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ANALYTICAL AND NUMERICAL ASPECT OF COINCIDENCE POINT PROBLEM OF QUASI-CONTRACTIVE OPERATORS

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Abstract. We propose a new Jungck-S iteration method for a class of quasi-contractive operators on a convex metric space and study its strong convergence, rate of convergence and stability. We also provide conditions under which convergence of this method is equivalent to Jungck–Ishikawa iteration method. Some numerical examples are provided to validate the theoretical findings obtained herein. Our results are refinement and extension of the corresponding ones existing in the current literature.

1. Introduction

The notion of convexity plays a fundamental role in the study of nonlinear problems arising in analysis on a linear domain, e.g., in fixed point theorems, topological degree theory and the construction of continuous selections (see, e.g., [**5**, **18**, **33**]). In a nonlinear domain, in the absence of a natural notion of convex set, several attempts have been made to introduce the notion of convexity (see, e.g., [**26**, **32**, **37**]). Among those, an important approach is due to Takahashi [**37**] and is formulated as under:

Let (X, d) be a metric space. A mapping $W\colon X^2 \times [0,1] \to X$ is said to be a W-convex structure (W-CS) in X if for all $(x_1, x_2, \gamma) \in X^2 \times [0,1]$ and $x \in X$, we have

$$d(x, W(x_1, x_2, \gamma)) \leqslant \gamma d(x, x_1) + (1-\gamma) d(x, x_2). \tag{1.1}$$

A metric space (X, d) endowed with a W-CS is called a convex metric space (CMS) and is denoted by (X, d, W). A nonempty subset C of a CMS is convex if $W(x_1, x_2, \gamma) \in C$ for all x_1, $x_2 \in C$ and $\gamma \in [0,1]$. From the definition of

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W-CS on X, it is obvious that

$$d(x, W(x_1, x_2, \gamma)) \geqslant (1-\gamma)d(x, x_2) - \gamma d(x, x_1) \tag{1.2}$$

for all x, x_1, $x_2 \in X$ and $\gamma \in [0,1]$ (see [**6**]).

A CMS (X, d, W) is referred to as a hyperbolic space (HS) (see [**27**]) if the following conditions hold:

$$d(W(x_1, x_2, \gamma_1), W(x_1, x_2, \gamma_2)) = |\gamma_1 - \gamma_2| d(x_1, x_2), \tag{1.3}$$

$$d(W(x_1, x_3, \gamma), W(x_2, x_4, \gamma)) \leqslant \gamma d(x_1, x_2) + (1-\gamma) d(x_3, x_4), \tag{1.4}$$

$$W(x_1, x_2, \gamma) = W(x_2, x_1, 1-\gamma),$$

for all $x_i \in X$, $i = 1, 2, 3, 4$ and γ, γ_1, $\gamma_2 \in [0,1]$.

For some other notions on HS, we refer the reader, for instance, to [**11**, **21**, **34**]. Clearly, every HS is CMS but the converse may not be true, in general. For example, if $X = \mathbb{R}$, $W(x_1, x_2, \gamma) = \gamma x_1 + (1-\gamma)x_2$ and define $d(x_1, x_2) = \frac{|x_1 - x_2|}{1 + |x_1 - x_2|}$ for x_1, $x_2 \in \mathbb{R}$, then (X, d, W) is a CMS but not a HS (the above condition (1.3) does not hold) (see [**22**, p. 7]). Indeed, all normed spaces and convex subsets thereof, as well as, $CAT(0)$ spaces in the sense of Gromov are properly contained in the class of HS as well as CMS. Several examples of CMS which are not imbedded in any normed space or Banach space were given in [**37**].

Metric fixed point theory of nonlinear operators in the setup of HS as well as CMS is an extremely dynamic area of research in nonlinear functional analysis (see, e.g., [**8**–**10**, **17**]). Moreover, iteration methods are the only viable tool to study fixed point problems of various classes of operators. In this connection, for the last 25 years, considerable efforts have been made to introduce various iteration methods and study their qualitative features like convergence, convergence rate, stability, data dependency etc. (see, e.g., [**4**, **6**, **7**, **12**, **14**, **20**, **23**–**25**, **30**, **31**]).

Let X be a CMS, Y an arbitrary set, and T, $S\colon Y \to X$ be two nonself operators such that $T(Y) \subseteq S(Y)$.

Definition 1.1. A point $x \in Y$ is called (i) a coincidence point of the pair (T, S) if $Tx = Sx$ and the set of all the coincidence points of T and S is denoted by $C(T, S)$; If $p = Sx = Tx$ for some x in Y, then p is called a coincidence value of S and T; (ii) common fixed point (provided that $Y = X$) of the pair (T, S) if $x = Tx = Sx$.

Definition 1.2. (See [**31**]) Let T, $S\colon Y \to X$ be nonself operators such that S is bijective, $T(Y) \subseteq S(Y)$ and $p = Tz = Sz$ a coincidence value of S and T. For $x_0 \in Y$, let $\{Sx_n\}_{n=0}^{\infty}$ be a sequence defined by the general iteration method of form

$$Sx_{n+1} = f_{T,\alpha_n}^{x_n}, \ \alpha_n \in [0,1] \text{ for all } n \in \mathbb{N}, \tag{1.5}$$

where $f_{T,\alpha_n}^{x_n}$ is a designated convex structure. Suppose that $\{Sx_n\}_{n=0}^{\infty}$ converges to p. Let $\{Sy_n\}_{n=0}^{\infty} \subseteq X$ be an arbitray sequence and set $\epsilon_n = d(Sy_{n+1}, f_{T,\alpha_n}^{y_n})$ for all $n \in \mathbb{N}$. Then, iteration method (1.5) is said to be (T, S)-stable or stable w.r.t. (T, S) if $\lim_{n\to\infty} \epsilon_n = 0$ implies $\lim_{n\to\infty} Sy_n = p$.

Recently, Olatinwo [**30**] introduced the Jungck–Mann and Jungck–Ishikawa iteration methods in terms of W-CS as follows;

$$\begin{cases} x_0^{(1)} \in Y, \\ Sx_{n+1}^{(1)} = W(Tx_n^{(1)}, Sx_n^{(1)}, \alpha_n) \text{ for all } n \in \mathbb{N}, \end{cases} \tag{1.6}$$

$$\begin{cases} x_0^{(2)} \in Y, \\ Sx_{n+1}^{(2)} = W(Ty_n^{(2)}, Sx_n^{(2)}, \alpha_n), \\ Sy_n^{(2)} = W(Tx_n^{(2)}, Sx_n^{(2)}, \beta_n) \text{ for all } n \in \mathbb{N}, \end{cases} \tag{1.7}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty} \subseteq [0,1]$ are real sequences satisfying certain control condition(s).

