PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 105(119) (2019), 131–136 DOI: https://doi.org/10.2298/PIM1919131G

APPROXIMATION WITH CERTAIN POST–WIDDER OPERATORS

Vijay Gupta and Prerna Maheshwari

Dedicated to Professor G. V. Milovanović on his 70-th birthday

Abstract. We consider a modification of well known Post–Widder operators, which preserve the exponential function. We estimate a direct estimate and a quantitative asymptotic formula for the modified operators.

1. Post–Widder Operators

In the last few decades many linear positive operators have been constructed or appropriately modified in order to achieve better approximation. In [**1**, **4**, **5**, **9**–**11**] and very recently in [**6**] several problems concerning approximation have been discussed for different operators. Post–Widder operators [**12**] are defined for $f \in C[0,\infty)$ as:

$$P_n(f,x) := \frac{1}{n!}\Big(\frac{n}{x}\Big)^{n+1} \int_0^\infty t^n e^{-\frac{nt}{x}} f(t)dt.$$

Let us consider $f(t) = e^{\theta t}$, $\theta \in \mathbb{R}$, then we have

$$P_n(e^{\theta t},x) = \frac{1}{n!}\Big(\frac{n}{x}\Big)^{n+1} \int_0^\infty t^n e^{-\frac{nt}{x}} e^{\theta t}dt = \frac{1}{n!}\Big(\frac{n}{x}\Big)^{n+1} \int_0^\infty t^n e^{-(\frac{n}{x}-\theta)t}dt.$$

Substituting $\left(\frac{n}{x}-\theta\right)t = u$, we can write

$$\begin{aligned} P_n(e^{\theta t},x) &= \frac{1}{n!}\Big(\frac{n}{x}\Big)^{n+1} \frac{1}{\left(\frac{n}{x}-\theta\right)^{n+1}} \int_0^\infty u^n e^{-u}du \\ &= \frac{1}{n!}\Big(\frac{n}{x}\Big)^{n+1} \frac{1}{\left(\frac{n}{x}-\theta\right)^{n+1}} \Gamma(n+1) \\ &= \Big(\frac{n}{x}\Big)^{n+1}\Big(\frac{n}{x}-\theta\Big)^{-(n+1)} = \Big(1-\frac{x\theta}{n}\Big)^{-(n+1)}. \end{aligned} \tag{1.1}$$

2010 *Mathematics Subject Classification:* 41A25; 41A30.
Key words and phrases: Post–Widder operators, exponential functions, moment generating function, quantitative results.
Communicated by Gradimir Milovanović.

Recently Gupta and Tachev [**7**] considered the Phillips operators (see [**3**] and references therein) and established some direct results which preserve exponential functions.

Let us consider that the Post–Widder operators preserve the test function e^{-x}, then we start with the following form

$$\tilde{P}_n(f,x) := \frac{1}{n!}\Big(\frac{n}{a_n(x)}\Big)^{n+1}\int_0^\infty t^n e^{-\frac{nt}{a_n(x)}} f(t)\,dt.$$

Then using (1.1), we have

$$\tilde{P}_n(e^{-t},x) = e^{-x} = \Big(1+\frac{a_n(x)}{n}\Big)^{-(n+1)},$$

implying

$$a_n(x) = n(e^{x/(n+1)}-1).$$

Thus our modified operators $\tilde{P}_n$ take the following form

$$\tilde{P}_n(f,x) := \frac{1}{n!}(e^{x/(n+1)}-1)^{-(n+1)}\int_0^\infty t^n e^{-\frac{t}{(e^{x/(n+1)}-1)}} f(t)\,dt, \tag{1.2}$$

$x\in(0,\infty)$ and $\tilde{P}_n(f,0)=f(0)$, which preserve constant and the test function e^{-x}. In [**2**] a Korovkin-type theorem for the function e^{-kt}, $k=0,1,2$ was considered for the class $C^*[0,\infty)$, which denote the linear space of real-valued continuous functions on $[0,\infty)$ with the property that $\lim_{x\to\infty} f(x)$ exists and is finite, equipped with uniform norm $\|.\|_\infty$.

THEOREM 1.1. [**2**, Theorem 2] *If $\{L_n f\}$ be the sequence of positive and linear operators defined on the space $C^*[0,\infty)$ and satisfies*

$$\lim_{n\to\infty} L_n(e^{-mt},x) = e^{-mx},\quad m=0,1,2,\quad \textit{uniformly in}\ \ [0,\infty),$$

then $L_n f$ converges uniformly to f for n sufficiently large.

