{Ann}_A(a)$.

In the above definitions, if one replaces the term b with a finitely generated ideal of A, then one gets the notions of quasi good extension, quasi f-extension, and quasi f^*-extension.

Example 4.2. Consider extension of an MV-algebra in Example 3.1. Since for all $x \in A$, there exists $y \in B_2$ such that $\mathrm{Ann}_A(x) = \mathrm{Ann}_A(y)$, we get $B_2 \hookrightarrow A$ is a good extension. But $\mathrm{Ann}_A(b) \neq \mathrm{Ann}_A(0)$ and $\mathrm{Ann}_A(b) \neq \mathrm{Ann}_A(1)$, we obtain $B_1 \hookrightarrow A$ is not a good extension. We have

$$A - I_1 = \{a, b, d, 1\}, \quad A - I_2 = \{c, d, 1\},$$
$$B_2 - I_1 = \{b, 1\}, \quad B_2 - I_2 = \{c, 1\}, \quad B_1 - I_2 = \{1\},$$
$$I_1 \cap B_1 = I_2 \cap B_1 = \{0\}, \quad I_1 \cap B_2 = \{0, c\}, \quad I_2 \cap B_2 = \{0, b\}.$$

Obviously, $B_2 \hookrightarrow A$ is an f-extension and since $\mathrm{Ann}_A(c) \nsubseteq \mathrm{Ann}_A(1)$, and $\mathrm{Ann}_A(0) = A \nsubseteq \mathrm{Ann}_A(a)$, we get $B_1 \hookrightarrow A$ is not f-extension and f^*-extension.

Theorem 4.4. *Let $B \hookrightarrow A$ and $A \hookrightarrow C$ be two inclusions of MV-algebras. Then $B \hookrightarrow C$ is a good extension if and only if $B \hookrightarrow A$ and $A \hookrightarrow C$ are good extensions.*

Proof. Let $B \hookrightarrow C$ be a good extension and $a \in A$. Since A is a subalgebra of C, so there exists $b \in B$ such that $\mathrm{Ann}_C(a) = \mathrm{Ann}_C(b)$. Since A is a subalgebra of C, we conclude $\mathrm{Ann}_A(a) = \mathrm{Ann}_A(b)$. Hence $B \hookrightarrow A$ is a good extension.

Now, let $x \in C$. Since $B \hookrightarrow C$ is a good extension, there exists $y \in B$ such that $\mathrm{Ann}_C(x) = \mathrm{Ann}_C(y)$. Since B is a subalgebra of A, we obtain $y \in A$, it follows that $A \hookrightarrow C$ is a good extension.

Conversely, let $B \hookrightarrow A$ and $A \hookrightarrow C$ be good extensions. Suppose that $x \in C$. Since $A \hookrightarrow C$ is a good extension, there exists $y \in A$ such that $\mathrm{Ann}_C(x) = \mathrm{Ann}_C(y)$. On the other hand $B \hookrightarrow A$ is a good extension, so there exists $z \in B$ such that $\mathrm{Ann}_A(y) = \mathrm{Ann}_A(z)$. Now we show that $\mathrm{Ann}_C(x) = \mathrm{Ann}_C(z)$. Let $\alpha \in \mathrm{Ann}_C(x)$ and $\alpha \notin \mathrm{Ann}_C(z)$. We have

$$\alpha \notin \mathrm{Ann}_C(z) \Rightarrow \alpha \notin \mathrm{Ann}_A(z) \Rightarrow \alpha \notin \mathrm{Ann}_A(y),$$

which is a contradiction. Thus $\mathrm{Ann}_C(x) \subseteq \mathrm{Ann}_C(z)$.