Olatinwo [**30**] and Olatinwo and Postolache [**31**] used the following contractive condition to establish some strong convergence and stability results for the Jungck–Mann (1.6) and Jungck–Ishikawa (1.7) iteration methods: there exist $\delta \in [0,1)$ and some $L \geqslant 0$ such that for all x, $y \in Y$, the following condition holds

$$d(Tx, Ty) \leqslant \delta d(Sx, Sy) + Lu(x, y), \tag{1.8}$$

where

$$u(x,y) = \min\{d(Sx,Tx), d(Sy,Ty), d(Sx,Ty), d(Sy,Tx), \\ \tfrac{1}{2}[d(Sx,Tx) + d(Sy,Ty)], \tfrac{1}{2}[d(Sx,Ty) + d(Sy,Tx)]\}.$$

More precisely, they proved:

THEOREM 1.1. *Let (X, d, W) be a complete CMS, Y a nonempty closed and convex subset of X and T, $S\colon Y \to X$ are nonself operators satisfying condition* (1.8) *such that $T(Y) \subseteq S(Y)$, where $S(Y)$ is a complete subspace of X, S is injective operator. Suppose that p is a coincidence value of T and S.*

(i) [**30**, Theorem 9.1] *Let $\{Sx_n^{(1)}\}_{n=0}^{\infty}$ be the Jungck–Mann iteration method* (1.6) *with $\{\alpha_n\}_{n=0}^{\infty} \subseteq [0,1]$, such that $0 < \alpha \leqslant \alpha_n$, for all n. Then, the sequence $\{Sx_n^{(1)}\}_{n=0}^{\infty}$ strongly converges to p.*

(ii) [**30**, Theorem 9.2] *Let $\{Sx_n^{(2)}\}_{n=0}^{\infty}$ be the Jungck–Ishikawa iteration method* (1.7) *with α_n, $\beta_n \in [0,1]$, such that $0 < \alpha \leqslant \alpha_n$, $0 < \beta \leqslant \beta_n$, for all n. Then, the sequence $\{Sx_n^{(2)}\}_{n=0}^{\infty}$ strongly converges to p.*

THEOREM 1.2. *Let (X, d, W) be a complete CMS, Y an arbitrary set, and T, $S\colon Y \to X$ be nonself operators satisfying condition* (1.8) *such that $T(Y) \subseteq S(Y)$, where $S(Y)$ is a complete subspace of X, S is injective operator. Suppose that p is a coincidence vaule of T and S.*

(i) [**31**, Theorem 3.1] *Let $\{Sx_n^{(1)}\}_{n=0}^{\infty}$ be the Jungck–Mann iteration method* (1.6) *converging to p, where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $[0,1]$ such that $0 < \alpha \leqslant \alpha_n$, for all n. Then, the Jungck–Mann iteration method* (1.6) *is (T, S)- stable.*

(ii) [**31**, Theorem 3.2] *Let $\{Sx_n^{(2)}\}_{n=0}^{\infty}$ be the Jungck–Ishikawa iteration method* (1.7) *converging to p, where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$ such* *that $0<\alpha\leqslant\alpha_n$ and $0<\beta\leqslant\beta_n$, for all n. Then, Jungck–Ishikawa iteration method (1.7) is (T,S)-stable.*

In almost all areas of computational and applied mathematics, an iteration method with a faster convergence rate, the simpler implementation and the lightest workload of computation under some conditions for solving certain nonlinear problems is always in demand. It is thus natural to ask whether an iteration method can be constructed which has higher convergence rate than (1.6) and (1.7) iteration methods and does not need the conditions (which are usually required to ensure convergence and stability of the Jungck–Mann and Jungck–Ishikawa iteration methods) "$0<\alpha\leqslant\alpha_n$ and $0<\beta\leqslant\beta_n$, for all n" used in Theorems 1.1 and 1.2. In this context, we would like to construct an iteration method that meets the above mentioned demands. For this we consider a simplified form of (1.7), namely

$$\begin{cases} x_0^{(3)}\in Y, \\ Sx_{n+1}^{(3)}=Ty_n^{(3)}, \\ Sy_n^{(3)}=W(Tx_n^{(3)},Sx_n^{(3)},\beta_n) \text{ for all } n\in\mathbb{N}. \end{cases} \tag{1.9}$$

REMARK 1.1. We call (1.9) "Jungck-S iteration method"; it reduces to:

(i) Jungck-Normal-S iteration method [**16**] if X is a normed space and $W(x_1,x_2,\gamma)=\gamma x_1+(1-\gamma)x_2$.
(ii) Normal-S iteration method [**35**] if X is a normed space, $W(x_1,x_2,\gamma)=\gamma x_1+(1-\gamma)x_2$, $X=Y$, and $S=I_d$ (identity mapping).

However, iteration method (1.9) is independent of Jungck–Mann (1.6) iteration method.

The following definitions and lemmas will be needed to realize our goals.

DEFINITION 1.3. [**28**] Let $\{\tau_n^i\}_{n=0}^\infty$, $i=1,2$ be two sequences converging to the same point η^*. We say that $\{\tau_n^1\}_{n=0}^\infty$ converges faster than $\{\tau_n^2\}_{n=0}^\infty$ to η^*, if

$$\lim_{n\to\infty}\frac{d(\tau_n^1,\eta^*)}{d(\tau_n^2,\eta^*)}=0.$$

DEFINITION 1.4. Let T, $\tilde{T}$, S, $\tilde{S}\colon Y\to X$ be operators. The pair $(\tilde{T},\tilde{S})$ is called an approximate operator pair of (T,S) if for all $x\in Y$, there exist fixed $\varepsilon_1>0$ and $\varepsilon_2>0$ such that $d(Tx,\tilde{T}x)\leqslant\varepsilon_1$, and $d(Sx,\tilde{S}x)\leqslant\varepsilon_2$.

