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IRREDUCIBILITY CRITERION AND 2-PISOT SERIES IN POSITIVE CHARACTER

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Abstract. Let $\mathbb{F}_q((X^{-1}))$ be the field of formal power series over a finite field $\mathbb{F}_q$. We characterize a pair of roots that lies outside the unit disc while all remaining conjugates have a modulus strictly less than 1. In particular, we provide a sufficient condition for a pair of formal power series to be a 2-Pisot series. We also give an irreducibility criterion over $\mathbb{F}_q[X]$.

1. Introduction

A Pisot number is an algebraic integer $\theta > 1$ having all its conjugates $\neq \theta$ of modulus < 1. It is known that the positive root $\theta_0 \simeq 1.3247$ of $z^3 - z - 1$ is the smallest Pisot number [**12**]. According to Dufresnoy and Pisot [**5**], the smallest limit point of the set S_1 of Pisot numbers is $\frac{1+\sqrt{5}}{2}$. The algorithm that they developed [**4**] was powerful and served to classify all Pisot numbers less than $\frac{1+\sqrt{5}}{2}$. In 1944, Salem proved that S_1 is closed [**10**].

A complex Pisot number is a non-real algebraic integer α, with $|\alpha| > 1$, whose remaining conjugates other than α lie in the open unit disk. Without loss of generality, we may require the real part of a complex Pisot number to be non-negative. The smallest complex Pisot number was provided by Chamfy [**3**] with modulus $\sqrt{\alpha_0} \cong 1.1509$, for which either $z^3 - z^2 + 1$ or $z^6 - z^2 + 1$ is a minimal polynomial.

A 2-Pisot number is a pair (α_1, α_2) of conjugate algebraic integers of modulus > 1 whose remaining conjugates have modulus < 1. The set S_2 of all 2-Pisot numbers can be partitioned as $S_2 = S_2' \cup S_2''$, with S_2' being the set of those 2-Pisot numbers with $\alpha_1, \alpha_2 \in \mathbb{R}$ and S_2'' being the set of 2-Pisot numbers with $\alpha_1, \alpha_2 = \overline{\alpha_1}$ non-real (so they are complex conjugates of each other).

In 1950, Kelley proved that the set $S_1 \cup S_2''$ is closed [**9**]. However, it is impossible that S_2' be closed, because it includes a dense, enumerable subset of real quadratic integers. He also proved that the set S_2' cannot be closed even if we omit the

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quadratic integers. According to Samet [**11**], the set S'_2 of degree $n > 2$ is dense in $|x| > 1$. The study of limit points of the set S_1 has attracted numerous researches and to a lesser extent the set S_2. In this regard, it is legitimate to figure out if the smallest limit point of S_2, the set of complex Pisot numbers, is the one with minimal polynomial $z^4 + z^2 - 1$, having modulus $\frac{\sqrt{1+\sqrt{5}}}{2} \cong 1.2720$. Also, Garth [**6**] discussed how to use Chamfy's algorithm to determine all complex Pisot numbers of modulus less than $\zeta = 1.17$.

In [**2**], Cantor studied certain k-tuples of algebraic integers which generalize these ideas. Distinctly, let $(\alpha_1, \ldots, \alpha_k)$ be a k-tuple of distinct algebraic integers with an absolute value strictly greater than 1, and let $P(z)$ be the monic polynomial of the lowest degree with integer coefficients which $\alpha_1, \ldots, \alpha_k$ all satisfy. If the remaining roots of $P(z)$ lie in the open unit disk, then $(\alpha_1, \ldots, \alpha_k)$ is called a Pisot k-tuple, and $P(z)$ is its defining polynomial. Moreover, if $P(z)$ is irreducible, then $(\alpha_1, \ldots, \alpha_k)$ is said to be an irreducible Pisot k-tuple. Cantor [**2**] focused on Pisot k-tuples whose defining polynomials are reciprocal. Recall that a polynomial $P(z)$ of degree d is reciprocal if $P(z) = z^d P(1/z)$. He ultimately claimed that a Pisot k-tuple is reciprocal if its defining polynomial is reciprocal.

In [**13**], Pisot showed that every reciprocal quadratic unit is a limit point of S_1. Samet [**11**], corroborated on this theory by asserting that every reciprocal biquadratic unit in S_2, i.e., every reciprocal unit of degree 4 in S_2 is a limit point of S_2.