Let $\beta \in \mathrm{Ann}_C(z)$ and $\beta \notin \mathrm{Ann}_C(x)$. We have

$$\beta \notin \mathrm{Ann}_C(x) \Rightarrow \beta \notin \mathrm{Ann}_C(y) \Rightarrow \beta \notin \mathrm{Ann}_A(y) \Rightarrow \beta \notin \mathrm{Ann}_A(z) \Rightarrow \beta \notin \mathrm{Ann}_C(z),$$

which is a contradition. Then $\mathrm{Ann}_C(z) \subseteq \mathrm{Ann}_C(x)$. Therefore $\mathrm{Ann}_C(x) = \mathrm{Ann}_C(z)$ and so $B \hookrightarrow C$ is a good extension. □

Theorem 4.5. *A (quasi) good extension is both (quasi) f-extension and f^*-extension.*

Proof. Let $B \hookrightarrow A$ be a good extension. So for every $a \in A$, there exists $b \in B$ such that $\mathrm{Ann}_A(a) = \mathrm{Ann}_A(b)$. Let $P \in \mathrm{Min}(A)$. We consider two cases:

Case 1. Let $a \in A - P$. It follows from Theorem 1.4 that $\mathrm{Ann}_A(b) = \mathrm{Ann}_A(a) \subseteq P$, by Theorem 1.4, we have $b \in B - P$, so $B \hookrightarrow A$ is an f-extension.

Case 2. Let $a \in P$. By Theorem 1.4, $\mathrm{Ann}_A(b) = \mathrm{Ann}_A(a) \nsubseteq P$, by Theorem 1.4, $b \in P$, we get $B \hookrightarrow A$ is a f^*-extension. □

Lemma 4.1. *Let A be an MV-algebra. For every $P \in \mathrm{Spec}(A)$, there exists $F \in \mathrm{Min}(A)$ such that $F \subseteq P$.*

Proof. Let $D = \{Q \in \mathrm{Spec}(A) : Q \subseteq P\}$. First we show that $D \neq \emptyset$. Since $P \in \mathrm{Spec}(A)$ and $P \subseteq P$, it follows that $D \neq \emptyset$. We define $\leqslant$ on D by

$$\forall Q_1, Q_2 \in D; \quad Q_1 \leqslant Q_2 \Leftrightarrow Q_2 \subseteq Q_1.$$

Obviously, $\leqslant$ is a partial order relation on D. Let $\{Q_i\}_{i\in I}$ be a chain of elements of D and $\bigcap_{i\in I} Q_i \in \mathrm{Spec}(A)$. We get $\bigcap_{i\in I} Q_i$ is an upper bounded of chain in D, so by Zorn's lemma, D has a maximal element F. We show that $F \in \mathrm{Min}(A)$. Let there exist $E \in \mathrm{Spec}(A)$ such that $E \subseteq F$. It follows that $F \leqslant E$. Since F is maximal of D, so $F = E$. Thus $F \in \mathrm{Min}(A)$. □

Theorem 4.6. *Let $B \hookrightarrow A$ be an f-extension of MV-algebras. Then map $\psi\colon \mathrm{Min}(A) \to \mathrm{Min}(B)$ by $\psi(P) = P \cap B$ is a bijection map.*

Proof. We first show that map ψ is a well defined map. Let $P \in \mathrm{Min}(A)$ but $P \cap B \notin \mathrm{Min}(B)$. By Lemma 4.1, there exists $Q \in \mathrm{Min}(B)$ such that $Q \subsetneqq P \cap B$. By Theorem 4.1, there exists $U \in \mathrm{Min}(A)$ such that $Q = U \cap B$. Since $U, P \in \mathrm{Min}(A)$, without loss of generality, we can let $U \nsubseteq P$, this means there exists $x \in U - P$ and by hypothesis, there exists $y \in B - P$ such that $\mathrm{Ann}_A(x) \subseteq \mathrm{Ann}_A(y)$. By Theorem 1.4, $\mathrm{Ann}_A(x) \nsubseteq U$, hence $\mathrm{Ann}_A(y) \nsubseteq U$. So there exists $e \in \mathrm{Ann}_A(y)$ such that $e \notin U$, but $0 = y \wedge e \in U$. Thus $y \in U$. Hence $y \in U \cap B = Q$, and so $y \in P$, which is a contradiction.

