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A NOTE ON SOME TOPOLOGICAL PROPERTIES OF KÖTHE SPACE $\lambda(P)$

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Abstract. We emphasize some topological properties of the Köthe space $\lambda(P)$ determined by a Köthe set P.

1. Introduction

Let ω be the vector space of all real or complex valued sequences. Any vector subspace of ω is called a *sequence space*. A complete metrizable locally convex space is called a *Fréchet space*. A sequence space λ with linear topology is called a *K-space* if each of the maps $r_n\colon \lambda \to \mathbb{C}$ defined by $r_n(x) = x_n$ is continuous for all $x = (x_n) \in \lambda$ and every $n \in \mathbb{N}$, where $\mathbb{C}$ and $\mathbb{N}$ denote the complex field and the set of natural numbers, respectively. A K-space λ is called an *FK-space* if λ is a complete linear metric space.

Let (λ, τ) be a K-space, $\varphi \subset \lambda$ the space of all sequences with only finitely many non-zero coordinates, and let $x = (x_k) \in \lambda$ be given. Then, $x^{[n]} = \sum_{k=0}^{n} x_k e^k$ is called the n^{th} *section* of the sequence $x = (x_k)$, where e^k is a sequence whose only nonzero term is 1 in the k^{th} place for each $k \in \mathbb{N}$. The distinguished subsets S_λ, W_λ, F_λ and L_λ of λ are given, as follows:

$$
\begin{aligned}
S_\lambda &:= \big\{x \in \lambda\colon x^{[n]} \to x \ \text{ in } \ \lambda\big\},\\
W_\lambda &:= \big\{x \in \lambda\colon x^{[n]} \to x \ \text{ in } \ \sigma(\lambda, \lambda')\big\},\\
F_\lambda &:= \Big\{x \in \lambda : \sum_{k=0}^{\infty} x_k f(e^k) \ \text{ exists for each } \ f \in \lambda'\Big\},\\
L_\lambda &:= \big\{x \in \lambda : \{x^{[n]} : n \in \mathbb{N}\} \ \text{ is bounded in } \ \lambda\big\},
\end{aligned}
$$

where λ' is the continuous dual of the space λ. Obviously, $S_\lambda \subset W_\lambda \subset F_\lambda \subset L_\lambda$ for a K-space λ with $\varphi \subset \lambda$. An FK-space λ satisfying $\lambda = S_\lambda$ and $\lambda = L_\lambda$, respectively, is called an *AK-space* and an *AB-space*.

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The α-, β-, γ- and f-duals λ^α, λ^β, λ^γ and λ^f of a sequence space λ are defined as follows:

$$\lambda^\alpha := \{x = (x_k) \in \omega : xy = (x_k y_k) \in \ell_1 \text{ for all } y = (y_k) \in \lambda\},$$
$$\lambda^\beta := \{x = (x_k) \in \omega : xy = (x_k y_k) \in cs \text{ for all } y = (y_k) \in \lambda\},$$
$$\lambda^\gamma := \{x = (x_k) \in \omega : xy = (x_k y_k) \in bs \text{ for all } y = (y_k) \in \lambda\},$$
$$\lambda^f := \{(f(e^k)) : f \in \lambda'\}.$$

Let λ be a sequence space and $A \subset \lambda$. A is called *normal* in λ [**18**] (or *a normal subset* of λ) if

$$|y_n| \leqslant |x_n|, \quad x = (x_n) \in A \quad \text{and} \quad y = (y_n) \in \lambda \quad \text{imply that} \quad y \in A.$$

A subset λ of ω is said to be *normal* if whenever it contains $x = (x_n)$ it also contains all vectors $y = (y_n)$ with $|y_n| \leqslant |x_n|$ for $n \in \mathbb{N}$, (see Köthe [**9**, p. 405]).

The intersection and the union of a family of normal subsets of a sequence space λ are clearly normal in λ. Therefore, for any subset A of λ, the *normal hull* of A in λ, denoted by $h(A)$, is defined to be the intersection of all normal subsets of λ containing A. The normal hull $h(A)$ of A in λ is the smallest normal subset of λ containing A [**18**] and

$$h(A) := \{y = (y_n) \in \lambda : |y_n| \leqslant |x_n| \quad \text{for some} \quad x = (x_n) \in A\}.$$

Let λ be a locally convex space. Then,

(i) λ is called *bornological* if every circled, convex subset $A \subset \lambda$ that absorbs every bounded set in λ is a neighborhood of 0 [**12**].