However, it is still not determined whether reciprocal Pisot k-tuples are limits of pisot k-tuples. Although this question is generally challenging, Garth [**7**] gave conditions for which the assertion turns out to be true.

This manuscript concerns an analogue of 2-Pisot numbers S'_2 defined over the ring $\mathbb{F}_q[X]$, where $\mathbb{F}_q$ is the finite field of q elements, with the field $\mathbb{R}$ being replaced by $\mathbb{F}_q((X^{-1}))$. The analogues of the Pisot numbers have been previously studied in this context by e.g., Bateman and Duquette [**1**]. An attempt is also made to extend some of their results to these numbers.

The paper is organized as follows: In Section 2, some preliminary definitions are given to present the field of formal power series over a finite field. We present some important results of Bateman and Duquette [**1**] which characterize Pisot elements in the field of formal power series. In Section 3, certain arithmetical proprieties of 2-Pisot are discussed and an irreducibility criterion in the case of formal power series is provided giving the series of 2-Pisot elements. We also formulate in Theorem 3.2, a criterion to construct 2-Pisot elements in the field of formal series over a finite field.

2. Field of formal series

For p a prime and q a power of p, let $\mathbb{F}_q$ be a field with q elements of characteristic p, $\mathbb{F}_q[X]$ the set of polynomials with coefficients in $\mathbb{F}_q$ and $\mathbb{F}_q(X)$ its field of fractions. The set $\mathbb{F}_q((X^{-1}))$ of Laurent series over $\mathbb{F}_q$ is defined as follows:

$$\mathbb{F}_q((X^{-1})) = \left\{ f = \sum_{i \geqslant n_0} f_i X^{-i} : n_0 \in \mathbb{Z} \quad \text{and} \quad f_i \in \mathbb{F}_q \right\}.$$

Let $f = \sum_{n \geqslant n_0} f_n X^{-n}$ be any formal power series, its polynomial part is denoted by $[f] \in \mathbb{F}_q[X]$ and $\{f\}$ its fractional part. We remark that $f = [f] + \{f\}$. If $f \neq 0$, then the polynomial degree of f is $\gamma(f) = \sup\{-i : f_i \neq 0\}$, the degree of the highest-degree nonzero monomial in f, and $\gamma(0) = -\infty$. Note that if $[f] \neq 0$ then $\gamma(f)$ is the degree of the polynomial $[f]$. Thus, we define $|f| = q^{\gamma(f)}$. Note that $|\,.\,|$ is a no Archimedean absolute value over $\mathbb{F}_q((X^{-1}))$. It is clear that, for all $P \in \mathbb{F}_q[X], |P| = q^{\deg P}$ and, for all $Q \in \mathbb{F}_q[X]$, such that $Q \neq 0, |\frac{Q}{P}| = q^{\deg P - \deg Q}$. We know that $\mathbb{F}_q((X^{-1}))$ is complete and locally compact with respect to the metric defined by this absolute value. We denote by $\overline{\mathbb{F}_q((X^{-1}))}$ an algebraic closure of $\mathbb{F}_q((X^{-1}))$. We note that the absolute value has a unique extension to $\overline{\mathbb{F}_q((X^{-1}))}$. Abusing the notation a little, we will use the same symbol $|\,.\,|$ for the two absolute values.

A Pisot series $w \in \mathbb{F}_q((X^{-1}))$ is an algebraic integer over $\mathbb{F}_q[X]$ such that $|w| > 1$ whose remaining conjugates in $\overline{\mathbb{F}_q((X^{-1}))}$ have an absolute value strictly smaller than 1. The set of all Pisot series is denoted S^*.

Since $\mathbb{F}_q[X] \subset \mathbb{F}_q((X^{-1}))$, every algebraic element over $\mathbb{F}_q[X]$ can be evaluated. However, $\mathbb{F}_q((X^{-1}))$ is not algebraically closed. Such an element is not necessarily expressed as a power series. For a full characterization of the algebraic closure of $\mathbb{F}_q[X]$, we refer to Kedlaya [**8**].

Definition 2.1. [**14**]. Let

$$f(X,Y) = A_m Y^m + A_{m-1} Y^{m-1} + \cdots + A_1 Y + A_0 \in \mathbb{F}_q[X,Y], \quad A_i \in \mathbb{F}_q[X] \tag{2.1}$$

be irreducible of $F_q[X,Y]$. To each monomial $A_i Y^i \neq 0$, we assign the point $(i, \deg(A_i)) \in \mathbb{Z}^2$. For $A_i = 0$, we ignore the corresponding point $(i, -\infty)$.