Now, we show that map ψ is an injection map. Let $P, Q \in \mathrm{Min}(A)$ such that $P \neq Q$. Without loss of generality choose $x \in P - Q$ and by hypothesis there exists $y \in B - Q$ such that $\mathrm{Ann}_A(x) \subseteq \mathrm{Ann}_A(y)$. But $x \in P$, by Theorem 1.4, we have $\mathrm{Ann}_A(x) \nsubseteq P$, hence $\mathrm{Ann}_A(y) \nsubseteq P$. So there exists $e \in \mathrm{Ann}_A(y)$ such that $e \notin P$ and since $0 = y \wedge e \in P$, thus $y \in P$, we get $y \in B \cap P$ but $y \notin B \cap Q$. Thus $P \cap B \neq Q \cap B$. Therefore map ψ is one to one. It follows from Theorem 4.1 that map ψ is surjective. $\square$

Corollary 4.1. *An f-extension of MV-algebras is a min-extension.*

Remark 4.1. Let A be an MV-algebra and $a \in A$. Obviously, $\mathrm{Ann}_A((a]) = \mathrm{Ann}_A(a)$.

Theorem 4.7. *An extension $B \hookrightarrow A$ is an f-extension if and only if it is a quasi f-extension.*

Proof. Let $B \hookrightarrow A$ be an f-extension and $P \in \mathrm{Min}(A)$ and $a \in A - P$. So there exists $b \in B - P$. Such that $\mathrm{Ann}_A(a) \subseteq \mathrm{Ann}_A(b)$. Let $I = (b]$. By Remark 4.1, we get $\mathrm{Ann}_A(I) = \mathrm{Ann}_A(b)$, hence $\mathrm{Ann}_A(a) \subseteq \mathrm{Ann}_A(I)$. Thus $B \hookrightarrow A$ is a quasi f-extension.

Conversely, let $B \hookrightarrow A$ be a quasi f-extension, $P \in \mathrm{Min}(A)$ and $a \in A - P$. Hence there exists finitely generated ideal I of B such that $I \nsubseteq P$ and $\mathrm{Ann}_A(a) \subseteq \mathrm{Ann}_A(I)$. Since $I \nsubseteq P$, so there exists $x \in I$ such that $x \notin P$, clearly $\mathrm{Ann}_A(I) \subseteq \mathrm{Ann}_A(x)$. Hence there exists $x \in B - P$. Such that $\mathrm{Ann}_A(a) \subseteq \mathrm{Ann}_A(x)$. Therefore $B \hookrightarrow A$ is an f-extension. $\square$

Remark 4.2. **[9]** A homeomorphism is a continuous and bijection function between topological space that has a continuous inverse function.

Theorem 4.8. *An extension $B \hookrightarrow A$ is an f-extension of MV-algebras if and only if $\psi\colon \mathrm{Min}(A) \to \mathrm{Min}(B)$ by $\psi(P) = P \cap B$ with respect to the spectral topology is a homeomorphism.*

Proof. Let $B \hookrightarrow A$ be an f-extension of MV-algebras. By Corollary 4.1 and Therorem 4.3, we get map ψ with respect to the spectral topology is continuous and by Theorem 4.6, map ψ is bijection. Let $a \in A$. We have $\psi(U_A(a)) = \{P \cap B : P \in \mathrm{Min}(A),\ a \notin P\}$. Let $P \cap B \in \psi(U_A(a))$. Then $a \notin P$. By hypothesis there exists $b \in B - P$. Such that $\mathrm{Ann}_A(a) \subseteq \mathrm{Ann}_A(b)$. Hence $P \cap B \in U_B(b)$ so $\psi(U_A(a)) \subseteq U_B(b)$. Now, let $M \in U_B(b)$. Then by Theorem 4.1, there exists $Q \in \mathrm{Min}(A)$ such that $M = Q \cap B$. Since $b \notin M$, we get $b \notin Q$. Hence $\mathrm{Ann}_A(b) \subseteq Q$. But $\mathrm{Ann}_A(a) \subseteq \mathrm{Ann}_A(b) \subseteq Q$, by Theorem 1.4, we obtain $a \notin Q$. Hence $Q \in U_A(a)$, so $M = Q \cap B \in \psi(U_A(a))$. We get $U_B(b) \subseteq \psi(U_A(a))$. Therefore $U_B(b) = \psi(U_A(a))$. Hence ψ is an open map.