(ii) A subset is called *barrel* if it is absolutely convex, absorbing and closed in λ. Moreover, λ is called a *barrelled space* if each barrel is a neighbourhood of zero [**5**].

Let λ be an FK-space. Then, we say that

(i) λ is a wedge (weak wedge) space if $e^n \to 0$ ($e^n \to 0$ weakly) in λ [**3**].

(ii) λ is a conservative space if $c \subset \lambda$ [**17**].

(ii) λ is a semiconservative space if $\lambda^f \subset cs$, or equivalently $c_0 \subset \lambda$ [**17**].

Let λ be a sequence space. Then a seminorm $q\colon \lambda \to \mathbb{R}$ is called *monotone* [**10**] if $q(x) = \sup_{m\in\mathbb{N}} q(x^{[m]})$ for each $x \in \lambda$.

Define ψ^n by

$$\psi^n = e - \sum_{k=0}^{n} e^k = (0, 0, \ldots, 0, 1, 1, 1, \ldots),$$

where $e = (1, 1, 1, \ldots, 1, \ldots)$. An FK-space λ is called *conull* if $\psi^n \to 0$ weakly in λ ; otherwise *coregular* [**13**].

A subset B of a Hausdorff space λ is called *relatively compact* if its closure $\bar{B}$ is compact. Every sequence on B then has an adherent point in λ [**9**].

A set B is called *Rosenthal* if every sequence (x_n) in B contains a weakly Cauchy subsequence. Let λ be a locally convex space. A bounded set $B \subseteq \lambda$ is called *strongly Rosenthal* if every operator $T\colon \lambda \to \ell_1$ maps B into a relatively compact set. A locally convex space λ is called *Schur*, if every $\sigma(\lambda,\lambda')$-convergent sequence also converges in λ [**4**].

Let λ be a nonempty subset of ω and $B := \{y \in bv : \|y\|_{bv} \leqslant 1\}$ be the unit ball of the BK-space $(bv, \|\cdot\|_{bv})$. Then, λ is called *B-invariant* if

$$\lambda = B \cdot \lambda := \{yx = (y_k x_k) : y = (y_k) \in B,\ x = (x_k) \in \lambda\}.$$

If B_0 denotes the unit ball of the BK-space $(bv_0, \|\cdot\|_{bv_0})$, then λ is called B_0-*invariant* if $\lambda = B_0 \cdot \lambda$ [**7, 6**].

A set of sequences of non-negative numbers P is called *Köthe set* [**15**] if it satisfies the following conditions:

(i) If $a = (a_n)$ and $b = (b_n)$ are two elements of P, then there is a $c = (c_n)$ in P with $\max\{a_n, b_n\} \leqslant c_n$ for each $n \in \mathbb{N}$.

(ii) There exists an $a = (a_n) \in P$ such that $a_n > 0$ for each $n \in \mathbb{N}$.

For any Köthe set P, the set $\lambda(P)$ defined by

$$\lambda(P) := \left\{x = (x_n) \in \omega : p_a(x) = \sum_{n=0}^{\infty} a_n |x_n| < \infty \quad \text{for all} \quad a = (a_n) \in P\right\},$$

is a vector subspace of ω, which is called the *Köthe space* or the *Köthe space determined by P*. $\lambda(P)$ is always normal and $\lambda(P) \supset \varphi$ [**15**].

Let P be a Köthe set. For any $a = (a_n) \in P$,

$$p_a(x) = \sum_{n=0}^{\infty} a_n |x_n| \quad \text{for all} \quad x = (x_n) \in \lambda(P)$$

is a seminorm on $\lambda(P)$. The family $\{p_a : a \in P\}$ determines a Hausdorff locally convex topology on $\lambda(P)$, which is denoted by τ_P and is called *the natural topology determined by P* [**15**] and the space $(\lambda(P), \tau_P)$ is complete [**18**]. Also, the sets $U_{a,\varepsilon} := \{x \in \lambda(P) : p_a(x) \leqslant \varepsilon\}$, where $a \in P$ and $\varepsilon > 0$, form a zero neighborhood basis in $\lambda(P)$ [**15**].