If we consider the upper convex hull of the set of points

$$\{(0, \deg(A_0)), \ldots, (m, \deg(A_m))\},$$

we obtain the so-called upper Newton polygon of $f(X,Y)$ with respect to Y. The polygon is a sequence of line segments $E_1, E_2, \ldots, E_t$, with monotonously decreasing slopes.

Proposition 2.1. *[**14**]. Let $f(X,Y) \in \mathbb{F}_q[X,Y] \subset \mathbb{F}_q((Y^{-1}))[X]$ be of the form (2.1). Since $\mathbb{F}_q((Y^{-1}))$ is complete with respect to $|\,.\,|$, there is a unique extension of $|\,.\,|$ to the splitting field L of $f(X,Y)$ over $K = \mathbb{F}_q((Y^{-1}))$.*

Let $1 = r < r+s = m$. We define E to be the line joining the points $(r, deg(A_r))$ and $(r+s, \deg(A_{r+s}))$, which has a slope

$$k = \frac{\deg(A_{r+s}) - \deg(A_r)}{s}.$$

Then $f(X,Y)$, as a polynomial in Y, has s roots $\alpha_1, \ldots, \alpha_s \in L$ with $|\alpha_1| = \cdots = |\alpha_s| = q^{-k}$.

Corollary 2.1. *There are no roots in $\mathbb{F}_q((X^{-1}))$ with absolute value > 1 of the polynomial $H(Y) = A_n Y^n + A_{n-1} Y^{n-1} + \cdots + A_0$, where $|A_n| = \sup_{0 \leqslant i \leqslant n} |A_i|$.*

Proof. Immediately proved by Proposition 2.1. ∎

COROLLARY 2.2. *Let* $P(Y) = A_nY^n + A_{n-1}Y^{n-1} + \cdots + A_0$, *with* $A_i \in \mathbb{F}_q[X]$, $A_n = 1$, $A_0 \neq 0$ *and* $|A_{n-1}| > |A_i|$, *for all* $i \neq n-1$. *Then,* P *has only one root* $f \in \mathbb{F}_q((X^{-1}))$ *satisfying* $|f| > 1$. *Moreover,* $[f] = -\left[\frac{A_{n-1}}{A_n}\right]$.

PROOF. The first part follows easily from Proposition 2.1. For the second part, we use the fact that $\left[\frac{A_{n-1}}{A_n}\right]$ is the sum of the roots of the polynomial P. $\square$

In 1962, Bateman and Duquette [**1**] introduced and characterized the Pisot elements in the field of Laurent series.

THEOREM 2.1. *[**1**]. An element* f *in* $\mathbb{F}_q((X^{-1}))$ *is a Pisot element if and only if its minimal polynomial can be written as* $P(Y) = Y^s + A_{s-1}Y^{s-1} + \cdots + A_0$, $A_i \in \mathbb{F}_q[X]$ *for* $i = 0, \ldots, s-1$ *with* $|A_{s-1}| > |A_i|$ *for* $i = 0, \ldots, s-2$.

THEOREM 2.2. *[**1**]. An element* $f \in \mathbb{F}_q((X^{-1}))$ *is a Pisot number if and only if there exists* $\lambda \in \mathbb{F}_q((X^{-1})) \smallsetminus \{0\}$ *such that* $\lim_{n\to+\infty}\{\lambda f^n\} = 0$*; Moreover* λ *can be chosen to belong to* $\mathbb{F}_q(X)(f)$.

A 2-Pisot series is a pair $(w_1, w_2) \in (\mathbb{F}_q((X^{-1})))^2$ of conjugate algebraic integers over $\mathbb{F}_q[X]$ of absolute value > 1 whose remaining conjugates in $\overline{\mathbb{F}_q((X^{-1}))}$ have an absolute value < 1. The set of all 2-Pisot series is denoted by S_2^* and $(S_2^*)''$ is the subset of 2-Pisot series (w_1, w_2) which are not in $(\mathbb{F}_q((X^{-1})))^2$ but (w_1^2, w_2^2) in $(\mathbb{F}_q((X^{-1})))^2$.

Note that a 2-Pisot element is necessarily separable over $\mathbb{F}_q(x)$ and also has an absolute value greater than 1. Another immediate consequence of the definition is a positive integral power of a 2-Pisot element.