Theorem 1.1 was extended to quantitative estimate in [**8**] as follows:

THEOREM 1.2. [**8**] *Let $f\in C^*[0,\infty)$ and $L_n : C^*[0,\infty)\to C^*[0,\infty)$ be a sequence of linear positive operators and satisfies the following three conditions:*

$$\|L_n 1 - 1\|_\infty = a_n,\quad \|L_n(e^{-t},x)-e^{-x}\|_\infty = b_n,\quad \|L_n(e^{-2t},x)-e^{-2x}\|_\infty = c_n,$$

where a_n, b_n and c_n tend to zero for n sufficiently large. Then, we have

$$\|L_n f - f\|_\infty \leqslant \|f\|_\infty a_n + (2+a_n)\cdot\omega^*(f,(a_n+2b_n+c_n)^{1/2}),$$

where the modulus of continuity $\omega^*(f,\delta) = \sup_{\substack{x,t\geqslant 0\\ |e^{-x}-e^{-t}|\leqslant\delta}} |f(x)-f(t)|$ for every $\delta\geqslant 0$.

2. Lemmas

LEMMA 2.1. *We have for* $\theta > 0$ *that*

$$\tilde{P}_n(e^{\theta t}, x) = (1 - (e^{x/(n+1)} - 1)\theta)^{-(n+1)}.$$

It may be observed that $\tilde{P}_n(e^{\theta t}, x)$ may be treated as moment generating function of the operators $\tilde{P}_n$, which may be utilized to obtain the moments of (1.2). Let $\mu_r^{\tilde{P}_n}(x) = \tilde{P}_n(e_r, x)$, where $e_r(t) = t^r$, $r \in \mathbb{N} \cup \{0\}$. The moments are given by

$$\mu_r^{\tilde{P}_n}(x) = \Big[\frac{\partial^r}{\partial \theta^r} \tilde{P}_n(e^{\theta t}, x)\Big]_{\theta=0} = \Big[\frac{\partial^r}{\partial \theta^r} \big\{(1 - (e^{x/(n+1)} - 1)\theta)^{-(n+1)}\big\}\Big]_{\theta=0}.$$

LEMMA 2.2. *The moments of arbitrary order, satisfy the following*

$$\mu_k^{\tilde{P}_n}(x) = (e^{x/(n+1)} - 1)^k (n+1)_k, \quad k = 0, 1, \ldots,$$

where the Pochhammer symbol is defined by

$$(c)_0 = 1, \quad (c)_k = c(c+1)\ldots(c+k-1).$$

Further, by linearity property and using Lemma 2.2, we have the following lemma:

LEMMA 2.3. *The central moments* $U_r^{\tilde{P}_n}(x) = \tilde{P}_n((t-x)^r, x)$ *are given below:*

$$U_k^{\tilde{P}_n}(x) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} x^{k-j} (e^{x/(n+1)} - 1)^j (n+1)_j, \quad k = 0, 1, \ldots.$$

Also, for each $k \in \mathbb{N}$ *we have*

$$\lim_{n\to\infty} n^k U_{2k}^{\tilde{P}_n}(x) = (2k-1)!!x^{2k},$$

$$\lim_{n\to\infty} n^k U_{2k-1}^{\tilde{P}_n}(x) = \frac{(2k-1)!!}{6} x^{2k-1}[4(k-1) + 3x].$$

3. Main Results

We first show the application of Theorem 1.1 to our operator (1.2)

THEOREM 3.1. *The sequence of modified Post–Widder operators* $\tilde{P}_n \colon C^*[0,\infty) \to C^*[0,\infty)$ *satisfy*

$$\|\tilde{P}_n f - f\|_\infty \leqslant 2\omega^*(f, \sqrt{c_n}), \quad f \in C^*[0,\infty).$$

Here the convergence takes place if n *is sufficiently large.*

PROOF. The operators $\tilde{P}_n$ preserve constant functions as well as e^{-x} so by Theorem 1.1, $a_n = b_n = 0$. We only have to evaluate c_n. In view of Lemma 2.1, we have

$$\tilde{P}_n(e^{-2t}; x) = (1 + 2(e^{x/(n+1)} - 1))^{-(n+1)}.$$

Let

$$f_n(x) = (2e^{x/(n+1)} - 1)^{-(n+1)} - e^{-2x}$$

Since $f_n(0) = f_n(\infty) = 0$, there exists a point $\xi_n \in (0, \infty)$ such that

$$\|f_n\|_\infty = f_n(\xi_n).$$

It follows that $f_n'(\xi_n)=0$, i.e.,

$$e^{-2\xi_n}=e^{\frac{\xi_n}{1+n}}\big(-1+2e^{\frac{\xi_n}{1+n}}\big)^{-n-2}$$

and

$$f_n(\xi_n)=\big(2e^{\frac{\xi_n}{1+n}}-1\big)^{-n-1}-e^{-2\xi_n}=\big(e^{\frac{\xi_n}{1+n}}-1\big)\big(2e^{\frac{\xi_n}{1+n}}-1\big)^{-2-n}.$$