LEMMA 1.1. [**3**] *Let $\{\varphi_n^{(i)}\}_{n=0}^\infty$ for $i=1,2$ be nonnegative sequences of real numbers satisfying $\varphi_{n+1}^{(1)}\leqslant\mu\varphi_n^{(1)}+\varphi_n^{(2)}$, for all $n\in\mathbb{N}$, where $\mu\in[0,1)$ and $\lim_{n\to\infty}\varphi_n^{(2)}=0$. Then, $\lim_{n\to\infty}\varphi_n^{(1)}=0$.*

LEMMA 1.2. [**36**] *Let $\{\varphi_n^{(i)}\}_{n=0}^\infty$ for $i=1,2,3$ be nonnegative real sequences with $\varphi_n^{(3)}\in(0,1)$, for all $n\in\mathbb{N}$, $\sum_{k=0}^\infty\varphi_n^{(3)}=\infty$. Suppose there exists $n_0\in\mathbb{N}$ such that for all $n\geqslant n_0$, one has the inequality $\varphi_{n+1}^{(1)}\leqslant(1-\varphi_n^{(3)})\varphi_n^{(1)}+\varphi_n^{(3)}\varphi_n^{(2)}$. Then $0\leqslant\limsup_{n\to\infty}\varphi_n^{(1)}\leqslant\limsup_{n\to\infty}\varphi_n^{(2)}$.*

2. Main Results

In the rest of the paper, we assume that (X, d, W) is a complete CMS, Y an arbitrary set, T, $S\colon Y \to X$ two nonself operators satisfying condition (1.8) such that $T(Y) \subseteq S(Y)$, where $S(Y)$ is a complete subspace of X, S is a bijective operator, and z is a coincidence point of T and S.

THEOREM 2.1. *Let* $\{Sx_n^{(3)}\}_{n=0}^{\infty}$ *be the iterative sequence* (1.9) *with real sequence* $\{\beta_n\}_{n=0}^{\infty}$ *in* $[0,1]$. *If* $C(T,S) \neq \varnothing$, *then there exists a unique coincidence point* z *and* $\{Sx_n^{(3)}\}_{n=0}^{\infty}$ *converges to* $p = Sz$.

PROOF. The proof of the uniqueness of the coincidence point of T and S was established in [**30**, Theorem 9.1]. We now prove that $\{Sx_n^{(3)}\}_{n=0}^{\infty}$ strongly converges to p. It follows from (1.1), (1.8), and (1.9) that

$$\begin{aligned} d\big(Sx_{n+1}^{(3)}, p\big) &= d(Ty_n^{(3)}, Tz) \\ &\leqslant \delta d(Sy_n^{(3)}, Sz) + L \min\big\{d(Sy_n^{(3)}, Ty_n^{(3)}), \\ &\qquad d(Sz, Tz), d(Sy_n^{(3)}, Tz), d(Sz, Ty_n^{(3)}), \\ &\qquad \tfrac{1}{2}\big[d(Sy_n^{(3)}, Ty_n^{(3)}) + d(Sz, Tz)\big], \\ &\qquad \tfrac{1}{2}\big[d(Sy_n^{(3)}, Tz) + d(Sz, Ty_n^{(3)})\big]\big\} \\ &= \delta d(Sy_n^{(3)}, p), \end{aligned} \tag{2.1}$$

$$\begin{aligned} d(Sy_n^{(3)}, p) &= d(W(Tx_n^{(3)}, Sx_n^{(3)}, \beta_n), p) \\ &\leqslant (1-\beta_n) d(Sx_n^{(3)}, p) + \beta_n d(Tx_n^{(3)}, Tz) \\ &\leqslant (1-\beta_n) d(Sx_n^{(3)}, p) + \beta_n \big\{\delta d(Sx_n^{(3)}, Sz) \\ &\qquad + L \min\big\{d(Sx_n^{(3)}, Tx_n^{(3)}), d(Sz, Tz), d(Sx_n^{(3)}, Tz), \\ &\qquad d(Sz, Tx_n^{(3)}), \tfrac{1}{2}\big[d(Sx_n^{(3)}, Tx_n^{(3)}) + d(Sz, Tz)\big], \\ &\qquad \tfrac{1}{2}\big[d(Sx_n^{(3)}, Tz) + d(Sz, Tx_n^{(3)})\big]\big\}\big\} \\ &= [1 - \beta_n(1-\delta)] d(Sx_n^{(3)}, p). \end{aligned} \tag{2.2}$$

Inserting (2.2) into (2.1), we get $d\big(Sx_{n+1}^{(3)}, p\big) \leqslant \delta[1-\beta_n(1-\delta)] d(Sx_n^{(3)}, p)$ which implies

$$d\big(Sx_{n+1}^{(3)}, p\big) \leqslant \delta^{n+1} \prod_{k=0}^{n} [1-\beta_k(1-\delta)] d(Sx_0^{(3)}, p). \tag{2.3}$$

Since $\delta \in [0,1)$ and $\beta_k \in [0,1]$ for all $k \in \mathbb{N}$, therefore $1 - \beta_k(1-\delta) < 1$ for all $k \in \mathbb{N}$ which implies that $\prod_{k=0}^{n}[1-\beta_k(1-\delta)] < 1$. Using this fact together with (2.3), we obtain

$$d\big(Sx_{n+1}^{(3)}, p\big) \leqslant \delta^{n+1} d(Sx_0^{(3)}, p). \tag{2.4}$$

Taking the limits of both sides of inequality (2.4), we have $\lim_{n\to\infty} d(Sx_n^{(3)}, p) = 0$. □

EXAMPLE 2.1. Let $Y = [0.1, 1] \times [0.1, 1]$ be endowed with the metric $d(x, y) = d((x_1, x_2), (y_1, y_2)) = \|(\ln \frac{x_1}{y_1}, \ln \frac{x_2}{y_2})\|_2$ where $\| \cdot \|_2$ is the Euclidean norm on $\mathbb{R}^2$. Consider the mapping $W : Y^2 \times [0, 1]$ defined by $W(x, y, \alpha) = x^\alpha y^{1-\alpha}$ where $x^\alpha = (x_1^\alpha, x_2^\alpha)$ and $xy = (x_1 y_1, x_2 y_2)$. The mapping W defines a convex structure on (Y, d) and hence the triplet (Y, d, W) is a convex metric space (see [**29**]). Define operators $S, T\colon Y \to [1, 8] \times [1, 32]$ by

$$Sx = S(x_1, x_2) = (x_1^3, x_2^5), \qquad Tx = T(x_1, x_2) = (x_1^2, x_2^4) \tag{2.5}$$

with a unique coincidence value $p = (p_1, p_2) = (1, 1) = S(1, 1) = T(1, 1)$. It can be easily checked that S is a bijective map and $T(Y) = [1, 4] \times [1, 16] \subset S(Y) = [1, 8] \times [1, 32]$.