The following original examples illustrate the previous definition.

EXAMPLE 2.1 (A 2-Pisot series of degree 3 on $\mathbb{F}_2((X^{-1}))$). Let the polynomial P be defined by

$$P(Y) = Y^3 + (X^2+1)Y^2 + (X^3+X)Y + 1 \in \mathbb{F}_2[X][Y].$$

Then it is irreducible over $\mathbb{F}_2[X]$ and has 3 roots defined by:

$$W_1 = \sum_{i=-2}^{\infty} w_i X^{-i}, \quad W_2 = \sum_{i=-1}^{\infty} w_i X^{-i} \quad \text{and} \quad W_3 = \sum_{i=1}^{\infty} w_i X^{-i}.$$

Then $W_i \in \mathbb{F}_2((X^{-1}))$ for $i = 1, 2$. Indeed

$$\begin{aligned} W_1 &= X^2 + X + \frac{1}{X^2} + \cdots = X^2 + X + \frac{1}{Z_1} && \text{such that } |Z_1| > 2; \\ W_2 &= X + 1 + \frac{1}{X} + \cdots = X + 1 + \frac{1}{Z_2} && \text{such that } |Z_2| > 1; \\ W_3 &= \frac{1}{X} + \cdots = \frac{1}{Z_3} && \text{such that } |Z_3| > 1. \end{aligned}$$

We have

$$W_1^3 + (X^2+1)W_1^2 + (X^3+X)W_1 + 1 = 0.$$

Then

$$Z_1^3 + (X^4 + X^3 + X^2 + X)Z_1^2 + (X+1)Z_1 + 1 = 0.$$

According to Corollary 2.2, $Z_1 \in \mathbb{F}_2((X^{-1}))$, so $W_1 \in \mathbb{F}_2((X^{-1}))$. In the same way, we can prove that $W_2 \in \mathbb{F}_2((X^{-1}))$. Therefore $(W_1, W_2) \in (S_2^*)$.

EXAMPLE 2.2 (A pair of $(S_2^*)''$). Let the polynomial P be defined by

$$P(Y) = Y^4 - 2X^2Y^2 - 2 \in \mathbb{F}_3[X][Y].$$

P is irreducible over $\mathbb{F}_3[X]$ and has two roots of modulus strictly greater than 1 defined by

$$(w_1, w_2) = \Big(a\Big(X - \frac{1}{X^3} + \dots\Big), -a\Big(X - \frac{1}{X^3} + \dots\Big)\Big)$$

by choosing a to be a square in $\mathbb{F}_3$, which (w_1, w_2) is not in $(\mathbb{F}_3((X^{-1})))^2$ since $\sqrt{2}$ is in $\mathbb{F}_9$ but not in $\mathbb{F}_3$ and the others have a modulus strictly less then 1. Thus $(w_1, w_2) \in (S_2^*)''$.

Let us remember that $\mathbb{F}_q((X^{-1}))$ contains 2-Pisot series of any degree over $\mathbb{F}_q(X)$. Indeed, consider the polynomial $Y^n - aX^2Y^{n-2} - b$, where $a, b \in \mathbb{F}_q^*$. It can be easily seen that the polynomial is irreducible over $\mathbb{F}_q[x]$. Furthermore if a is a square in $\mathbb{F}_q$, then the polynomial has two roots $(w_1, w_2) \in \mathbb{F}_q((X^{-1}))$ such that $|w_1| > 1, |w_1| > 1$ and all of its conjugates in $\overline{\mathbb{F}_q((X^{-1}))}$ have an absolute value strictly smaller than 1.

3. Results

3.1. 2-Pisot series arithmetical proprieties. In this section, we discuss some analogous results to those known about 2-Pisot numbers in the real case. Our main results are the following propositions.

PROPOSITION 3.1. *Let* $(w_1, w_2) \in S_2^*$, *then* $(w_1^n, w_2^n) \in S_2^*$, *for all* $n \in \mathbb{N}^*$.

PROOF. Let $(w_1, w_2) \in S_2^*$ and $M \in \mathbb{F}_q[X][Y]$ the minimal polynomial of w and $w = w_1, \dots, w_d$ the conjugates of w. Then there exists exactly 2 conjugates $w = w_1, w_2$ of w that lie outside the unit disc. Let $w_3, \dots, w_d$ denote the other roots of M.

