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A NEW STUDY ON GENERALIZED QUASI POWER INCREASING SEQUENCES

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Abstract. We prove a general theorem dealing with an application of quasi β-power increasing sequences. This theorem also includes several new and known results.

1. Introduction

A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leqslant b_n \leqslant Bc_n$ (see [**1**]). Obviously, every increasing sequence is almost increasing. However, the converse need not be true as can be seen by taking an example, say, $b_n = ne^{(-1)^n}$. Let $\sum a_n$ be a given infinite series with the partial sums (s_n). By (u_n) and (t_n) we denote the n-th $(C,1)$ means of the sequences (s_n) and (na_n), respectively. The series $\sum a_n$ is said to be summable $|C,1|_k$, $k \geqslant 1$, if (see [**11**,**13**])

$$\sum_{n=1}^{\infty} n^{k-1}|u_n - u_{n-1}|^k = \sum_{n=1}^{\infty} \frac{1}{n}|t_n|^k < \infty.$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty; \quad \text{as } n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geqslant 1).$$

The sequence-to-sequence transformation $\sigma_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v s_v$ defines the sequence (σ_n) of the Riesz mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [**12**]).

The series $\sum a_n$ is said to be summable $|\bar{N}, p_n|_k$, $k \geqslant 1$, if (see [**2**])

$$\sum_{n=1}^{\infty} \Big(\frac{P_n}{p_n}\Big)^{k-1} |\Delta\sigma_{n-1}|^k < \infty,$$

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where

$$\Delta\sigma_{n-1} = -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v, \quad n \geqslant 1.$$

In the special case, when $p_n = 1$ for all values of n (resp. $k = 1$), $|\bar{N}, p_n|_k$ summability reduces to $|C, 1|_k$ (resp. $|\bar{N}, p_n|$) summability. Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. We associate two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, \dots$$

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, \dots$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \bar{a}_{nv} a_v, \quad \bar{\Delta} A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv} a_v. \tag{1.1}$$

Let $A = (a_{nv})$ be a normal matrix. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, \dots$$

Let (φ_n) be any sequence of positive real numbers. The series $\sum a_n$ is said to be summable $\varphi - |A; \delta|_k$, $k \geqslant 1$ and $\delta \geqslant 0$, if (see [**20**])

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k + k - 1} |\bar{\Delta} A_n(s)|^k < \infty, \text{ where } \bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s).$$

If we take $\delta = 0$ and $\varphi_n = \frac{P_n}{p_n}$, then $\varphi - |A; \delta|_k$ summability reduces to $|A, p_n|_k$ summability (see [**22**]). Also, if we take $\delta = 0$, $\varphi_n = \frac{P_n}{p_n}$ and $a_{nv} = \frac{p_v}{P_n}$, then we get $|\bar{N}, p_n|_k$ summability. If we take $\delta = 0$, $\varphi_n = n$, $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n, $\varphi - |A; \delta|_k$ summability reduces to $|C, 1|_k$ summability. If we take $\delta = 0$ and $\varphi_n = n$, then we get $|A|_k$ summability (see [**23**]). Finally, if we take $\delta = 0$, $\varphi_n = n$ and $a_{nv} = \frac{p_v}{P_n}$, then we get $|R, p_n|_k$ summability (see [**4**]).

2. Known Result

Many works dealing with absolute summability and absolute matrix summability factors of infinite series have been done (see [**3**–**10**, **15**–**21**]). Among them, in [**6**], Bor has proved the following theorem for $|\bar{N}, p_n|_k$ summability factors of infinite series by using almost increasing sequences.

Theorem 2.1. *Let (X_n) be an almost increasing sequence and let there be sequences (β_n) and (λ_n) such that*

$$|\Delta \lambda_n| \leqslant \beta_n, \quad \beta_n \to 0 \text{ as } n \to \infty, \tag{2.1}$$

(2.2) $$\sum_{n=1}^{\infty} n|\Delta\beta_n|X_n < \infty, \quad |\lambda_n|X_n = O(1).$$

If

(2.3) $$\sum_{n=1}^{m} \frac{|\lambda_n|}{n} = O(1) \quad \text{as} \quad m \to \infty,$$

$$\sum_{n=1}^{m} \frac{1}{n}|t_n|^k = O(X_m) \quad \text{as} \quad m \to \infty,$$

and (p_n) *is a sequence such that* $\sum_{n=1}^{m} \frac{p_n}{P_n}|t_n|^k = O(X_m)$ *as* $m \to \infty$*, then the series* $\sum a_n\lambda_n$ *is summable* $|\bar{N}, p_n|_k, k \geqslant 1$*.*

3. Main Result

A positive sequence (γ_n) is said to be quasi β-power increasing sequence if there exists a constant $K = K(\beta, \gamma) \geqslant 1$ such that $Kn^\beta\gamma_n \geqslant m^\beta\gamma_m$ holds for all $n \geqslant m \geqslant 1$ (see [**14**]). It should be noted that every almost increasing sequence is quasi β-power increasing sequence for any nonnegative β, but the converse need not be true as can be seen by taking the example, say $\gamma_n = n^{-\beta}$ for $\beta > 0$.