The Wolfram Mathematica 9 software package implies that

$$\begin{aligned} d(Tx, Ty) &= d\big((x_1^2, x_2^4), (y_1^2, y_2^4)\big) = \left\| \left(\ln \frac{x_1^2}{y_1^2}, \ln \frac{x_2^4}{y_2^4} \right) \right\|_2 \\ &= \sqrt{(\ln x_1^2 - \ln y_1^2)^2 + (\ln x_2^4 - \ln y_2^4)^2} \\ &\leqslant \sqrt{\begin{array}{c} (0.6|\ln x_1^3 - \ln y_1^3| + 40|\ln x_1^2 - \ln y_1^3|)^2 \\ + (0.65|\ln x_2^5 - \ln y_2^5| + 40|\ln x_2^4 - \ln y_2^5|)^2 \end{array}} \end{aligned} \tag{2.6}$$

Now applying the well-known facts

$$(a + b)^p = \begin{cases} 2^{p-1}(a^p + b^p) & \text{if } 1 \leqslant p < \infty \\ a^p + b^p & \text{if } 0 < p < 1 \end{cases}$$

and $\sqrt{a + b} \leqslant \sqrt{a} + \sqrt{b}$ for $a, b > 0$ to (2.6), we obtain

$$\begin{aligned} d(Tx, Ty) &\leqslant \sqrt{\begin{array}{c} 2\big[(0.6|\ln x_1^3 - \ln y_1^3|)^2 + (40|\ln x_1^2 - \ln y_1^3|)^2\big] \\ + 2\big[(0.65|\ln x_1^5 - \ln y_1^5|)^2 + (40|\ln x_2^4 - \ln y_2^5|)^2\big] \end{array}} \\ &\leqslant \sqrt{2(0.6|\ln x_1^3 - \ln y_1^3|)^2 + 2(0.65|\ln x_1^5 - \ln y_1^5|)^2} \\ &\quad + \sqrt{2(40|\ln x_1^2 - \ln y_1^3|)^2 + 2(40|\ln x_2^4 - \ln y_2^5|)^2} \\ &\leqslant \sqrt{2(0.65|\ln x_1^3 - \ln y_1^3|)^2 + 2(0.65|\ln x_1^5 - \ln y_1^5|)^2} \\ &\quad + \sqrt{2(40|\ln x_1^2 - \ln y_1^3|)^2 + 2(40|\ln x_2^4 - \ln y_2^5|)^2} \\ &\leqslant 0.92\sqrt{(|\ln x_1^3 - \ln y_1^3|)^2 + |\ln x_1^5 - \ln y_1^5|^2} \\ &\quad + 56.57\sqrt{|\ln x_1^2 - \ln y_1^3|^2 + |\ln x_2^4 - \ln y_2^5|^2} \\ &= 0.92\left\| \left(\ln \frac{x_1^3}{y_1^3}, \ln \frac{x_1^5}{y_1^5} \right) \right\|_2 + 56.57\left\| \left(\ln \frac{x_1^2}{y_1^3}, \ln \frac{x_2^4}{y_2^5} \right) \right\|_2 \\ &= 0.92d\big((x_1^3, x_1^5), (y_1^3, y_1^5)\big) + 56.57d\big((x_1^2, x_2^4)(y_1^3, y_2^5)\big) \\ &= 0.92d(Sx, Sy) + 56.57d(Tx, Sy). \end{aligned}$$

TABLE 1. Convergence behavior of iteration method (2.7).

# of Iter.	$\left(x_1^{(n+1)}, x_2^{(n+1)}\right)$	$S\left(\left(x_1^{(n+1)}, x_2^{(n+1)}\right)\right)$
0	$(1.5, 1.9)$	$(1.5, 1.9)$
1	$(1.2869684, 1.6371285)$	$(2.1315899, 11.760177)$
2	$(1.1792693, 1.4765607)$	$(1.6399816, 7.0186975)$
3	$(1.1152028, 1.3637682)$	$(1.3869524, 4.7174010)$
4	$(1.0751053, 1.2810032)$	$(1.2426620, 3.4494597)$
5	$(1.0493659, 1.2188232)$	$(1.1555290, 2.6896983)$
⋮	⋮	⋮
15	$(1.0008333, 1.0214332)$	$(1.0025020, 1.1118593)$
16	$(1.0005554, 1.0171096)$	$(1.0016671, 1.0885259)$
⋮	⋮	⋮
30	$(1.0000019, 1.0007462)$	$(1.0000057, 1.0037366)$
⋮	⋮	⋮

which shows that the operators S and T satisfy condition (1.8) with $\delta = 0.92 \in (0, 1)$, $L \geqslant 56.57$, $u(x, y) = d(Tx, Sy)$.

If we put $\beta_n = \frac{1}{2n^3+10n+5}$ for all $n \in \mathbb{N}$, the operators S and T, respectively defined by (2.5), in (1.9), then we have

$$\begin{cases} x_0 \in Y \\ S((x_1^{(n+1)}, x_2^{(n+1)})) = \left((y_1^{(n)})^2, (y_2^{(n)})^4\right) \\ S((y_1^{(n)}, y_2^{(n)})) = \left((x_1^{(n)})^{3-\frac{1}{2n^3+10n+5}}, (x_2^{(n)})^{5-\frac{1}{2n^3+10n+5}}\right) \end{cases} \quad \text{for all } n \in \mathbb{N},$$

which gives

(2.7)
$$\begin{cases} x_0 \in Y \\ (x_1^{(n+1)}, x_2^{(n+1)}) = \left((x_1^{(n)})^{\frac{2}{9}(3-\frac{1}{2n^3+10n+5})}, (x_2^{(n)})^{\frac{4}{25}(5-\frac{1}{2n^3+10n+5})}\right) \\ S((x_1^{(n+1)}, x_2^{(n+1)})) = \left((x_1^{(n)})^{\frac{2}{3}(3-\frac{1}{2n^3+10n+5})}, (x_2^{(n)})^{\frac{4}{5}(5-\frac{1}{2n^3+10n+5})}\right) \end{cases} \quad \text{for all } n \in \mathbb{N}.$$

The convergence results for iteration method (2.7) to $z = (1, 1)$ and $p = (1, 1) = T(1, 1) = S(1, 1)$, respectively, are listed in Table 1.

THEOREM 2.2. *Let (X, d, W) be an HS. Let $\left\{Sx_n^{(2)}\right\}_{n=0}^{\infty}$ be defined by* (1.7) *for $x_0^{(2)} \in Y$ with $\{\alpha_n\}_{n=0}^{\infty} \subseteq [0, 1]$ satisfying $0 < \alpha = \inf_{n\in\mathbb{N}} \alpha_n$, and let $\left\{Sx_n^{(3)}\right\}_{n=0}^{\infty}$ be defined by* (1.9) *for $x_0^{(3)} \in Y$. Then, the following statements are equivalent:*

(i) $\left\{Sx_n^{(3)}\right\}_{n=0}^{\infty}$ *strongly converges to $p = Sz$;*
(ii) $\left\{Sx_n^{(2)}\right\}_{n=0}^{\infty}$ *strongly converges to $p = Sz$.*

TABLE 2. Convergence behavior of $d\big((x_1^{(n+1)}, x_2^{(n+1)}), z\big)$ and $d\big((S(x_1^{(n+1)}, x_2^{(n+1)})), p\big)$ for (2.7).

