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ON I-STATISTICALLY ROUGH CONVERGENCE

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Abstract. We introduce rough I-statistical convergence as an extension of rough convergence. We define the set of rough I-statistical limit points of a sequence and analyze the results with proofs.

1. Introduction

The idea of statistical convergence was introduced by Steinhaus [**24**] and Fast [**7**] independently for real and complex sequences and some applications of statistical convergence in number theory and mathematical analysis have been shown by various researchers [**2**, **3**, **8**, **9**, **12**, **22**]. The concept of I-convergence was introduced by Kostyrko et al. [**11**] which generalizes and unifies different notion of convergence of sequences including usual convergence and statistical convergence. They used the notion of an ideal I of subsets of the set N to define such a concept [**5**, **10**, **23**].

The idea of I-statistical convergence was introduced by Savas and Das [**20**] as an extension of ideal convergence. Later on it was further investigated by Savas and Das [**21**], Savas [**18**, **19**], Debnath and Debnath [**4**], Et et al. [**6**] and many others.

The idea of rough convergence was first introduced by Phu [**17**] in finite dimensional normed linear spaces and studied the basic properties of this interesting concept [**15**, **16**]. It should be mentioned that the idea of rough convergence occurs very naturally in numerical analysis and has interesting application there. Recently Aytar [**1**] and Pal et al. [**14**] generalize the idea of rough convergence into rough statistical convergence and rough ideal convergence.

In this paper, we introduce the notion of rough I-statistical convergence with an aim to study some elementary properties of the set of rough I-statistical limit points.

It is to be noted that though our method of proofs is similar to those in [**1**, **14**, **17**], in comparison to previous studies we present our results in the most generalized form. This also enhances the applicability of these concepts.

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2. Definitions and Preliminaries

Definition 2.1. [**17**] Let $\{x_n\}_{n\in N}$ be a sequence in some normed linear space $(X, \|.\|)$ and r be a non-negative real number. Then $\{x_n\}_{n\in N}$ is said to be r-convergent to x_*, denoted by $x_n \xrightarrow{r} x_*$ if for all $\varepsilon > 0$ there exists $n_0 \in N$ such that $n \geqslant n_0 \Rightarrow \|x_n - x_*\| < r + \varepsilon$. Or equivalently if $\limsup \|x_n - x_*\| \leqslant r$. This is the rough convergence with r as roughness degree.

Definition 2.2. [**1**] A sequence $x = \{x_n\}_{n\in N}$ is said to be r-statistically convergent to x_*, denoted by $x_n \xrightarrow{r-st} x_*$, provided that the set $\{n \in N : \|x_n - x_*\| \geqslant r + \varepsilon\}$ has natural density zero for every $\varepsilon > 0$.

Definition 2.3. [**14**] A sequence $x = \{x_n\}_{n\in N}$ is said to be r-I convergent to x_*, denoted by $x_n \xrightarrow{r-I} x_*$, provided that the set $\{n \in N : \|x_n - x_*\| \geqslant r + \varepsilon\} \in I$ for any $\varepsilon > 0$.

Definition 2.4. [**20**] A real number sequence $x = \{x_n\}_{n\in N}$ is said to be I-statistically convergent to x_0 if for every $\varepsilon > 0$ and every $\delta > 0$,

$$\Big\{n \in N : \frac{1}{n}|\{k \leqslant n : \|x_k - x_0\| \geqslant \varepsilon\}| \geqslant \delta\Big\} \in I.$$

The number x_0 is called I-statistical limit of the sequence x and we write, $I\text{-st}\lim x_n = x_0$.

Through out the paper we consider I as an admissible ideal.

Definition 2.5. [**13**] The real number sequence $x = \{x_n\}$ is said to be I-st bounded if there is a number G such that $\{n \in N : \frac{1}{n}|\{k \leqslant n : |x_k| > G\}| > \delta\} \in I$.

Definition 2.6. [**13**] An element $\gamma \in X$ is called I-statistical cluster point of a sequence $x = \{x_n\}_{n\in N}$ if for each $\varepsilon > 0$ and $\delta > 0$ the set

$$\Big\{n \in N : \frac{1}{n}|\{k \leqslant n : \|x_k - \gamma\| \geqslant \varepsilon\}| < \delta\Big\} \notin I$$

The set of all I-statistical cluster points of x will be denoted by I-$S(\Gamma_x)$.