A sequence (λ_n) is said to be of bounded variation, denoted by $(\lambda_n) \in \mathcal{BV}$, if $\sum_{n=1}^{\infty}|\Delta\lambda_n| = \sum_{n=1}^{\infty}|\lambda_n - \lambda_{n+1}| < \infty$.

We generalize Theorem 2.1 to $\varphi - |A;\delta|_k$ summability by using quasi β-power increasing sequences instead of almost increasing sequences, and prove the following theorem.

Theorem 3.1. *Let* $A = (a_{nv})$ *be a positive normal matrix such that*

$$\overline{a}_{n0} = 1, \;\; n = 0, 1, \dots, \qquad a_{n-1,v} \geqslant a_{nv}, \; \text{for } n \geqslant v+1, \qquad a_{nn} = O\Big(\frac{p_n}{P_n}\Big).$$

Let (X_n) *be a quasi* β*-power increasing sequence for some* $0 < \beta < 1$ *and* $\varphi_n p_n = O(P_n)$*. If* $(\lambda_n) \in \mathcal{BV}$*, the conditions* (2.1)–(2.3) *of Theorem* 2.1 *and*

$$\sum_{n=v+1}^{m+1} \varphi_n^{\delta k}|\Delta_v \hat{a}_{nv}| = O(\varphi_v^{\delta k-1}) \quad \text{as} \quad m \to \infty,$$

$$\sum_{n=v+1}^{m+1} \varphi_n^{\delta k}|\hat{a}_{n,v+1}| = O(\varphi_v^{\delta k}) \quad \text{as} \quad m \to \infty,$$

$$\sum_{n=1}^{m} \varphi_n^{\delta k}\frac{1}{n}|t_n|^k = O(X_m) \quad \text{as} \quad m \to \infty,$$

$$\sum_{n=1}^{m} \varphi_n^{\delta k-1}|t_n|^k = O(X_m) \quad \text{as} \quad m \to \infty$$

are satisfied, then the series $\sum a_n\lambda_n$ *is summable* $\varphi - |A;\delta|_k$*,* $k \geqslant 1$ *and* $0 \leqslant \delta < 1/k$*.*

If we take $\delta = 0$, $\varphi_n = \frac{P_n}{p_n}$, $a_{nv} = \frac{p_v}{P_n}$ and (X_n) as an almost increasing sequence, then we get Theorem 2.1. In this case the condition $(\lambda_n) \in \mathcal{BV}$ is not needed.

We need the following lemma for the proof of Theorem 3.1.

Lemma 3.1. [**14**] *Under the conditions on* (X_n)*,* (β_n) *and* (λ_n) *as taken in the statement of Theorem* 3.1*, the following conditions hold*

$$n\beta_n X_n = O(1) \text{ as } n \to \infty, \qquad \sum_{n=1}^{\infty} \beta_n X_n < \infty.$$

4. Proof of Theorem 3.1

Let (I_n) denotes A-transform of the series $\sum a_n \lambda_n$. Then, by (1.1), we have

$$\bar{\Delta} I_n = \sum_{v=0}^{n} \hat{a}_{nv} a_v \lambda_v = \sum_{v=1}^{n} \frac{\hat{a}_{nv}\lambda_v}{v} v a_v.$$

Applying Abel's transformation to this sum, we get that

$$\begin{aligned}
\bar{\Delta} I_n &= \sum_{v=1}^{n-1} \Delta_v\Big(\frac{\hat{a}_{nv}\lambda_v}{v}\Big) \sum_{r=1}^{v} r a_r + \frac{\hat{a}_{nn}\lambda_n}{n} \sum_{r=1}^{n} r a_r \\
&= \frac{n+1}{n} a_{nn}\lambda_n t_n + \sum_{v=1}^{n-1} \frac{v+1}{v} \Delta_v(\hat{a}_{nv}) \lambda_v t_v \\
&\quad + \sum_{v=1}^{n-1} \frac{v+1}{v} \hat{a}_{n,v+1} \Delta\lambda_v t_v + \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \lambda_{v+1} \frac{t_v}{v} \\
&= I_{n,1} + I_{n,2} + I_{n,3} + I_{n,4}.
\end{aligned}$$