If P is countable and (ε_n) is any null sequence of positive numbers, then the sets U_{a,ε_n} for $a \in P$ and $n \in \mathbb{N}$, form a countable zero neighborhood basis, i.e., τ_P is metrizable [**8**].

Let P be a Köthe set. Then, the topological dual of $(\lambda(P), \tau_P)$ is isomorphic to the vector subspace of $\mathbb{C}^{\mathbb{N}}$ consisting of sequences $y = (y_n)$ with the following property $|y_n| \leqslant C a_n$, for all $n \in \mathbb{N}$, for some $C > 0$ and $a = (a_n) \in P$. Consequently, $\lambda(P)'$ is the normal hull in $\mathbb{C}^{\mathbb{N}}$ of P, (see Wong [**18**, p. 192]).

The set P defined by

$$P := \bigl\{\{(n+1)^k\}_n : k = 1, 2, \dots\bigr\}, \tag{1.1}$$

is obviously a (countable) Köthe set, the natural topology on $\lambda(P)$ is determined by the family $\{p_k : k = 1, 2, \dots\}$ of seminorms, where $p_k(x) = \sum_{n=1}^{\infty} (n+1)^k |x_n|$ for any $x = (x_n) \in \lambda(P)$. Then, the space $(\lambda(P), \tau_P)$ is called the *Fréchet space of rapidly decreasing sequences*, and denoted by s [**15**]. Also, one can find topological properties of the sequence spaces in the recent papers [**1, 2**] and of a Köthe space in [**16, 19**].

Now, we may quote some required lemmas, below.

Lemma 1.1. [**17**, Theorem 7.2.7, p. 106] *Let $\lambda \supset \phi$ be an FK-space. Then, the following statements hold:*

(i) $\lambda^\beta \subset \lambda^\gamma \subset \lambda^f$.
(ii) *If λ has AK-property, then $\lambda^\beta = \lambda^f$*
(iii) *If λ has AD-property, then $\lambda^\beta = \lambda^\gamma$.*

Lemma 1.2. [**12**, p. 60] *Every Fréchet space is a barrelled space.*

Lemma 1.3. [**12**, p. 61] *Every Fréchet space is bornological.*

Lemma 1.4. [**18**, p. 191] *Let P be a Köthe set and*

$$K_p := \{x = (x_n) \in \lambda(P) : x_n \geqslant 0 \quad \text{for all} \quad n \in \mathbb{N}\},$$

which is a Köthe set. Then, α-dual of the space $\lambda(P)$ is $\lambda(K_p)$.

Lemma 1.5. [**11**, Remark 1.27, p. 157] *If λ is a normal subset of ω, then $\lambda^\alpha = \lambda^\beta$.*

Lemma 1.6. [**3**] *An FK-space λ is a weak wedge space if and only if λ contains ℓ_1 and the inclusion mapping is weakly compact.*

Lemma 1.7. [**5**, Theorem 6.3.14, p. 285] *If (X, p) and (Y, q) are semi-normed spaces and $T : X \to Y$ is a linear map, then the following statements are equivalent:*

(i) *T is continuous.*
(ii) *T is continuous at zero.*
(iii) *There exists an $M > 0$ for all $x \in X$ such that $q(Tx) \leqslant Mp(x)$.*

Lemma 1.8. [**6**] *If λ is a barrelled AK-space, then $\lambda' = \lambda^\beta = \lambda^\gamma$.*

Lemma 1.9. [**5**, Theorem 11.4.13, p. 555] *If λ is an $FK - AK$-space, then $B_0 \cdot \lambda = S_\lambda$.*

In this paper, we use standard terminology and notation due to [**9**] and [**15**].

2. Main results

Theorem 2.1. *The space $\lambda(P)$ is an FK-space.*

Proof. Let $x = (x_n) \in \lambda(P)$. Then, there exists an $M > 0$ such that

$$|r_n(x)| = |x_n| \leqslant M\left(\sum_{n=0}^{\infty} a_n|x_n|\right) = Mp_a(x),$$

where $r_n : \lambda(P) \to \mathbb{C}$ for each $n \in \mathbb{N}$. Therefore, one can see by Lemma 1.7 that each of the linear maps P_n is continuous. So, the space $\lambda(P)$ is a K-space. Since it is a complete space with the natural topology determined by P, it is an FK-space. □

Since $(\lambda(P), \tau_P)$ is complete and τ_P is metrizable whenever P is countable, under this condition the space $\lambda(P)$ has the following properties:

(i) It is barrelled by Lemma 1.2.
(ii) It is bornological by Lemma 1.3.
(iii) $\lambda(P)' = \lambda(P)^\beta = \lambda(P)^\gamma$ by Lemma 1.8 and Theorem 2.2.