Conversely, let $P \in \mathrm{Min}(A)$, $a \in A - P$. Then $P \in U_A(a)$. Since map ψ is a homeomorphism, $\psi(U_A(a))$ is a subset open of $\mathrm{Min}(B)$. Hence there exists $I \in \mathrm{Id}(B)$ such that $U_B(I) = \psi(U_A(a))$. But $P \in U_A(a)$, so $\psi(P) = P \cap B \in U_B(I)$, thus $I \nsubseteq P$, so there exists $b \in I$ such that $b \notin P$. We show that $\mathrm{Ann}_A(a) \subseteq \mathrm{Ann}_A(b)$. Let $x \in \mathrm{Ann}_A(a)$. Then $x \in A$ and $x \wedge a = 0$. Suppose $Q \in \mathrm{Min}(A)$. We consider two cases:

Case 1. Let $Q \in V_A(b)$. Then $b \in Q$, Since $b \wedge x \leqslant b$, we obtain $b \wedge x \in Q$.

Case 2. Let $Q \in U_A(b)$. Then $b \notin Q$, hence $I \nsubseteq Q$. Since $I \subseteq B$ and $I \nsubseteq Q$, we have $I \nsubseteq Q \cap B$. Thus $\psi(Q) = Q \cap B \in U_B(I) = \psi(U_A(a))$. But map ψ is bijection, then $a \notin Q$. Since $a \wedge x = 0$, we get $x \in Q$. Since $x \in Q$ and $b \wedge x \leqslant x$, we obtain $b \wedge x \in Q$. Hence for every $Q \in \mathrm{Min}(A)$, $b \wedge x \in Q$. By Lemma 1.6, we obtain $b \wedge x = 0$. Therefore $x \in \mathrm{Ann}_A(b)$. Hence $B \hookrightarrow A$ is an f-extension of MV-algebras. □

Corollary 4.2. *$B \hookrightarrow A$ is a quasi f-extension of MV-algebras if and only if $\psi\colon \mathrm{Min}(A) \to \mathrm{Min}(B)$ by $\psi(P) = P \cap B$ with respect to the spectral topology is a homeomorphism.*

Theorem 4.9. *Suppose that $B \hookrightarrow A$ is an f-extension of MV-algebras. $B \hookrightarrow A$ is a good extension if and only if $\psi\colon \mathrm{Min}(A) \to \mathrm{Min}(B)$ by $\psi(P) = P \cap B$, maps basis open sets to basis open sets, with respect to the spectral topology.*