Definition 2.7. [**13**] For a real sequence $x = \{x_n\}_{n\in N}$, let B_x denote the set

$$B_x = \Big\{b \in R : \{n \in N : \frac{1}{n}|\{k \leqslant n : x_k > b\}| > \delta\} \notin I\Big\}.$$

Similarly, $A_x = \{a \in R : \{n \in N : \frac{1}{n}|\{k \leqslant n : x_k < a\}| > \delta\} \notin I\}$.

Let x be a real number sequence. Then I-statistical limit superior of x is given by,

$$I\text{-st}\limsup x = \begin{cases} \sup B_x, & \text{if } B_x \neq \emptyset \\ -\infty & \text{if } B_x = \emptyset \end{cases}.$$

Also, I-statistical limit inferior of x is given by,

$$I\text{-st}\liminf x = \begin{cases} \inf A_x, & \text{if } A_x \neq \emptyset \\ \infty & \text{if } A_x = \emptyset \end{cases}.$$

Theorem 2.1. [**13**] *If a I-statistically bounded sequence has one cluster point then it is I-statistically convergent.*

3. Main Results

Definition 3.1. A sequence $x=\{x_n\}_{n\in N}$ in X is said to be *rough-I-statistically convergent* to x_*, denoted by $x_n \overset{r-I-\text{st}}{\longrightarrow} x_*$, provided that the set

$$\left\{n\in N : \tfrac{1}{n}|\{k\leqslant n : \|x_k-x_*\|\geqslant r+\varepsilon\}|\geqslant\delta\right\}\in I$$

for any $\varepsilon>0$, $\delta>0$. Or equivalently we can say I-st $\limsup\|x_n-x_*\|\leqslant r$. Here r is called the roughness degree. If we take $r=0$ we obtain the notion of I-statistical convergence.

For instance assume that the sequence $y=\{y_n\}_{n\in N}$ is I-statistically convergent which can not be calculated exactly, one has to do with an approximated sequence $x=\{x_n\}_{n\in N}$ satisfying $\|x_n-y_n\|\leqslant r$ for all n. Then I-statistical convergence of the sequence x is not assured, but as the inclusion,

$$\left\{n\in N : \tfrac{1}{n}|\{k\leqslant n : \|x_k-x_*\|\geqslant r+\varepsilon\}|\geqslant\delta\right\}
\subseteq\left\{n\in N : \tfrac{1}{n}|\{k\leqslant n : \|y_k-x_*\|\geqslant \varepsilon\}|\geqslant\delta\right\}$$

holds, so the sequence $\{x_n\}_{n\in N}$ is r-I-statistically convergent.

In general the rough I-st limit of a sequence may not be unique for the roughness degree $r>0$. We define I-st $LIM^r x$ = set of all rough I-st limit of $x=\{x_*\in X : x_n \overset{r-I\text{-st}}{\longrightarrow} x_*\}$.

The sequence x is said to be r-I-statistically convergent provided I-st $LIM^r x\neq\emptyset$. It is clear that if I-st $LIM^r x\neq\emptyset$ for a sequence $x=\{x_n\}_{n\in N}$ of real number, then we have I-st $LIM^r x=[I\text{-st}\limsup x-r,\ I\text{-st}\liminf x+r]$.

Theorem 3.1. *For a sequence $x=\{x_n\}_{n\in N}$, diam(I-st$LIM^r x$) $\leqslant 2r$. In general diam(I-st$LIM^r x$) has no smaller bound.*

Proof. Assume that, diam(I-st $LIM^r x$) $>2r$. Then there exists $y,z\in I$-st $LIM^r x$ such that $\|y-z\|>2r$. Choose $0<\varepsilon<\frac{\|y-z\|}{2}-r$ and $\delta>0$.