# of Iter.	$d\big((x_1^{(n+1)}, x_2^{(n+1)}), z\big)$	$d\big(S(x_1^{(n+1)}, x_2^{(n+1)}), p\big)$
0	0.759195867	0.759195867
1	0.553754017	2.578311286
2	0.423164932	2.010390053
3	0.328854019	1.585371124
4	0.258015055	1.257132831
5	0.203668086	0.999933445
$\vdots$	$\vdots$	$\vdots$
15	0.021223091	0.106063100
16	0.016973963	0.084840749
$\vdots$	$\vdots$	$\vdots$
30	0.000745924	0.003729641
$\vdots$	$\vdots$	$\vdots$

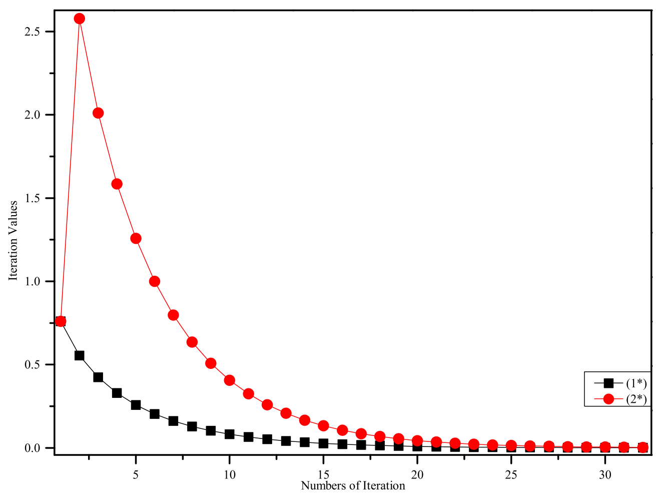

FIGURE 1. (1*) and (2*) represent, respectively, convergence behaviors of $d\big((x_1^{(n+1)}, x_2^{(n+1)}), z\big)$ and $d\big((S(x_1^{(n+1)}, x_2^{(n+1)})), p\big)$ for (2.7).

PROOF. (i)⇒(ii): Suppose that $\lim_{n\to\infty} Sx_n^{(3)} = p$. Now, by using (1.1), (1.4), (1.7)–(1.9), we have

$$d(Sx_{n+1}^{(3)}, Sx_{n+1}^{(2)}) = d(Ty_n^{(3)}, W(Ty_n^{(2)}, Sx_n^{(2)}, \alpha_n))$$

$$\begin{aligned}
&\leqslant \alpha_n d(Ty_n^{(3)}, Ty_n^{(2)}) + (1-\alpha_n) d(Ty_n^{(3)}, Sx_n^{(2)}) \\
&\leqslant \alpha_n \delta d(Sy_n^{(3)}, Sy_n^{(2)}) + (1-\alpha_n) d(Ty_n^{(3)}, Sx_n^{(2)}) + \alpha_n L u_2(x,y) \\
&\leqslant \alpha_n \delta d(W(Tx_n^{(3)}, Sx_n^{(3)}, \beta_n), W(Tx_n^{(2)}, Sx_n^{(2)}, \beta_n)) \\
&\quad + (1-\alpha_n) d(Ty_n^{(3)}, Sx_n^{(3)}) + (1-\alpha_n) d(Sx_n^{(3)}, Sx_n^{(2)}) + \alpha_n L u_2(x,y) \\
&\leqslant \alpha_n \delta \beta_n d(Tx_n^{(3)}, Tx_n^{(2)}) + \alpha_n \delta (1-\beta_n) d(Sx_n^{(3)}, Sx_n^{(2)}) \\
&\quad + (1-\alpha_n) d(Ty_n^{(3)}, Sx_n^{(3)}) + (1-\alpha_n) d(Sx_n^{(3)}, Sx_n^{(2)}) + \alpha_n L u_2(x,y) \\
&\leqslant [\alpha_n \beta_n \delta^2 + \alpha_n \delta (1-\beta_n) + (1-\alpha_n)] d(Sx_n^{(3)}, Sx_n^{(2)}) \\
&\quad + (1-\alpha_n) d(Ty_n^{(3)}, Sx_n^{(3)}) + \alpha_n \delta \beta_n L u_1(x,y) + \alpha_n L u_2(x,y) \\
&\leqslant [1-\alpha_n + \alpha_n \beta_n \delta^2 + \alpha_n \delta (1-\beta_n)] d(Sx_n^{(3)}, Sx_n^{(2)}) \\
&\quad + (1-\alpha_n)(1 + \delta[1-\beta_n(1-\delta)]) d(Sx_n^{(3)}, p) \\
&\quad + \alpha_n \delta \beta_n L u_1(x,y) + \alpha_n L u_2(x,y),
\end{aligned} \tag{2.8}$$

where

$$u_1(x,y) = \min\{d_{11}, d_{12}, d_{13}, d_{14}, d_{15}, d_{16}\}, \tag{2.9}$$

$$u_2(x,y) = \min\{d_{21}, d_{22}, d_{23}, d_{24}, d_{25}, d_{26}\}, \tag{2.10}$$

$$\begin{aligned}
&d_{11} = d(Sx_n^{(3)}, Tx_n^{(3)}), \quad d_{12} = d(Sx_n^{(2)}, Tx_n^{(2)}), \\
&d_{13} = d(Sx_n^{(3)}, Tx_n^{(2)}), \quad d_{14} = d(Sx_n^{(2)}, Tx_n^{(3)}), \\
&d_{15} = \tfrac{1}{2}[d(Sx_n^{(3)}, Tx_n^{(3)}) + d(Sx_n^{(2)}, Tx_n^{(2)})], \\
&d_{16} = \tfrac{1}{2}[d(Sx_n^{(3)}, Tx_n^{(2)}) + d(Sx_n^{(2)}, Tx_n^{(3)})], \\
&d_{21} = d(Sy_n^{(3)}, Ty_n^{(3)}), \quad d_{22} = d(Sy_n^{(2)}, Ty_n^{(2)}), \\
&d_{23} = d(Sy_n^{(3)}, Ty_n^{(2)}), \quad d_{24} = d(Sy_n^{(2)}, Ty_n^{(3)}), \\
&d_{25} = \tfrac{1}{2}[d(Sy_n^{(3)}, Ty_n^{(3)}) + d(Sy_n^{(2)}, Ty_n^{(2)})], \\
&d_{26} = \tfrac{1}{2}[d(Sy_n^{(3)}, Ty_n^{(2)}) + d(Sy_n^{(2)}, Ty_n^{(3)})].
\end{aligned}$$