We know that the product of any two algebraics is, itself, an algebraic. Since w_1 is an algebraic, then for all $n \in \mathbb{N}$, w_1^n a is also an algebraic. Let $P \in \mathbb{F}_q[X][Y]$ be the minimal polynomial of w_1^n. Now, we consider embedding σ_i of $\mathbb{F}_q(X)(w_1)$ into $\overline{\mathbb{F}_q((X^{-1}))}$, which fixes $\mathbb{F}_q(X)$ and maps w_1 to w_i.

$$P(w_i^n) = P((\sigma_i(w_1)^n)) = P(\sigma_i(w_1^n)) = \sigma_i(P(w_1^n)) = \sigma_i(0) = 0.$$

So for all $i \leqslant d$, w_i^n satisfies $P(Y) = 0$. We have,

$$[\mathbb{F}_q(X)(w_1^n) : \mathbb{F}_q(X)] \leqslant [\mathbb{F}_q(X)(w_1) : \mathbb{F}_q(X)].$$

This shows that $\deg(P) \leqslant \deg(M)$. So $w_1^n, w_2^n, \dots, w_d^n$ are all the roots of P. If $3 \leqslant i \leqslant d$, then $|w_i^n| = |w_i|^n < 1$ and $|w_1^n| = |w_i|^n > 1$ for $i = 1, 2$. Therefore $(w_1^n, w_2^n) \in S_2^*$ for all $n \in \mathbb{N}^*$. □

PROPOSITION 3.2. *Let* $(w_1, w_2) \in (S_2^*)$; *then* $\lim_{n\to+\infty}\{w_1^n + w_2^n\} = 0$.

Proof. Let (w_1, w_2) be a 2-Pisot series and $w_1, \dots, w_d$ its conjugates. From the preceding propositions results, for all $n \in \mathbb{N}$; w_1^n, w_2^n are the roots of the same degree d irreducible polynomial, P_n in $\mathbb{F}_q[X]$. Also, $\mathrm{Tr}(P_n) = \sum_{i=1}^d w_i^n \in \mathbb{F}_q[X]$. So $\{\mathrm{Tr}(P_n)\} = 0$. The above can be rewritten as

$$\Big\{ \mathrm{Tr}(P_n) = \sum_{i=1}^{d} w_i^n \Big\} = \Big\{ w_1^n + w_2^n + \sum_{i=3}^{d} w_i^n \Big\}$$

Since, for $3 \leqslant i \leqslant d$, by definition $|w_i| < 1$, therefore $w_i^n \to 0$. Thus $\{\sum_{i=3}^d w_i^n\} \to 0$. Therefore $\lim_{n\to+\infty}\{w_1^n + w_2^n\} = 0$. □

Proposition 3.3. *Let $(w_1, w_2) \in S_2^*$ with a minimal polynomial $P \in \mathbb{F}_q[X][Y]$ of degree 3 and $w = w_1, w_2, w_3$ the conjugates of w. If w is unit and $P(0) = c \in \mathbb{F}_q^*$, then $\frac{(-1)}{c} w_1 w_2 \in S^*$.*

Proof. Let $(w_1, w_2) \in S_2^*$ with a minimal polynomial P of degree 3 and $P(0) = c \in \mathbb{F}_q^*$. Let w_3 be the third root of P. Since

$$P(Y) = (Y - w_1)(Y - w_2)(Y - w_3)$$

consider

$$Q(Y) = \frac{Y^3}{c} P\Big(\frac{1}{Y}\Big).$$

Clearly, Q is an irreducible unit over $\mathbb{F}_q[X][Y]$, and has roots

$$\frac{1}{w_1}; \frac{1}{w_2}; \frac{1}{w_3} = \frac{(-1)}{c} w_1, w_2.$$

We have $\left|\frac{1}{w_3}\right| = \left|\frac{-w_1 w_2}{c}\right| = |w_1 w_2| > 1$ and $\left|\frac{1}{w_i}\right| < 1$ for $i = 1, 2$. Therefore $\frac{(-1)}{c} w_1 w_2$ is a Pisot series. □

3.2. Irreducibility criterion and 2-Pisot series. In this subsection, we give an irreducibility criterion over $\mathbb{F}_q[X]$ in the case of a formal power series. Let us begin by:

Theorem 3.1. *Let the polynomial P be defined by*

$$P(Y) = A_n Y^n + A_{n-1} Y^{n-1} + A_{n-2} Y^{n-2} + \cdots + A_1 Y + A_0$$

where $A_0 \neq 0$, $A_n \in \mathbb{F}_q^*$, $A_i \in \mathbb{F}_q[X]$. P has exactly 2 roots that lie outside the unit disc and all remaining roots have a modulus strictly less than 1 if and only if

$$|A_{n-2}| > \sup_{i \neq n-2} |A_i|.$$

Proof. The first part is trivial.