Put, $A_1=\{n\in N : \frac{1}{n}|\{k\leqslant n : \|x_k-y\|\geqslant r+\varepsilon\}|\geqslant\delta\}\in I$ and $A_2=\{n\in N : \frac{1}{n}|\{k\leqslant n : \|x_k-z\|\geqslant r+\varepsilon\}|\geqslant\delta\}\in I$.

Hence, $M=N\setminus(A_1\cup A_2)\in F(I)$. So $M\neq\emptyset$, let $m\in M$ then for infinitely many $k\leqslant m$, $\|y-z\|\leqslant\|x_k-y\|+\|x_k-z\|<2(r+\varepsilon)$, which is a contradiction. Thus diam(I-st $LIM^r x$) $\leqslant 2r$.

To prove the second part, consider a sequence $x=\{x_n\}_{n\in N}$ such that I-st $\lim x = x_*$. Let $\varepsilon>0$ and $\delta>0$ then by definition of I-statistical convergence $A=\{n\in N : \frac{1}{n}|\{k\leqslant n : \|x_k-x_*\|\geqslant\varepsilon\}|<\delta\}\in F(I)$. Let $m\in A$ then $\frac{1}{m}|\{k\leqslant m : \|x_k-x_*\|\geqslant\varepsilon\}|<\delta$ i.e., for maximum $k\leqslant m$, $\|x_k-x_*\|<\varepsilon$.

Now for each $y\in\bar{B}_r(x_*)=\{y\in X : \|y-x_*\|\leqslant r\}$ we have, $\|x_k-y\|\leqslant\|x_k-x_*\|+\|x_*-y\|<r+\varepsilon$ for maximum $k\leqslant m\in A$ i.e., $\{n\in N : \frac{1}{n}|\{k\leqslant n : \|x_k-y\|\geqslant r+\varepsilon\}|<\delta\}\supseteq A\in F(I)$ which shows that $y\in I$-st $LIM^r x$ and consequently, I-st $LIM^r x=\bar{B}_r(x_*)$. This shows that in general upper bound $2r$ of the diameter of the set I-st $LIM^r x$ can not be decreased any more. □

Theorem 3.2. *A sequence $x=\{x_n\}_{n\in N}$ is I-st bounded if and only if there exists a non-negative real number r such that I-st$LIM^r x\neq\emptyset$.*

PROOF. Let $x = \{x_n\}_{n\in N}$ be an I-st bounded sequence. Then there exists a positive real number G such that $A = \{n \in N : \frac{1}{n}|\{k \leqslant n : \|x_k\| > G\}| > \delta\} \in I$. Let $\bar{r} = \sup\{\|x_k\|$ for almost all $k \leqslant m \in M = N \setminus A\}$. The set I-st $LIM^{\bar{r}}x$ contains the origin of X and so I-st $LIM^{\bar{r}}x \neq \emptyset$.

Conversely, suppose that I-st $LIM^r x \neq \emptyset$ for some $r \geqslant 0$, then there exists $x_* \in I$-st $LIM^r x$ i.e., $\{n \in N : \frac{1}{n}|\{k \leqslant n : \|x_k - x_*\| \geqslant r + \varepsilon\}| \geqslant \delta\} \in I$ for each $\varepsilon > 0$ and $\delta > 0$. This implies that $x = \{x_n\}_{n\in N}$ is I-st bounded. □

THEOREM 3.3. *The set I-st$LIM^r x$ of a sequence $x = \{x_n\}_{n\in N}$ is a closed set.*

PROOF. If I-st $LIM^r x = \emptyset$, Then we have nothing to prove.