Let $x_n:=e^{\frac{\xi_n}{1+n}}-1>0$. It follows that

$$f_n(\xi_n)=\frac{x_n}{(2x_n+1)^{n+2}}\leqslant\min\Big\{x_n,\frac{1}{(2x_n+1)^{n+1}}\Big\}\to 0\ \text{ as }\ n\to\infty. \qquad\square$$

Next, we prove the quantitative asymptotic formula.

THEOREM 3.2. *Let $f,f''\in C^*[0,\infty)$, then, for $x\in[0,\infty)$, the following inequality holds:*

$$\Big|n[\tilde{P}_n(f,x)-f(x)]-\frac{x^2}{2}[f'(x)+f''(x)]\Big|\leqslant|p_n(x)||f'|+|q_n(x)||f''| + \tfrac{1}{2}(2q_n(x)+x^2+r_n(x))\omega^*(f'',n^{-1/2}),$$

where

$$p_n(x)=n[(n+1)(e^{x/(n+1)}-1)-x]-\frac{x^2}{2},$$
$$q_n(x)=\tfrac{1}{2}\big(n[(e^{x/(n+1)}-1)^2(n+1)(n+2)+x^2-2x(n+1)(e^{x/(n+1)}-1)]-x^2\big),$$
$$r_n(x)=n^2\big[\tilde{P}_n((e^{-x}-e^{-t})^4,x)U_4^{\tilde{P}_n}(x)\big]^{1/2}.$$

PROOF. By the Taylor's formula, there exist ξ lying between x and t such that

$$f(t)=f(x)+(t-x)f'(x)+(t-x)^2\frac{f''(x)}{2}+h(\xi,x)(t-x)^2,$$

where $h(t,x):=\frac{f''(t)-f''(x)}{2}$ is a continuous function and ξ is between x and t. Applying the operator $\tilde{P}_n$ to above equality and using Lemma 2.3, we can write that

$$\Big|\tilde{P}_n(f,x)-f(x)-U_1^{\tilde{P}_n}(x)f'(x)-\frac{f''(x)}{2}U_2^{\tilde{P}_n}(x)\Big|\leqslant\tilde{P}_n(|h(\xi,x)|(t-x)^2,x).$$

Again using Lemma 2.3, we get

$$\Big|n[\tilde{P}_n(f,x)-f(x)]-\frac{x^2}{2}[f'(x)+f''(x)]\Big|$$
$$\leqslant\Big|nU_1^{\tilde{P}_n}(x)-\frac{x^2}{2}\Big||f'(x)|+\frac{1}{2}\Big|nU_2^{\tilde{P}_n}(x)-x^2\Big||f''(x)|+\big|n\tilde{P}_n(h(\xi,x)(t-x)^2,x)\big|.$$

Let $p_n(x):=nU_1^{\tilde{P}_n}(x)-\frac{x^2}{2}$ and $q_n(x):=\frac{1}{2}(nU_2^{\tilde{P}_n}(x)-x^2)$. Then,

$$\Big|n\ [\tilde{P}_n(f,x)-f(x)]-\frac{x^2}{2}[f'(x)+f''(x)]\Big|$$
$$\leqslant|p_n(x)||f'(x)|+|q_n(x)||f''(x)|+\big|n\ \tilde{P}_n(h(\xi,x)\ (t-x)^2,x)\big|.$$

Also, from Lemma 2.3, we have $p_n(x) \to 0$ and $q_n(x) \to 0$ for n sufficiently large. Now, we just have to compute the last estimate: $n\tilde{P}_n(h(\xi,x)(t-x)^2,x)$. Using the property of $\omega^*(\cdot,\delta)$: $|f(t)-f(x)| \leqslant \big(1+\frac{(e^{-x}-e^{-t})^2}{\delta^2}\big)\omega^*(f,\delta)$, $\delta>0$, we get that

$$|h(\xi,x)| \leqslant \frac{1}{2}\Big(1+\frac{(e^{-x}-e^{-t})^2}{\delta^2}\Big)\omega^*(f'',\delta).$$

Hence, we get

$$n\tilde{P}_n(|h(\xi,x)|(t-x)^2,x) \leqslant \frac{1}{2}n\omega^*(f'',\delta)U_2^{\tilde{P}_n}(x) + \frac{n}{2\delta^2}\omega^*(f'',\delta)\tilde{P}_n((e^{-x}-e^{-t})^2(t-x)^2,x).$$