As $\delta \in [0,1)$ and α_n, $\beta_n \in [0,1]$ for all $n \in \mathbb{N}$, so we have

$$1 - \alpha_n + \alpha_n \beta_n \delta^2 + \alpha_n \delta (1-\beta_n) < 1 - \alpha_n (1-\delta)$$

for all $n \in \mathbb{N}$. Using this fact in (2.8), we obtain

$$\begin{aligned}
d(Sx_{n+1}^{(3)}, Sx_{n+1}^{(2)}) &\leqslant [1-\alpha_n(1-\delta)] d(Sx_n^{(3)}, Sx_n^{(2)}) \\
&\quad + (1-\alpha_n)(1 + \delta[1-\beta_n(1-\delta)]) d(Sx_n^{(3)}, p) \\
&\quad + \alpha_n \delta \beta_n L u_1(x,y) + \alpha_n L u_2(x,y).
\end{aligned} \tag{2.11}$$

Following the above lines, by (1.1), (1.8), and (1.9), we have the following estimates:

$$\begin{cases} d_{11} \leqslant (1+\delta)d(Sx_n^{(3)}, p) = d'_{11}, \\ d_{12} \leqslant (1+\delta)d(Sx_n^{(3)}, p) + (1+\delta)d(Sx_n^{(2)}, Sx_n^{(3)}) = d'_{12}, \\ d_{13} \leqslant (1+\delta)d(Sx_n^{(3)}, p) + \delta d(Sx_n^{(3)}, Sx_n^{(2)}) = d'_{13}, \\ d_{14} \leqslant (1+\delta)d(Sx_n^{(3)}, p) + d(Sx_n^{(3)}, Sx_n^{(2)}) = d'_{14}, \\ d_{15} \leqslant (1+\delta)d(Sx_n^{(3)}, p) + \frac{1}{2}(1+\delta)d(Sx_n^{(3)}, Sx_n^{(2)}) = d'_{15}, \\ d_{16} \leqslant (1+\delta)d(Sx_n^{(3)}, p) + \frac{1}{2}(1+\delta)d(Sx_n^{(3)}, Sx_n^{(2)}) = d'_{16}, \end{cases} \tag{2.12}$$

$$\begin{cases} d_{21} \leqslant (1+\delta)d(Sy_n^{(3)}, p) = d'_{21}, \\ d_{22} \leqslant (1+\delta)d(Sy_n^{(3)}, p) + (1+\delta)d(Sy_n^{(3)}, Sy_n^{(2)}) = d'_{22}, \\ d_{23} \leqslant (1+\delta)d(Sy_n^{(3)}, p) + \delta d(Sy_n^{(3)}, Sy_n^{(2)}) = d'_{23}, \\ d_{24} \leqslant (1+\delta)d(Sy_n^{(3)}, p) + d(Sy_n^{(3)}, Sy_n^{(2)}) = d'_{24}, \\ d_{25} \leqslant (1+\delta)d(Sy_n^{(3)}, p) + \frac{1}{2}(1+\delta)d(Sy_n^{(3)}, Sy_n^{(2)}) = d'_{25}, \\ d_{26} \leqslant (1+\delta)d(Sy_n^{(3)}, p) + \frac{1}{2}(1+\delta)d(Sy_n^{(3)}, Sy_n^{(2)}) = d'_{26}. \end{cases} \tag{2.13}$$

Define

$$u'_1(x,y) = \min\{d'_{11}, d'_{12}, d'_{13}, d'_{14}, d'_{15}, d'_{16}\}, \tag{2.14}$$

$$u'_2(x,y) = \min\{d'_{21}, d'_{22}, d'_{23}, d'_{24}, d'_{25}, d'_{26}\}. \tag{2.15}$$

It is clear from (2.9)–(2.15) that

$$u_1(x,y) \leqslant u'_1(x,y) = d'_{11} = (1+\delta)d(Sx_n^{(3)}, p), \tag{2.16}$$

$$\begin{aligned} u_2(x,y) \leqslant u'_2(x,y) = d'_{21} &= (1+\delta)d(Sy_n^{(3)}, p) \\ &\leqslant (1+\delta)[1-\beta_n(1-\delta)]d(Sx_n^{(3)}, p). \end{aligned} \tag{2.17}$$

By the assumption $\alpha = \inf_{n\in\mathbb{N}} \alpha_n$, we have $1-\alpha(1-\delta) \geqslant 1-\alpha_n(1-\delta)$ for all $n \in \mathbb{N}$. Using (2.16) and (2.17) in (2.11), we get that

$$\begin{aligned} d(Sx_{n+1}^{(3)}, Sx_{n+1}^{(2)}) \leqslant{}& [1-\alpha(1-\delta)]d(Sx_n^{(3)}, Sx_n^{(2)}) \\ &+ \{(1-\alpha_n)(1+\delta[1-\beta_n(1-\delta)]) \\ &\quad + \alpha_n L(1+\delta)[1-\beta_n(1-2\delta)]\}d(Sx_n^{(3)}, p). \end{aligned} \tag{2.18}$$

Set

$$\varphi_n^{(1)} = d(Sx_n^{(3)}, Sx_n^{(2)}) \geqslant 0, \quad \varphi_n^{(3)} = \alpha(1-\delta) \in (0,1),$$

$$\varphi_n^{(2)} = \{(1-\alpha_n)(1+\delta[1-\beta_n(1-\delta)]) + \alpha_n L(1+\delta)[1-\beta_n(1-2\delta)]\}d(Sx_n^{(3)}, p).$$

Hence, (2.18) satisfies all the requirements of Lemma 1.1 and so we get that

$$\lim_{n\to\infty} d(Sx_n^{(3)}, Sx_n^{(2)}) = 0. \tag{2.19}$$

Also, we have

$$0 \leqslant d(Sx_n^{(2)}, p) \leqslant d(Sx_n^{(3)}, Sx_n^{(2)}) + d(Sx_n^{(3)}, p), \quad \text{for all } n \in \mathbb{N}. \tag{2.20}$$

Taking the limits on both sides of (2.20) and then using (2.19) together with the assumption $\lim_{n\to\infty} d(Sx_n^{(3)}, p) = 0$, we have $\lim_{n\to\infty} d(Sx_n^{(1)}, p) = 0$.