Assume that I-st $LIM^r x \neq \emptyset$. Suppose $\{y_n\}_{n\in N} \subseteq I$-st $LIM^r x$ and $y_n \to y_*$ as $n \to \infty$. Let $\varepsilon > 0$ given then there exist $n_{\frac{\varepsilon}{2}} \in N$ such that $\|y_n - y_*\| < \frac{\varepsilon}{2}$ for all $n > n_{\frac{\varepsilon}{2}}$. Let $n_0 \in N$ such that $y_{n_0} \in \{y_n\}_{n\in N} \subseteq I$-st $LIM^r x$. So, $A = \{n \in N : \frac{1}{n}|\{k \leqslant n : \|x_k - y_{n_0}\| \geqslant r + \frac{\varepsilon}{2}\}| < \delta\} \in F(I)$. Let $m \in A$ then $\frac{1}{m}|\{k \leqslant m : \|x_k - y_{n_0}\| \geqslant r + \frac{\varepsilon}{2}\}| < \delta$. i.e., for maximum $k \leqslant m$, $\|x_k - y_{n_0}\| < r + \frac{\varepsilon}{2}$. Choose an $n_0 > n_{\frac{\varepsilon}{2}}$ we have, $\|x_k - y_*\| \leqslant \|x_k - y_{n_0}\| + \|y_{n_0} - y_*\| < r + \varepsilon$ for maximum $k \leqslant m \in A$. i.e., $\{n \in N : \frac{1}{n}|\{k \leqslant n : \|x_k - y_*\| \geqslant r + \varepsilon\}| < \delta\} \supseteq A \in F(I)$. So, $y_* \in I$-st $LIM^r x$. i.e., I-st $LIM^r x$ is a closed set. □

THEOREM 3.4. *The set I-st$LIM^r x$ of a sequence $x = \{x_n\}_{n\in N}$ is convex.*

PROOF. Let $y_0, y_1 \in I$-st $LIM^r x$. Let $\varepsilon > 0$ and $\delta > 0$. So,

$$A_1 = \{n \in N : \tfrac{1}{n}|\{k \leqslant n : \|x_k - y_0\| \geqslant r + \varepsilon\}| \geqslant \delta\} \in I$$
$$A_2 = \{n \in N : \tfrac{1}{n}|\{k \leqslant n : \|x_k - y_1\| \geqslant r + \varepsilon\}| \geqslant \delta\} \in I.$$

$M = N \setminus (A_1 \cup A_2) \in F(I)$ and so M must be a infinite set. Let $m \in M$ then $d(B_1) = 0$, where $B_1 = \{k \leqslant m : \|x_k - y_0\| \geqslant r + \varepsilon\}$ and $d(B_2) = 0$, where $B_2 = \{k \leqslant m : \|x_k - y_1\| \geqslant r + \varepsilon\}$. Now for each $k \in B_1^c \cap B_2^c$ and each $\lambda \in [0, 1]$,

$$\|x_k - ((1-\lambda)y_0 + \lambda y_1)\| = \|(1-\lambda)(x_k - y_0) + \lambda(x_k - y_1)\| < r + \varepsilon.$$

Since, $d(B_1^c \cap B_2^c) = 1$, we get $\frac{1}{m}|\{k \leqslant m : \|x_k - ((1-\lambda)y_0 + \lambda y_1)\| \geqslant r + \varepsilon\}| < \delta$. So, $\big\{n \in N : \frac{1}{n}\big|\big\{k \leqslant n : \|x_k - ((1-\lambda)y_0 + \lambda y_1)\| \geqslant r + \varepsilon\big\}\big| < \delta\big\} \supseteq M \in F(I)$, which shows that $(1-\lambda)y_0 + \lambda y_1 \in I - st\,LIM^r x$, for any $\lambda \in [0, 1]$. Hence the set I-st $LIM^r x$ is convex. □

THEOREM 3.5. *Let $x = \{x_n\}_{n\in N}$ then for an arbitrary $c \in I$-$S(\Gamma_x)$, $\|x_* - c\| \leqslant r$ for all $x_* \in I$-st$LIM^r x$.*

PROOF. If possible suppose that there exists $c \in I$-$S(\Gamma_x)$ and $x_* \in I$-st $LIM^r x$ such that $\|x_* - c\| > r$. Let $\varepsilon = \frac{\|x_* - c\| - r}{2}$, we have $A = \big\{n \in N : \frac{1}{n}|\{k \leqslant n : \|x_k - c\| \geqslant \varepsilon\}| < \delta\big\} \notin I$.