THEOREM 2.2. *The space $\lambda(P)$ is an AK-space.*

PROOF. Let $x = (x_n) \in \lambda(P)$. Then, we derive that

$$\lim_{m\to\infty} p_a\big(x - x^{[m]}\big) = \lim_{m\to\infty} \left(\sum_{n\geqslant m+1} a_n|x_n| \right) = 0,$$

as desired. □

Since $S_{\lambda(P)} = \lambda(P)$ by Theorem 2.2, $W_{\lambda(P)} = F_{\lambda(P)} = L_{\lambda(P)} = \lambda(P)$. Hence, every $\sigma(\lambda(P), \lambda(P)')$-convergent sequence also converges in $\lambda(P)$. Thus, the space $\lambda(P)$ is a Schur space.

THEOREM 2.3. *If P consists of null sequences, the space $\lambda(P)$ is a wedge space.*

PROOF. Suppose that P consists of null sequences. Then, we have

$$p_a(e^n - 0) = \sum_{n=0}^{\infty} a_n|e^n| = a_n \to 0, \quad \text{as} \quad n \to \infty.$$ □

THEOREM 2.4. *The space $\lambda(P)$ has monotone seminorm.*

PROOF. Let $x \in \lambda(P)$. Then, we have

$$p_a(x) = \sum_{n=0}^{\infty} a_n|x_n| = \sup_{m\in\mathbb{N}} \sum_{n=0}^{m} a_n|x_n| = \sup_{m\in\mathbb{N}} p_a(x^{[m]}),$$

where $p_a\colon \lambda(P) \to \mathbb{R}$'s are seminorms for all $a \in P$. □

REMARK 2.1. Let the Köthe set P be as in (1.1). Then, we have the following results:

(i) Take the sequence $x = (x_n)$ defined by $x_n = 1$ for each $n \in \mathbb{N}$. It is immediate that $x \in c$ but since $\sum_{n=0}^{\infty} a_n|x_n| = \sum_{n=0}^{\infty}(n+1)^k = \infty$,, $x \notin \lambda(P)$. So, the space $\lambda(P)$ is not conservative for every Köthe set P.

(ii) Take the sequence $x = (x_n)$ defined by $x_n = 1/(n+1)^2$ for each $n \in \mathbb{N}$, which is in ℓ_1 but since $\sum_{n=0}^{\infty} a_n|x_n| = \sum_{n=0}^{\infty}(n+1)^{k-2} = \infty$, $x \notin \lambda(P)$. So, by Lemma 1.6 the space $\lambda(P)$ is not a weak wedge for every Köthe set P. Also, since $x \in c_0\backslash\lambda(P)$, the inclusion $c_0 \subset \lambda(P)$ does not hold. Hence, $\lambda(P)$ is not semiconservative space for every Köthe set P.

THEOREM 2.5. *If $P \subset \ell_1$, then the space $\lambda(P)$ is conull.*

PROOF. Let $a = (a_n) \in P \subset \ell_1$ and $u \in \lambda(P)'$. Then, for all $x \in \lambda(P)$

$$u(x) = \sum_{n=0}^{\infty} x_n u(e^n) = \sum_{n=0}^{\infty} x_n u_n, \tag{2.1}$$

where $u_n = u(e^n)$ for each $n \in \mathbb{N}$. Since $u \in \lambda(P)'$, $|u_n| \leqslant |a_n|$ for some $a \in P$. Then, by (2.1) we have that

$$|u(\psi^n)| = \left| \sum_{k\geqslant n+1} u_n \right| \leqslant \sum_{k\geqslant n+1} |u_n| \leqslant \sum_{k\geqslant n+1} |a_n|.$$

Thus, $u(\psi^n) \to u(0)$ as $n \to \infty$, as desired. □

Corollary 2.1. *Since conull spaces are semiconservative* (see Snyder [**14**]), *the space* $\lambda(P)$ *is semiconservative whenever* $P \subset \ell_1$.