(ii)⇒(i): Its proof is very similar to that of (i)⇒ (ii), and hence is omitted. □

THEOREM 2.3. *Let* $\{Sx_n^{(1)}\}_{n=0}^{\infty}$, $\{Sx_n^{(2)}\}_{n=0}^{\infty}$, $\{Sx_n^{(3)}\}_{n=0}^{\infty}$ *be the iterative sequences of Jungck-Mann* (1.6)*, Jungck–Ishikawa* (1.7)*, and Jungck-S* (1.9) *iteration methods, respectively. If* $C(T,S) \neq \varnothing$ *and* $\{\alpha_n\}_{n=0}^{\infty}$ *and* $\{\beta_n\}_{n=0}^{\infty}$ *are sequences in* $[0,1]$ *such that* $0 \leqslant \alpha_n < \frac{1}{1+\delta}$ *for all* $n \in \mathbb{N}$, $\lim_{n\to\infty} \alpha_n = 0$*, then Jungck-S iteration method* (1.9) *converges faster than Jungck–Mann* (1.6) *and Jungck–Ishikawa* (1.7) *iteration methods to* p*, provided that the initial point is the same for all iterations, that is,* $x_0^{(1)} = x_0^{(2)} = x_0^{(3)}$.

PROOF. We know from (2.4) that

$$d\big(Sx_{n+1}^{(3)}, p\big) \leqslant \delta^{n+1} d(Sx_0^{(3)}, p). \tag{2.21}$$

Also, by (1.1), (1.2), (1.6), (1.7), and (1.8), we have the following estimates:

$$\begin{aligned}
d(Sx_{n+1}^{(1)}, p) &= d(W(Tx_n^{(1)}, Sx_n^{(1)}, \alpha_n), p)\\
&\geqslant (1-\alpha_n)d(Sx_n^{(1)}, p) - \alpha_n d(Tx_n^{(1)}, Tz)\\
&\geqslant (1-\alpha_n)d(Sx_n^{(1)}, p) - \alpha_n\big\{\delta d(Sx_n^{(1)}, Sz)\\
&\quad + L\min\big\{d(Sx_n^{(1)}, Tx_n^{(1)}), d(Sz,Tz), d(Sx_n^{(1)}, Tz), d(Sz, Tx_n^{(1)}),\\
&\quad \tfrac{1}{2}\big[d(Sx_n^{(1)}, Tx_n^{(1)}) + d(Sz,Tz)\big], \tfrac{1}{2}\big[d(Sx_n^{(1)}, Tz) + d(Sz, Tx_n^{(1)})\big]\big\}\big\}\\
&= [1-\alpha_n(1+\delta)]d(Sx_n^{(1)}, p) \geqslant \ldots \geqslant \prod_{k=0}^{n}[1-\alpha_k(1+\delta)]d(Sx_0^{(1)}, p),
\end{aligned} \tag{2.22}$$

$$\begin{aligned}
d(Sx_{n+1}^{(2)}, p) &= d(W(Ty_n^{(2)}, Sx_n^{(2)}, \alpha_n), p) \geqslant (1-\alpha_n)d(Sx_n^{(2)}, p) - \alpha_n d(Ty_n^{(2)}, Tz)\\
&\geqslant (1-\alpha_n)d(Sx_n^{(2)}, p) - \alpha_n\big\{\delta d(Sy_n^{(2)}, Sz)\\
&\quad + L\min\big\{d(Sy_n^{(2)}, Ty_n^{(2)}), d(Sz,Tz), d(Sy_n^{(2)}, Tz), d(Sz, Ty_n^{(2)}),\\
&\quad \tfrac{1}{2}[d(Sy_n^{(2)}, Ty_n^{(2)}) + d(Sz,Tz)], \tfrac{1}{2}[d(Sy_n^{(2)}, Tz) + d(Sz, Ty_n^{(2)})]\big\}\big\}\\
&= (1-\alpha_n)d(Sx_n^{(2)}, p) - \alpha_n\delta d(Sy_n^{(2)}, p)\\
&= (1-\alpha_n)d(Sx_n^{(2)}, p) - \alpha_n\delta d(p, W(Tx_n^{(2)}, Sx_n^{(2)}, \beta_n))\\
&\geqslant (1-\alpha_n)d(Sx_n^{(2)}, p) - \alpha_n\delta\big\{\beta_n d(Tx_n^{(2)}, p) + (1-\beta_n)d(Sx_n^{(2)}, p)\big\}\\
&\geqslant (1-\alpha_n)d(Sx_n^{(2)}, p) - \alpha_n\delta\big\{\beta_n\big\{\delta d(Sx_n^{(2)}, Sz)\\
&\quad + L\min\big\{d(Sx_n^{(2)}, Tx_n^{(2)}), d(Sz,Tz), d(Sx_n^{(2)}, Tz), d(Sz, Tx_n^{(2)}),\\
&\quad \tfrac{1}{2}[d(Sx_n^{(2)}, Tx_n^{(2)}) + d(Sz,Tz)], \tfrac{1}{2}[d(Sx_n^{(2)}, Tz) + d(Sz, Tx_n^{(2)})]\big\}\big\}\\
&\quad + (1-\beta_n)d(Sx_n^{(2)}, p)\big\}\\
&= [1-\alpha_n(1+\delta[1-\beta_n(1-\delta)])]d(Sx_n^{(2)}, p)\\
&\geqslant [1-\alpha_n(1+\delta)]d(Sx_n^{(2)}, p) \mathrm{as} 1-\beta_n(1-\delta) < 1, \quad \forall n \in \mathbb{N}
\end{aligned} \tag{2.23}$$

$$\geqslant \dots \geqslant \prod_{k=0}^{n}[1-\alpha_k(1+\delta)]d(Sx_0^{(2)},p).$$

Combining (2.22) and (2.23), we have

$$d\big(Sx_{n+1}^{(i)},p\big) \geqslant \prod_{k=0}^{n}[1-\alpha_k(1+\delta)]d(Sx_0^{(i)},p) \quad \text{where } i=1,2. \tag{2.24}$$

It follows by (2.21), (2.24) and the assumption $x_0^{(1)}=x_0^{(2)}=x_0^{(3)}$ that

$$\frac{d\big(Sx_{n+1}^{(3)},p\big)}{d\big(Sx_{n+1}^{(i)},p\big)} \leqslant \frac{\delta^{n+1}}{\prod_{k=0}^{n}[1-\alpha_k(1+\delta)]} \quad \text{where } i=1,2. \tag{2.25}$$