Proof. Let $B \hookrightarrow A$ be an f-extension of MV-algebras. Hence for every $a \in A$, there exists $b \in B$ such that $\mathrm{Ann}_A(a) = \mathrm{Ann}_A(b)$. We show that $\psi(U_A(a)) = U_A(b)$. Let $P \in \mathrm{Min}(A)$ and $a \in A - P$. Then $P \in U_A(a)$ and $\psi(P) = P \cap B \in \mathrm{Min}(B)$. By Theorem 1.4, we get $\mathrm{Ann}_A(a) = \mathrm{Ann}_A(b) \subseteq P$, and so by Theorem 1.4, we conclude $b \notin P$. Then $b \notin P \cap B$, that is, $P \cap B \in U_B(b)$. Hence $\psi(U_A(a)) \subseteq U_B(b)$. Let $M \in U_B(b)$. We get $M \in \mathrm{Min}(B)$ and $b \notin M$, so by Theorem 4.1, there exists $Q \in \mathrm{Min}(A)$ such that $\psi(Q) = Q \cap B = M$. If $a \in Q$, then by Theorem 1.4, $\mathrm{Ann}_A(a) \nsubseteq Q$ and Since $\mathrm{Ann}_A(a) \subseteq \mathrm{Ann}_A(b)$, we have $\mathrm{Ann}_A(a) \subseteq \mathrm{Ann}_A(b) \nsubseteq Q$ and by Theorem 1.4, we have $b \in Q$. Hence $b \in M$, which is a contradiction. So $a \notin Q$, that is, $Q \in U_A(a)$. But $\psi(Q) = M$, we get $M \in \psi(U_A(a))$, hence $U_B(b) \subseteq \psi(U_A(a))$. Therefore $\psi(U_A(a)) = U_B(b)$.

Conversely, let $a \in A$. By hypothesis, there exists $b \in B$ such that $\psi(U_A(a)) = U_B(b)$. We show that $\mathrm{Ann}_A(a) = \mathrm{Ann}_A(b)$. Let $P \in \mathrm{Min}(A)$. We consider two cases:

Case 1. Let $P \in U_A(a)$. By hypothesis, we get $\psi(P) = P \cap B \in U_B(b)$, so $b \notin P \cap B$, by Theorem 1.4, we get $\mathrm{Ann}_A(b) \subseteq P$ and $\mathrm{Ann}_A(a) \subseteq P$. Since for all $x \in \mathrm{Ann}_A(a)$ and $y \in \mathrm{Ann}_A(b)$, we have $x \wedge b \leqslant x$ and $y \wedge a \leqslant y$, we obtain $a \wedge \mathrm{Ann}_A(b) \subseteq P$ and $b \wedge \mathrm{Ann}_A(a) \subseteq P$.

Case 2. Let $P \in V_A(a)$. By hypothesis, we conclude $\psi(P) = P \cap B \in V_B(b)$, hence $b \in P \cap B$. By Theorem 1.4, we get $\mathrm{Ann}_A(b) \nsubseteq P$ and $\mathrm{Ann}_A(a) \nsubseteq P$. Since for all $x \in \mathrm{Ann}_A(a)$ and $y \in \mathrm{Ann}_A(b)$, we have $x \wedge b \leqslant b$ and $y \wedge a \leqslant a$, hence $a \wedge \mathrm{Ann}_A(b) \subseteq P$ and $b \wedge \mathrm{Ann}_A(a) \subseteq P$.

We imply that for all $P \in \mathrm{Min}(A)$, $a \wedge \mathrm{Ann}_A(b) \subseteq P$ and $b \wedge \mathrm{Ann}_A(a) \subseteq P$. By Lemma 1.6, we obtain $a \wedge \mathrm{Ann}_A(y) = b \wedge \mathrm{Ann}_A(a) = 0$. Let $x \in \mathrm{Ann}_A(a)$. Then

$b \wedge x = 0$, and so $x \in \mathrm{Ann}_A(b)$, thus $\mathrm{Ann}_A(a) \subseteq \mathrm{Ann}_A(b)$. By similar way, we can obtain $\mathrm{Ann}_A(b) \subseteq \mathrm{Ann}_A(a)$. We get $\mathrm{Ann}_A(a) = \mathrm{Ann}_A(b)$. Therefore $B \hookrightarrow A$ is a good extension of MV-algebras. $\square$

LEMMA 4.2. *Let $B \hookrightarrow A$ be a extension of MV-algebras. For every $P \in \mathrm{Min}(A)$ and finitely generated ideal I of $B, I \subseteq P \cap B$ if and only if $\mathrm{Ann}_A(I) \nsubseteq P$.*

PROOF. Let $I \subseteq P \cap B$ and $J = I^e$. Hence $J \subseteq P$, It follows from Theorem 1.4 that $\mathrm{Ann}_A(J) \nsubseteq P$. By Remark 4.1, we get $\mathrm{Ann}_A(J) = \mathrm{Ann}_A(I)$, and so $\mathrm{Ann}_A(I) \nsubseteq P$.