For the second part, suppose first that P has no roots of absolute value greater than 1, which is absurd because the absolute value of the leading coefficient of the polynomial P is superior or equal to 1.

Suppose now that P has k exact roots ($k \neq 2$) that lie outside the unit disc and all the remaining roots have a modulus strictly less than 1. Let $w = w_1, w_2, \dots, w_n$

be the roots of $P(Y)$ such that $|w_1| \geqslant \cdots \geqslant |w_k| > 1 > |w_{k+1}| \geqslant \cdots \geqslant |w_n|$. By the symmetric relations of the roots of a polynomial, we obtain

$$\begin{aligned}\left|\frac{A_{n-k}}{A_n}\right| &= \left|\sum_{1\leqslant i_1<i_2<\ldots\leqslant n} w_{i_1}w_{i_2}\ldots w_{i_k}\right| \\ &= |w_1w_2\ldots w_k| > \sup(|w_1w_2|,\ldots,|w_{k-1}w_k|) \geqslant \left|\frac{A_{n-2}}{A_n}\right|.\end{aligned}$$

Then $\sup_{k\neq 2}|A_{n-k}| > |A_{n-2}|$, which is also absurd. □

THEOREM 3.2. *Let $(w_1, w_2) \in \mathbb{F}_q((X^{-1}))^2$ are the roots of the polynomial*

$$\Lambda(Y) = Y^n + \lambda_{n-1}Y^{n-1} + \lambda_{n-2}Y^{n-2} + \cdots + \lambda_1 Y + \lambda_0$$

such that $\lambda_i \in \mathbb{F}_q[X]$, $\lambda_0 \neq 0$ and $|\lambda_{n-2}| > \sup_{i<n-2}|\lambda_i|$. If $\deg\lambda_{n-2} \geqslant 2\deg\lambda_{n-1}$ and $\deg\lambda_{n-2}$ is odd, then Λ is irreducible and has a pair of 2-Pisot series.

PROOF. According to Theorem 3.1, Λ has exactly 2 roots that lie outside the unit disc and all the remaining roots have a modulus that is strictly less than 1. Let $w_1, w_2, \ldots, w_n$ be the roots of Λ such that $|w_1| \geqslant |w_2| > 1 > |w_i|$, for $i = 3, 4, \ldots, n$. Taking into account the Theorem condition, $\Lambda(0) = \lambda_0 \neq 0$; hence, all roots of the polynomial $\Lambda(Y)$ are not equal to 0. Let $\Lambda(Y) = \Lambda_1(Y), \Lambda_2(Y)$, where the coefficients of $\Lambda_i(Y)$, for $i = 1, 2$, are in $\mathbb{F}_q[X]$.

Suppose first that w_1 and w_2 are the roots of Λ_1 and the other roots are of Λ_2. Clearly, the absolute value of the leading coefficient of the polynomial Λ_2 is superior or equal to 1, which is absurd because the roots of Λ_2 are exactly w_i for $i = 3, 4, \ldots, n$ such that $0 < |w_i| < 1$.

Suppose secondly that Λ_1 is the polynomial of the series w_1 and Λ_2 the polynomial of the series w_2. Then we have

$$\begin{aligned}\Lambda(Y) &= \Lambda_1(Y).\Lambda_2(Y) \\ &= (Y^s + A_{s-1}Y^{s-1} + \cdots + A_1Y + A_0)(Y^m + B_{m-1}Y^{m-1} + \cdots + B_1Y + B_0).\end{aligned}$$

Therefore

$$\begin{aligned}\lambda_{n-1} &= A_{s-1} + B_{m-1}, \\ \lambda_{n-2} &= B_{m-2} + A_{s-1}B_{m-1} + A_{s-2}.\end{aligned}$$

This gives us

$$\begin{aligned}\deg\lambda_{n-1} &\leqslant \sup(\deg A_{s-1}; \deg B_{m-1}), \\ \deg\lambda_{n-2} &= \deg A_{s-1} + \deg B_{m-1}.\end{aligned}$$