Let $B = \big\{n \in N : \frac{1}{n}|\{k \leqslant n : \|x_k - x_*\| \geqslant r + \varepsilon\}| \geqslant \delta\big\}$. Let $m \in A$, i.e., $\frac{1}{m}|\{k \leqslant m : \|x_k - c\| \geqslant \varepsilon\}| < \delta$. So for maximum $k \leqslant m$, $\|x_k - c\| < \varepsilon$. Now $\|x_k - x_*\| \geqslant \|x_* - c\| - \|x_k - c\| > r + \varepsilon$, for all $k \leqslant m \in A$ Therefore, $B \supseteq A$ implies that $B \notin I$, which contradicts the fact that $x_* \in I$-st $LIM^r x$. Thus $\|x_* - c\| \leqslant r$ for all $x_* \in I$-st $LIM^r x$ and $c \in I$-$S(\Gamma_x)$. □

THEOREM 3.6. *A sequence $x = \{x_n\}_{n\in N}$ is I-statistically convergent to x_* if and only if I-st$LIM^r x = \bar{B}_r(x_*)$.*

PROOF. The necessary part of the theorem is already proved in the 2nd part of Theorem 3.1.

For the sufficiency, let I-st $LIM^r x = \bar{B}_r(x_*) \neq \emptyset$. Thus the sequence $x = \{x_n\}_{n\in N}$ is I-statistically bounded. Suppose that x has another I-statistical cluster point x'_* different from x_*. The point $\bar{x}_* = x_* + \frac{r}{\|x_* - x'_*\|}(x_* - x'_*)$ satisfies, $\|\bar{x}_* - x'_*\| > r$. Since, $x'_* \in I\text{-}S(\Gamma_x)$, by Theorem 3.5, $\bar{x}_* \notin I$-st $LIM^r x$. But this is impossible as $\|\bar{x}_* - x_*\| = r$ and I-st $LIM^r x = \bar{B}_r(x_*)$. Therefore x_* is the unique I-statistical cluster point of x. Also x is I-statistically bounded. So by Theorem 2.1, x is I-statistically convergent to x_*. □

THEOREM 3.7. *Let $r > 0$. Then a sequence $x = \{x_n\}_{n\in N}$ is r-I-statistically convergent to x_* if and only if there exists a sequence $y = \{y_n\}_{n\in N}$ such that I-st$\lim y = x_*$ and $\|x_n - y_n\| \leqslant r$ for all $n \in N$.*

PROOF. Necessity, let $x_n \overset{r-I_{st}}{\longrightarrow} x_*$. Then we have,

$$I\text{-st}\limsup \|x_n - x_*\| \leqslant r \tag{3.1}$$

Now we define, $y_n = \begin{cases} x_*, & \text{if } \|x_n - x_*\| \leqslant r \\ x_n + r\frac{x_* - x_n}{\|x_n - x_*\|}, & \text{otherwise} \end{cases}$

Then,

$$\|y_n - x_*\| = \begin{cases} 0, & \text{if } \|x_n - x_*\| \leqslant r \\ \|x_n - x_*\| - r, & \text{otherwise} \end{cases} \tag{3.2}$$

by the definition of y_n we have $\|x_n - y_n\| \leqslant r$, for all $n \in N$. Now by (3.1) and (3.2) we get, $\{n \in N : \frac{1}{n}|\{k \leqslant n : \|y_k - x_*\| \geqslant \varepsilon\}| \geqslant \delta\} \in I$ which implies that I-st $\lim y_n = x_*$.

Sufficiency, suppose that the given condition holds. For any $\varepsilon > 0$, $\delta > 0$ the set

$$A = \{n \in N : \tfrac{1}{n}|\{k \leqslant n : \|y_k - x_*\| \geqslant \varepsilon\}| < \delta\} \in I$$

and $\|x_n - y_n\| \leqslant r$ for $n \in N$. Therefore, $\|x_k - x_*\| \leqslant \|x_k - y_k\| + \|y_k - x_*\| < r + \varepsilon$ for maximum $k \leqslant m \in A^c$

This shows that

$$\{n \in N : \tfrac{1}{n}|\{k \leqslant n : \|x_k - x_*\| \geqslant r + \varepsilon\}| < \delta\} \supseteq A^c \in F(I)$$

and so, $x_n \overset{r-I_{st}}{\longrightarrow} x_*$. □

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