Conversely, let $\mathrm{Ann}_A(I) \nsubseteq P$. Then there exists $\alpha \in \mathrm{Ann}_A(I)$ such that $\alpha \notin P$. We get $\alpha \wedge I = 0$. Thus $I \subseteq P$. Therefore $I \subseteq P \cap B$. $\square$

THEOREM 4.10. *Let $B \hookrightarrow A$ be a quasi f^*-extension of MV-algebras. Then map $\psi\colon \mathrm{Min}(A) \to \mathrm{Min}(B)$ by $\psi(P) = P \cap B$ is a bijection.*

PROOF. First, we show that map ψ is well defined. Let $P \in \mathrm{Min}(A)$. Then $\psi(P) = P \cap B \in \mathrm{Spec}(B)$. Suppose that there exists $Q \in \mathrm{Min}(B)$ such that $Q \subsetneqq P \cap B$. By Theorem 4.1, there exists $M \in \mathrm{Min}(A)$ such that $Q = M \cap B$. If $M \neq P$, then by hypothesis for every $a \in M - P$, there exists finitely generated ideal I of B such that $I \subseteq M \cap B$ and $\mathrm{Ann}_A(I) \subseteq \mathrm{Ann}_A(a)$, since $I \subseteq Q \subseteq P$, by Lemma 4.2, we obtain $\mathrm{Ann}_A(I) \nsubseteq P$. Hence $\mathrm{Ann}_A(a) \nsubseteq P$, so there exists $x \in \mathrm{Ann}_A(a)$ such that $x \notin P$. Since $x \wedge a = 0$ and $x \notin P$, we conclude that $a \in P$, which is a contradiction. Thus $M = P$ and $\psi(P) = P \cap B \in \mathrm{Min}(B)$. Now, we show that map ψ is injection. Let $P, Q \in \mathrm{Min}(A)$ such that $P \neq Q$. Without loss of generality, let $a \in P - Q$. By hypothesis, there exists finitely generated ideal I of B such that $I \subseteq P \cap B$ and $\mathrm{Ann}_A(I) \subseteq \mathrm{Ann}_A(a)$. Since $a \notin Q$, by Theorem 1.4, $\mathrm{Ann}_A(a) \subseteq Q$. Hence $\mathrm{Ann}_A(I) \subseteq Q$ and by Lemma 4.2, we deduce that $I \nsubseteq Q \cap B$. Hence there exists $r \in I - Q$ such that $r \in P \cap B - Q \cap B$. So $P \cap B \neq Q \cap B$. It is follows $\psi(P) \neq \psi(Q)$. By Theorem 4.1, we imply that ψ is surjective. Therefore map ψ is bijective. $\square$

COROLLARY 4.3. *Every quasi f^*-extension $B \hookrightarrow A$ of MV-algebras, is a min-extension.*

THEOREM 4.11. *Suppose $B \hookrightarrow A$ is a extension of MV-algebras. A is a quasi f^*-extension of B if and only if map $\psi\colon \mathrm{Min}(A) \to \mathrm{Min}(B)$ by $\psi(P) = P \cap B$ with respect to the inverse topology is a homeomorphism.*