Considering its Newton polygon, it can be seen that the polynomial Λ has exactly 2 roots w_1 and w_2 such that $|w_1| = |w_2| = q^{-k} > 1$ where $k = -\frac{\deg(A_{s-1})}{2}$ and all the remaining roots have a modulus that is strictly less than 1. As $\deg(A_{s-1}) = \deg(B_{m-1})$, then $\deg\lambda_{n-2} = \deg A_{s-1} + \deg B_{m-1} = 2\deg A_{s-1}$ is even, which is the desired contradiction. Therefore, we conclude that Λ is irreducible and it is the minimal polynomial of w_1 and w_2. So $(w_1, w_2) \in S_2^*$. □

Example 3.1. Let $H(Y) = Y^d + AY^{d-1} + BY^{d-2} + C$, $A, B \in \mathbb{F}_q[X] \setminus \{0\}$, such that $\deg B > 2 \deg A > 1$, $C \in \mathbb{F}\backslash\{0\}$ and $\deg(B)$ is odd. Then H is irreducible over $\mathbb{F}_q[X]$.

Remark 3.1. The inverse case is not always true. Indeed, consider the polynomial

$$P(Y) = Y^3 + (X^2 + X)Y^2 + X^3 Y + 1 \in \mathbb{F}_2[X][Y].$$

$P(Y)$ is irreducible in $\mathbb{F}_2[X][Y]$ and has two roots of a modules strictly greater than 1 defined by

$$\begin{aligned} w_1 &= X^2 + \frac{1}{Z_1} \quad \text{such that } |Z_1| > 2; \\ w_2 &= X^2 + \frac{1}{Z_2} \quad \text{such that } |Z_2| > 1; \\ w_3 &= \frac{1}{Z_3} \qquad\quad\ \ \text{such that } |Z_3| > 1. \end{aligned}$$

It is easy to prove that Z_1 and Z_2 are a formal series according to Corollary 2.2; then $w_1, w_2 \in \mathbb{F}_2((X^{-1}))$. As P is unit, so (w_1, w_2) is a 2-Pisot but

$$\deg(X^3) < 2\deg(X^2 + X).$$

Theorem 3.3. *Let the polynomial* Λ *be defined by*

$$\Lambda(Y) = Y^n + \lambda_{n-1}Y^{n-1} + \lambda_{n-2}Y^{n-2} + \cdots + \lambda_1 Y + \lambda_0$$

such that $\lambda_i \in \mathbb{F}_q[X]$, $\lambda_0 \neq 0$, $|\lambda_{n-2}| > \sup_{i<n-2} |\lambda_i|$ *and* $\deg \lambda_{n-2} < 2 \deg \lambda_{n-1}$, *then*

1) *If* Λ *is irreducible, then there exists a pair of* 2*-Pisot series* (w_1, w_2) *and* Λ *is the minimal polynomial of* (w_1, w_2).

2) *If* $\Lambda = \Lambda_1\Lambda_2$ *such that* $\deg(\Lambda_1) \geqslant 1$ *and* $\deg(\Lambda_2) \geqslant 1$*, then there exists two Pisot series* w_1 *and* w_2 *such that* Λ_k *is the minimal polynomial of* w_k *for* $k = 1, 2$.

Proof. 1) Considering the Newton polygon of Λ, then Λ has exactly 2 roots $w_1, w_2 \in \mathbb{F}_q((X^{-1}))$ that lie outside the unit disc of different absolute value such that

$$\begin{aligned} |w_1| &= q^{-k_1} > 1 \text{ where } k_1 = -\deg(\lambda_{n-1}) \\ |w_2| &= q^{-k_2} > 1 \text{ where } k_2 = \deg(\lambda_{n-1}) - \deg(\lambda_{n-2}) \end{aligned}$$

and all the remaining roots have a modulus that is strictly less than 1.

2) Trivial. □

Before concluding, we would like to suggest the following example.

Example 3.2. Let the polynomial

$$P(Y) = Y^3 + X^3Y^2 + X^4Y + 1 \in \mathbb{F}_2[X][Y].$$

It is easy to prove that P is irreducible over $\mathbb{F}_2[X][Y]$. Then using Theorem 3.3, P has two roots $w_1, w_2 \in \mathbb{F}_2((X^{-1}))$ such that $|w_1| > 1, |w_2| > 1$ and $|w_3| < 1$. As w_1 is an algebraic integer, then (w_1, w_2) is a pair of 2-Pisot series.

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