Applying the Cauchy–Schwarz inequality, we obtain

$$n\tilde{P}_n(|h(\xi,x)|(t-x)^2,x) \leqslant \frac{1}{2}n\omega^*(f'',\delta)U_2^{\tilde{P}_n}(x) + \frac{n}{2\delta^2}\omega^*(f'',\delta)\big[\tilde{P}_n((e^{-x}-e^{-t})^4,x)\cdot U_4^{\tilde{P}_n}(x)\big]^{1/2}.$$

Considering

$$r_n(x) := [n^2\tilde{P}_n((e^{-x}-e^{-t})^4,x)]^{1/2}\cdot[n^2U_4^{\tilde{P}_n}(x)]^{1/2}.$$

and choosing $\delta = n^{-1/2}$, we finally get the desired result. □

REMARK 3.1. The convergence of modified Post–Widder operators $\tilde{P}_n$ in the above theorem takes place for n sufficiently large. Using the software Mathematica, we find that

$$\begin{aligned}
&\lim_{n\to\infty} n^2\tilde{P}_n((e^{-x}-e^{-t})^4,x)\\
&\quad= \lim_{n\to\infty} n^2(\tilde{P}_n(e^{-4t},x) - 4e^{-x}\tilde{P}_n(e^{-3t},x)\\
&\qquad + 6e^{-2x}\tilde{P}_n(e^{-2t},x) - 4e^{-3x}\tilde{P}_n(e^{-t},x) + e^{-4x})\\
&\quad= \lim_{n\to\infty} n^2\big((1+4(e^{x/(n+1)}-1))^{-(n+1)} - 4e^{-x}(1+3(e^{x/(n+1)}-1))^{-(n+1)}\\
&\qquad + 6e^{-2x}(1+2(e^{x/(n+1)}-1))^{-(n+1)} - 4e^{-3x}(1+(e^{x/(n+1)}-1))^{-(n+1)} + e^{-4x}\big)\\
&\quad= 3x^4e^{-4x}.
\end{aligned}$$

and by Lemma 2.3 $\lim_{n\to\infty} n^2U_4^{\tilde{P}_n}(x) = 3x^4$.

References

1. U. Abel, V. Gupta, R. N. Mohapatra, *Local approximation by a variant of Bernstein–Durrmeyer operators*, Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods **68**(11) (2008), 3372–3381.
2. B. D. Boyanov, V. M. Veselinov, *A note on the approximation of functions in an infinite interval by linear positive operators*, Bull. Math. Soc. Sci. Math. Répub. Soc. Roum., Nouv. Sér. **14**(62) (1970), 9–13.
3. Z. Finta, V. Gupta, *Direct and inverse estimates for Phillips type operators*, J. Math. Anal. Appl. **303**(2) (2005), 627–642.
4. V. Gupta, *An estimate on the convergence of Baskakov–Bézier operators*, J. Math Anal. Appl. **12**(1) (2005), 280–288.

5. V. Gupta, R. P. Agarwal, *Convergence Estimates in Approximation Theory*, Springer, 2014.
6. V. Gupta, G. Tachev, *Approximation with positive linear operators and linear combinations*, Dev. Math. **50**, Springer, Dordrecht, 2017.
7. V. Gupta, Tachev, *On approximation properties of phillips operators preserving exponential functions*, Mediterr. J. Math. **14**(4) (2017), Article 177.
8. A. Holhoş, *The rate of approximation of functions in an infinite interval by positive linear operators*, Stud. Univ. Babeş-Bolyai, Math. **55**(2) (2010), 133–142.
9. H. Karsli, V. Gupta, *Some approximation properties of q-Chlodowsky operators*, Appl. Math. Comput. **195**(1) (2008), 220–229.
10. L. M. Kocić, G. V. Milovanović, *Shape preserving approximations by polynomials and splines*, Comput. Math. Appl. **33**(11) (1997), 59–97.
11. G. V. Milovanović, Th. M. Rassias (eds.), *Analytic Number Theory, Approximation Theory, and Special Functions* (*In Honor of Hari M. Srivastava*), Springer-Verlag, New York, 2014.
12. D. V. Widder, *The Laplace Transform*, Princeton Mathematical Series **6**, Princeton Univ. Press, Princeton, N. J., 1941.

(Received 28 08 2017)

Department of Mathematics
Netaji Subhas University of Technology
New Delhi
India
vijaygupta2001@hotmail.com

Department of Mathematics
SRM University
Modinagar
India
mprerna_anand@yahoo.com