PROOF. Let $B \hookrightarrow A$ be a quasi f^*-extension of MV-algebras. By Theorem 4.10, map ψ is bijection and by Theorem 4.3 map ψ is continuous respect to inverse topology. Now we prove that ψ^{-1} is continuous. Let $a \in A$ and $Q \in \psi(V_A(a))$. Then there exists $P \in V_A(a)$ such that $\psi(P) = P \cap B = Q$. By hypothesis there exists finitely generated ideal I of B such that $I \subseteq P \cap B = Q$ and $\mathrm{Ann}_A(I) \subseteq \mathrm{Ann}_A(a)$. Hence $Q \in V_B(I)$, that is, $\psi(V_A(a)) \subseteq V_B(I)$. Let $M \in V_B(I)$. We get $M \in \mathrm{Min}(B)$ and $I \subseteq M$. By Theorem 4.1, there exists $T \in \mathrm{Min}(A)$ such that $T \cap B = M$, so $I \subseteq T$. It follows from Theorem 1.4 that $\mathrm{Ann}_B(b) \subseteq T$. So $\mathrm{Ann}_A(a) \nsubseteq T$, by Theorem 1.4, we get $a \in T$. Hence $M = T \cap B \in \psi(V_A(a))$, and so $V_B(I) \subseteq \psi(V_A(a))$. Therefore $\psi(V_A(a)) = V_B(I)$.

Conversely, let map ψ respect to the inverse topology be a homeomorphism. Suppose $P \in \mathrm{Min}(A)$ and $a \in P$. Since $V_A(a)$ is an open subset of $\mathrm{Min}^{-1}(A)$, so $\psi(V_A(a))$ is a open subset of $\mathrm{Min}^{-1}(B)$ such that $P \cap B \in \psi(V_A(a))$. Since map ψ is open, there exists finitely generated ideal I of B such that $P \cap B \in V_B(I) = \psi(V_A(a))$. We show that $\mathrm{Ann}_A(I) \subseteq \mathrm{Ann}_A(a)$. Let $t \in \mathrm{Ann}_A(I)$ and $Q \in \mathrm{Min}(A)$. We consider two cases:

CASE 1. Let $a \in Q$, since $a \wedge t \leqslant a$, we get $a \wedge t \leqslant Q$.

CASE 2. Let $a \notin Q$, hence $Q \notin V_A(a)$. Since map ψ is bijection, we obtain $Q \cap B \notin \psi(V_A(a))$. Hence we get $Q \cap B \notin V_B(I)$. Thus $I \nsubseteq Q \cap B$. Then $I \nsubseteq Q$. If $x \in \mathrm{Ann}_A(I)$ then $0 = x \wedge I \subseteq Q$, that is, $x \in Q$. Hence $\mathrm{Ann}_A(I) \subseteq Q$, so $t \in Q$. Since $a \wedge t \leqslant t$, we have $a \wedge t \in Q$.

Therefore for all $Q \in \mathrm{Min}(A)$, $a \wedge t \in Q$. By Lemma 4.2, we have $a \wedge t = 0$. So $t \in \mathrm{Ann}_A(a)$. Hence $\mathrm{Ann}_A(I) \subseteq \mathrm{Ann}_A(a)$. □

THEOREM 4.12. *Let $B \hookrightarrow A$ be a quasi f^*-extension of MV-algebras. Then $B \hookrightarrow A$ is a quasi good extension if and only if ψ: $\mathrm{Min}(A) \to \mathrm{Min}(B)$ by $\psi(P) = P \cap B$, maps basis open sets to inverse topology basis open sets, with respect to the inverse topology.*