To complete the proof of Theorem 3.1, by Minkowski's inequality, it is enough to show that $\sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1} |I_{n,r}|^k < \infty$, for $r = 1, 2, 3, 4$. First, by using Abel's transformation, we have that

$$\begin{aligned}
\sum_{n=1}^{m} \varphi_n^{\delta k+k-1} |I_{n,1}|^k &= O(1) \sum_{n=1}^{m} \varphi_n^{\delta k+k-1} a_{nn}^k |\lambda_n|^k |t_n|^k \\
&= O(1) \sum_{n=1}^{m} \varphi_n^{\delta k-1} \Big(\frac{\varphi_n p_n}{P_n}\Big)^k |\lambda_n|^{k-1} |\lambda_n| |t_n|^k \\
&= O(1) \sum_{n=1}^{m} \varphi_n^{\delta k-1} |\lambda_n| |t_n|^k \\
&= O(1) \sum_{n=1}^{m-1} \Delta|\lambda_n| \sum_{r=1}^{n} \varphi_r^{\delta k-1} |t_r|^k + O(1) |\lambda_m| \sum_{n=1}^{m} \varphi_n^{\delta k-1} |t_n|^k \\
&= O(1) \sum_{n=1}^{m-1} |\Delta\lambda_n| X_n + O(1) |\lambda_m| X_m
\end{aligned}$$

$$= O(1)\sum_{n=1}^{m-1}\beta_n X_n + O(1)|\lambda_m|X_m$$
$$= O(1) \quad \text{as} \quad m \to \infty,$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.1.

Now, applying Hölder's inequality with indices k and k', where $k > 1$ and $\frac{1}{k} + \frac{1}{k'} = 1$, as in $I_{n,1}$, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}|I_{n,2}|^k &= O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_v||t_v|\right)^k\\
&= O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_v|^k|t_v|^k \times \left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right)^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}a_{nn}^{k-1}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_v|^k|t_v|^k\right)\\
&= O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k}\left(\frac{\varphi_n p_n}{P_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_v|^k|t_v|^k\right)\\
&= O(1)\sum_{v=1}^{m}|\lambda_v|^k|t_v|^k\sum_{n=v+1}^{m+1}\varphi_n^{\delta k}|\Delta_v(\hat{a}_{nv})|\\
&= O(1)\sum_{v=1}^{m}\varphi_v^{\delta k-1}|\lambda_v||t_v|^k\\
&= O(1) \quad \text{as} \quad m \to \infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.1.

Now, using Hölder's inequality we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}|I_{n,3}|^k &= O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v||t_v|\right)^k\\
&= O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v||t_v|^k \times \left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v|\right)^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}a_{nn}^{k-1}\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\beta_v|t_v|^k \times \left(\sum_{v=1}^{n-1}|\Delta\lambda_v|\right)^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k}\left(\frac{\varphi_n p_n}{P_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\beta_v|t_v|^k\right)\\
&= O(1)\sum_{v=1}^{m}\beta_v|t_v|^k\sum_{n=v+1}^{m+1}\varphi_n^{\delta k}|\hat{a}_{n,v+1}|
\end{aligned}$$

$$
\begin{aligned}
&= O(1)\sum_{v=1}^{m}\varphi_v^{\delta k}v\beta_v\frac{|t_v|^k}{v}\\
&= O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{r=1}^{v}\varphi_r^{\delta k}\frac{1}{r}|t_r|^k + O(1)m\beta_m\sum_{v=1}^{m}\varphi_v^{\delta k}\frac{1}{v}|t_v|^k\\
&= O(1)\sum_{v=1}^{m-1}|\Delta(v\beta_v)|X_v + O(1)m\beta_m X_m\\
&= O(1)\sum_{v=1}^{m-1}v|\Delta\beta_v|X_v + O(1)\sum_{v=1}^{m-1}\beta_v X_v + O(1)m\beta_m X_m\\
&= O(1) \quad \text{as} \quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.1.

Again, using Hölder's inequality, we have that

$$
\begin{aligned}
\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}|I_{n,4}|^k &\leqslant \sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\Big(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\lambda_{v+1}|\frac{|t_v|}{v}\Big)^k\\
&\leqslant \sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\Big(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\lambda_{v+1}|\frac{|t_v|^k}{v}\Big)\times\Big(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\frac{|\lambda_{v+1}|}{v}\Big)^{k-1}\\
&\leqslant \sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}a_{nn}^{k-1}\Big(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\lambda_{v+1}|\frac{|t_v|^k}{v}\Big)\times\Big(\sum_{v=1}^{n-1}\frac{|\lambda_{v+1}|}{v}\Big)^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}a_{nn}^{k-1}\Big(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\lambda_{v+1}|\frac{|t_v|^k}{v}\Big)\\
&= O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k}\Big(\frac{\varphi_n p_n}{P_n}\Big)^{k-1}\Big(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\lambda_{v+1}|\frac{|t_v|^k}{v}\Big)\\
&= O(1)\sum_{v=1}^{m}|\lambda_{v+1}|\frac{|t_v|^k}{v}\sum_{n=v+1}^{m+1}\varphi_n^{\delta k}|\hat{a}_{n,v+1}|\\
&= O(1)\sum_{v=1}^{m}\varphi_v^{\delta k}\frac{1}{v}|\lambda_{v+1}||t_v|^k\\
&= O(1)\sum_{v=1}^{m-1}|\Delta\lambda_{v+1}|\sum_{r=1}^{v}\varphi_r^{\delta k}\frac{1}{r}|t_r|^k + O(1)|\lambda_{m+1}|\sum_{v=1}^{m}\varphi_v^{\delta k}\frac{1}{v}|t_v|^k\\
&= O(1)\sum_{v=1}^{m-1}\beta_{v+1}X_{v+1} + O(1)|\lambda_{m+1}|X_{m+1}\\
&= O(1) \quad \text{as} \quad m\to\infty,
\end{aligned}
$$