With a similar way in Boos [**5**], we have that: if λ is a normal subset of ω, then $\lambda^\gamma \subset \lambda^\alpha$. Also, always the inclusion $\lambda^\alpha \subset \lambda^\beta$ holds. Then, by Part (i) of Lemma 1.1 and Lemma 1.5 we obtain that $\lambda^\alpha = \lambda^\beta = \lambda^\gamma$ for a normal subset of ω.

Corollary 2.2. *We have the following results:*

(i) $\{\lambda(P)\}^\alpha = \{\lambda(P)\}^\beta = \{\lambda(P)\}^\gamma = \lambda(K_p)$.

(ii) *Since* $\lambda(P)$ *is an* AK*-space, the equality* $\{\lambda(P)\}^f = \lambda(K_p)$ *holds by Part* (ii) *of Lemma* 1.1.

Theorem 2.6. *The following statements hold:*

(i) $\lambda(P) = \lambda(P) \cdot c_0 := \{zx : z \in \lambda(P),\ x \in c_0\}$.

(ii) *If* P *consists of unbounded sequences, then* $\ell_1 = \lambda(P) \cdot c_0$.

Proof. (i) Let $y = (y_n) \in \lambda(P) \cdot c_0$. Then, $y_n = z_n x_n$ for all $n \in \mathbb{N}$, where $(z_n) \in \lambda(P)$ and $(x_n) \in c_0$. Therefore, we see that

$$\sum_{n=0}^{\infty} a_n |y_n| = \sum_{n=0}^{\infty} a_n |z_n x_n| \leqslant \|x\|_\infty \sum_{n=0}^{\infty} a_n |z_n| < \infty,$$

where $a = (a_n) \in P$. Therefore, $y \in \lambda(P)$.

Let $y = (y_n) \in \lambda(P)$. Then, $\sum_{n=0}^{\infty} a_n |y_n| < \infty$ for $a = (a_n) \in P$. We have to show that there exist a $z = (z_n) \in \lambda(P)$ and a $x = (x_n) \in c_0$ such that $y_n = z_n x_n$ for all $n \in \mathbb{N}$. Let (n_k) be an increasing sequence of positive integers such that $\sum_{j=n_k}^{n_{k+1}} a_j |y_j| < 1/4^k$ for all $k \in \mathbb{N}$.

Take the sequence $x = (x_j) \in c_0$ defined by Wong [**18**] as

$$x_j := \begin{cases} 1, & 0 \leqslant j < n_1, \\ 1/4^k, & n_{k+1} \leqslant j < n_{k+2} \end{cases}$$

for all $k \in \mathbb{N}$. So, we have

$$\sum_{j=0}^{\infty} a_j |y_j| \frac{1}{x_j} < \sum_{j=0}^{\infty} a_j |y_j| + \frac{1}{2} < \infty.$$

Thus, the sequence $z = (z_n)$ defined by $z_n = y_n / x_n$ for all $n \in \mathbb{N}$ is in $\lambda(P)$.

(ii) This is obvious from the proof of part (i). □

Theorem 2.7. *The space* $\lambda(P)$ *is* B*-invariant.*

Proof. Let $y = (y_n) \in B \cdot \lambda(P)$. Then, $y_n = z_n x_n$ for all $n \in \mathbb{N}$, where $(x_n) \in B$ and $(z_n) \in \lambda(P)$. Since $bv \subset \ell_\infty$, we derive for $a = (a_n) \in P$ that

$$\sum_{n=0}^{\infty} a_n |y_n| = \sum_{n=0}^{\infty} a_n |z_n x_n| \leqslant \|x\|_\infty \sum_{n=0}^{\infty} a_n |z_n| < \infty.$$

Hence, $y \in \lambda(P)$.

Let $y = (y_n) \in \lambda(P)$. Then, $\sum_{n=0}^{\infty} a_n|y_n| < \infty$ for $a = (a_n) \in P$. We want to show that there exist a $x = (x_n) \in B$ and $z = (z_n) \in \lambda(P)$ such that $y_n = z_n x_n$ for all $n \in \mathbb{N}$. Take the sequence $x = e \in B$. Then, we have

$$\sum_{n=0}^{\infty} a_n|y_n|\frac{1}{x_n} = \sum_{j=n}^{\infty} a_n|y_n| < \infty.$$

Thus, the sequence $z = (z_n)$ defined by $z_n = y_n/x_n$ for all $n \in \mathbb{N}$ is in $\lambda(P)$. This completes the proof. □