PROOF. Let $B \hookrightarrow A$ be a quasi f^*-extension. By Corollary 4.3, map ψ is well defined. Let $J = (a_1, a_2, \ldots, a_n]$ be a finitely generated ideal of A. So for every $1 \leqslant i \leqslant n$, there exists a finitely generated ideal I_i of B such that $\mathrm{Ann}_A(I_i) = \mathrm{Ann}_A(a_i)$. Let $I = I_1 \vee I_2 \vee \cdots \vee I_n$. By Remark 1.5, I is a finitely generated ideal. We show that $\psi(V_A(J)) = V_B(I)$. Let $P \in V_A(J)$, hence $\psi(P) = P \cap B \in \psi(V_A(J))$ and $J \subseteq P$. By Theorem 1.4, for every $1 \leqslant i \leqslant n$, $\mathrm{Ann}_A(a_i) \nsubseteq P$, so $\mathrm{Ann}_A(I_i) \nsubseteq P$, we obtain there exists $x \in \mathrm{Ann}_A(I_i)$ such that $x \notin P$. Since $x \wedge I_i = 0$ and $x \notin P$, we have $I_i \subseteq P$. Hence $I \subseteq P \cap B$. We get $P \cap B \in V_B(I)$, it follows that $\psi(V_A(J)) \subseteq V_B(I)$. Let $Q \in V_B(I)$. Then $Q \in \mathrm{Min}(B)$ and $I \subseteq Q$. By Theorem 4.1, there exists $P \in \mathrm{Min}(A)$ such that $Q = P \cap B$, hence $I \subseteq P$. By Theorem 1.4, $\mathrm{Ann}_A(I) \nsubseteq P$, so for every $1 \leqslant i \leqslant n, \mathrm{Ann}_A(I_i) \nsubseteq P$, thus $\mathrm{Ann}_A(a_i) \nsubseteq P$. We get that there exists $x \in \mathrm{Ann}_A(a_i)$ such that $x \notin P$. Obviously $a_i \in P$, hence $J \subseteq P$. So we have $P \in V_A(J)$ and $\psi(P) = P \cap B = Q$. Hence $V_B(I) \subseteq \psi(V_A(J))$. Therefore $V_B(I) = \psi(V_A(J))$.

Conversely, let $a \in A$. Put $J = (a]$. By hypothesis, there exists finitely generated ideal I of B such that $\psi(V_A(J)) = V_B(I)$. We show that $\mathrm{Ann}_A(a) = \mathrm{Ann}_A(I)$. Let $P \in \mathrm{Min}(A)$. We consider two cases:

CASE 1. Let $P \in U_A(I)$. Then $\psi(P) = P \cap B \in U_B(I)$, we get $a \notin P$ and $I \nsubseteq P \cap B$. By Theorem 1.4, we obtain $\mathrm{Ann}_A(a) \subseteq P$ and $\mathrm{Ann}_A(I) \subseteq P$. We know that for every $x \in \mathrm{Ann}_A(I)$, $\alpha \in I$ and $y \in \mathrm{Ann}_A(a)$, we have $a \wedge x \leqslant x$ and $\alpha \wedge y \leqslant y$. Hence $a \wedge x \in P$ and $\alpha \wedge y \in P$.

CASE 2. Let $P \in V_A(I)$. Then $\psi(P) = P \cap B \in V_B(I)$, we deduce that $a \in P$ and $I \subseteq P \cap B$. By Theorem 1.4, we get $\mathrm{Ann}_A(a) \nsubseteq P$ and $\mathrm{Ann}_A(I) \nsubseteq P$. We know that for every $x \in \mathrm{Ann}_A(I)$, we have $\alpha \in I$ and $y \in \mathrm{Ann}_A(a)$, $a \wedge x \leqslant a$ and $\alpha \wedge y \leqslant \alpha$. Hence $a \wedge x \in P$ and $\alpha \wedge y \in P$.

Therefore for every $P \in \mathrm{Min}(A)$, we get $a \wedge \mathrm{Ann}_A(I) \subseteq P$ and $I \wedge \mathrm{Ann}_A(a) \subseteq P$. By Lemma 1.6, we have $a \wedge \mathrm{Ann}_A(I) = 0$ and $I \wedge \mathrm{Ann}_A(a) = 0$. Now, we show that $\mathrm{Ann}_A(a) = \mathrm{Ann}_A(I)$. Let $x \in \mathrm{Ann}_A(a)$. Then $x \wedge I = 0$, we get $x \in \mathrm{Ann}_A(I)$. Hence $\mathrm{Ann}_A(a) \subseteq \mathrm{Ann}_A(I)$. Similarly, $\mathrm{Ann}_A(I) \subseteq \mathrm{Ann}_A(a)$, we conclude $\mathrm{Ann}_A(a) = \mathrm{Ann}_A(I)$. □

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