by virtue of hypotheses of Theorem 3.1 and Lemma 3.1.

This completes the proof of Theorem 3.1.

5. Conclusions

We note that, if we take $\delta = 0$ and $\varphi_n = \frac{P_n}{p_n}$, then we get a theorem dealing with $|A, p_n|_k$ summability. If we take (X_n) as an almost increasing sequence, $\delta = 0$ and $\varphi_n = \frac{P_n}{p_n}$, then we get another theorem dealing with $|A, p_n|_k$ summability (see [**16**]). If we take $\delta = 0$, $\varphi_n = n$, $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n, then we get a result for $|C, 1|_k$ summability.

References

1. N. K. Bari, S. B. Stečkin, *Best approximations and differential properties of two conjugate functions*, Tr. Mosk. Mat. Obšč. **5** (1956), 483–522. [in Russian]
2. H. Bor, *On two summability methods*, Math. Proc. Camb. Philos. Soc. **97** (1985), 147–149.
3. ______, *A note on $|\bar{N}, p_n|_k$ summability factors of infinite series*, Indian J. Pure Appl. Math. **18** (1987), 330–336.
4. ______, *On the relative strength of two absolute summability methods*, Proc. Am. Math. Soc. **113** (1991), 1009–1012.
5. ______, *On absolute summability factors*, Proc. Am. Math. Soc. **118** (1993), 71–75.
6. ______, *On absolute Riesz summability factors*, Adv. Stud. Contemp. Math., Pusan **3**(2) (2001), 23–29.
7. ______, *A new application of quasi power increasing sequences. I*, J. Inequal. Appl. **69** (2013), 5 pp.
8. ______, *A new result on the quasi power increasing sequences*, Appl. Math. Comput. **248** (2014), 426–429.
9. H. Bor, H. S. Özarslan, *On the quasi power increasing sequences*, J. Math. Anal. Appl. **276** (2002), 924–929.
10. ______, *A study on quasi power increasing sequences*, Rocky Mt. J. Math. **38** (2008), 801–807.
11. T. M. Flett, *On an extension of absolute summability and some theorems of Littlewood and Paley*, Proc. Lond. Math. Soc. **7** (1957), 113–141.
12. G. H. Hardy, *Divergent Series*, Oxford University Press, Oxford, 1949, 1–396.
13. E. Kogbetliantz, *Sur les séries absolument sommables par la méthode des moyennes arithmétiques*, Bull. Sci. Math. **49** (1925), 234–256.
14. L. Leindler, *A new application of quasi power increasing sequences*, Publ. Math. **58** (2001), 791–796.
15. H. S. Özarslan, *A new application of almost increasing sequences*, Miskolc Math. Notes **14** (2013), 201–208 .
16. ______, *A new application of absolute matrix summability*, C. R. Acad. Bulg. Sci. **68** (2015), 967–972.
17. ______, *On generalized absolute matrix summability methods*, Int. J. Anal. Appl. **12** (2016), 66–70.
18. ______, *A new study on generalised absolute matrix summability methods*, Maejo Int. J. Sci. Technol. **12**(3) (2018), 199–205.
19. ______, *A new factor theorem for absolute matrix summability*, Quaest. Math. (2018), 1–7.
20. H. S. Özarslan, T. Ari, *Absolute matrix summability methods*, Appl. Math. Lett. **24**(12) (2011), 2102–2106.
21. H. S. Özarslan, E. Yavuz, *A new note on absolute matrix summability*, J. Inequal. Appl. **474** (2013), 7 pp.
22. W. T. Sulaiman, *Inclusion theorems for absolute matrix summability methods of an infinite series. IV*, Indian J. Pure Appl. Math. **34**(11) (2003), 1547–1557.
23. N. Tanović-Miller, *On strong summability*, Glas. Mat., III. Ser. **14**(34) (1979), 87–97.

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