Corollary 2.3. *Since $S_{\lambda(P)} = \lambda(P)$ from Theorem* 2.2*, the space $\lambda(P)$ is B_0-invariant by Lemma* 1.9*.*

Theorem 2.8. *Every subset of $\lambda(P)$ is* Rosenthal*.*

Proof. Let B be a subset of $\lambda(P)$, $z = (z_n) \in B$ and $u = (u_n) \in \lambda(P)'$. Hence, $\sum_{n=0}^{\infty} a_n|z_n| < \infty$ for all $a = (a_n) \in P$ and $|u_n| \leqslant |a_n|$ for some $a \in P$. So, we have that

$$\begin{aligned} |u(x_n) - u(x_{2n})| &= \left|\sum_{n=0}^{\infty} x_n u_n - \sum_{n=0}^{\infty} x_{2n}u_{2n}\right| \leqslant \sum_{n=0}^{\infty} |x_n u_n| + \sum_{n=0}^{\infty} |x_{2n}u_{2n}| \\ &\leqslant \sum_{n=0}^{\infty} a_n|x_n| + \sum_{n=0}^{\infty} a_{2n}|x_{2n}| \leqslant 2\sum_{n=0}^{\infty} a_n|x_n|. \end{aligned} \tag{2.2}$$

Letting $n \to \infty$ in (2.2), we have $u(x_n) \to u(x_{2n})$, as desired. □

Remark 2.2. If P consists of unbounded sequences, there exists a subset of $\lambda(P)$ such that it is not strongly Rosenthal.

Let P consists of unbounded sequences, $I\colon \lambda(P) \to \ell_1$ be the identity map and B be a bounded subset of $\lambda(P)$. So, the identity map I is well defined. We want to show that $I(B)$ is not relatively compact in ℓ_1, that is, its closure $\overline{I(B)}$ is not compact in ℓ_1. The closure of $I(B)$ is the set

$$\overline{I(B)} := \left\{x \in \lambda(P) : \sum_{n=0}^{\infty} |x_n| < \infty \text{ or } \sum_{n=0}^{\infty} |x_n - z_n| < \varepsilon_i \text{ for } i \in \mathbb{N} \text{ and } z \in I(B)\right\}.$$

Assume that $(t_n) \subset \overline{I(B)}$. Then, there exist the following two cases:

(i) $\sum_{n=0}^{\infty} |t_n| < \infty$.
(ii) $\sum_{n=0}^{\infty} |t_n - z_n| < \varepsilon_i$ for $i \in \mathbb{N}$ and $z \in I(B)$.

Let (t_{n_k}) be a subsequence of (t_n) and $\xi \in I(B) \subset \ell_1$. Then, we have that

For part (i): Since $(t_n) \subset \overline{I(B)}$,

$$\sum_{k=0}^{\infty} |t_{n_k} - \xi| \leqslant \sum_{n=0}^{\infty} |t_n - \xi| \leqslant \sum_{n=0}^{\infty} (|t_n| + |\xi|).$$

Then, the convergence of the series $\sum_{n=0}^{\infty} |t_n|$ implies that $|t_n| \to 0$, as $n \to \infty$. Thus, $|t_n| + |\xi| \not\to 0$, as $n \to \infty$, that is, the series $\sum_{n=0}^{\infty}(|t_n| + |\xi|)$ is divergent. So, $t_{n_k} \not\to t$, as $k \to \infty$. Hence, $\overline{I(B)}$ is not compact in ℓ_1 and so is not strongly Rosenthal.

For part (ii): With the similar way used in Part (i), we derive that

$$\sum_{k=0}^{\infty}|t_{n_k}-\xi| \leqslant \sum_{n=0}^{\infty}|t_n-\xi| \leqslant \sum_{n=0}^{\infty}(|t_n-z_n|+|z_n-\xi|),$$

where $z \in I(B)$. Since $z \in I(B)$, $|z_n| \to 0$, as $n \to \infty$, and so $(|t_n-z_n|+|z_n-\xi|) \not\to 0$, as $n \to \infty$. That is to say that the series $\sum_{n=0}^{\infty}[|t_n-z_n|+|z_n-\xi|]$ is not convergent. So, $t_{n_k} \not\to t$, as $k \to \infty$. Hence, $\overline{I(B)}$ is not compact in ℓ_1 and so is not strongly Rosenthal.

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