Set

$$\tau_n = \frac{\delta^{n+1}}{\prod_{k=0}^{n}[1-\alpha_k(1+\delta)]} \quad \text{for all } n\in\mathbb{N},$$

which implies that

$$\frac{\tau_{n+1}}{\tau_n} = \frac{\delta^{n+2}}{\prod_{k=0}^{n+1}[1-\alpha_k(1+\delta)]}\frac{\prod_{k=0}^{n}[1-\alpha_k(1+\delta)]}{\delta^{n+1}} = \frac{\delta}{1-\alpha_{n+1}(1+\delta)}$$

for all $n\in\mathbb{N}$. By the assumption $\lim_{n\to\infty}\alpha_n=0$, we obtain

$$\ell = \lim_{n\to\infty}\frac{\tau_{n+1}}{\tau_n} = \lim_{n\to\infty}\frac{\delta}{1-\alpha_{n+1}(1+\delta)} = \delta < 1.$$

Since $l=\delta<1$, the ratio test tells us that the series $\sum_{n=0}^{\infty}\tau_n$ converges. This allows us to conclude that $\lim_{n\to\infty}\tau_n=0$. It follows from (2.25) that

$$\lim_{n\to\infty}\frac{d\big(Sx_{n+1}^{(3)},p\big)}{d\big(Sx_{n+1}^{(i)},p\big)} = 0 \quad \text{where } i=1,2.$$

Hence, by Definition 1.3, $\{Sx_{n+1}^{(3)}\}_{n=0}^{\infty}$ converges faster than $\{Sx_{n+1}^{(1)}\}_{n=0}^{\infty}$ and $\{Sx_{n+1}^{(2)}\}_{n=0}^{\infty}$ to p. □

EXAMPLE 2.2. Let $Y=[0,1.5]$ be endowed with the usual metric. Define operators T, $S\colon[0,1.5]\to[0.025,0.9625]$ by

$$Tx = 0.25 + \frac{x^2\sqrt{1000+\frac{x\cos(\frac{2\pi}{7}x)}{10+5x^3}}}{181+2\exp\big(\operatorname{arcsinh}\big(\frac{2\pi}{5}\sqrt{x}\big)\big)}, \qquad Sx=\frac{1}{4}(x^2+x+0.1)$$

with the unique point of coincidence $p=0.34045488=T(0.7295607)=S(0.7295607)$. It is easy to check that $T([0,1.5])=[0.25,0.628973]\subset S([0,1.5])=[0.025,0.9625]$, $S([0,1.5])$ is a complete subspace of $[0.025,0.9625]$, and the operator S is invertible. The Wolfram Mathematica 9 software package implies that the pair (T,S) satisfies condition (1.8) with $\delta=0.47\in(0,1]$, $L\geqslant1.7$, $u(x,y)=d(Sx,Ty)=|Sx-Ty|$, that is, the inequality

$$|Tx-Ty| \leqslant 0.47|Sx-Sy| + 1.7|Sx-Ty| \tag{2.26}$$

holds for all $x, y \in [0, 1.5]$ as shown in Figure 2 below. Take $\alpha_n = \frac{\sqrt[10]{n}+1}{\sqrt[5]{n}+2}$, $\beta_n = \frac{1}{2n^3+10n+5}$ for all $n \in \mathbb{N}$, and $x_0 = 0.5$, the convergence results for various Jungck-type iteration methods to $z = 0.7295607$ and $p = 0.34045488 = T(0.7295607) = S(0.7295607)$ are listed in Table 1 (Figures 3 and 4).

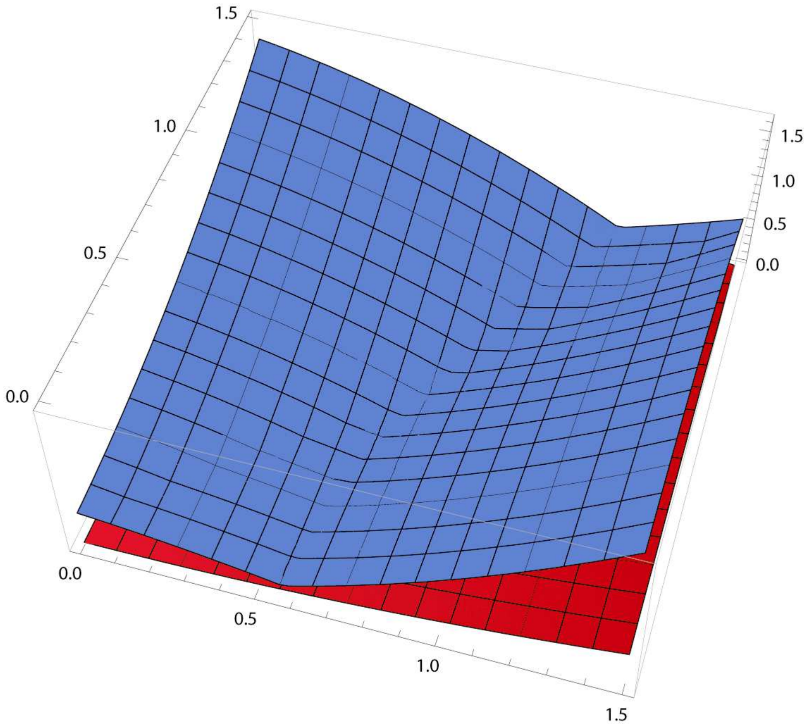

FIGURE 2. red: $|Tx - Ty|$; blue: $0.47|Sx - Sy| + 1.7|Sx - Ty|$.

THEOREM 2.4. *Let* $\{Sx_n^{(3)}\}_{n=0}^{\infty}$ *be the iterative sequence* (1.9) *with real sequence* $\{\beta_n\}_{n=0}^{\infty}$ *in* $[0, 1]$. *Suppose that* $C(T, S) \neq \emptyset$. *Let* $\{q_n\}_{n=0}^{\infty} \subset X$ *be an arbitrary sequence and define a sequence* $\{\epsilon_n\}_{n=0}^{\infty}$ *in* $\mathbb{R}^+$ *by*

$$\epsilon_n = d(Sq_{n+1}, Tu_n), \quad Su_n = W(Tq_n, Sq_n, \beta_n) \text{ for all } n \in \mathbb{N}$$

Then the Jungck-S iteration method (1.9) *is* (T, S)*-stable.*

PROOF. Assume that $\lim_{n\to\infty} \epsilon_n = 0$ and p a coincidence value of T and S such that $p = Sz$. In order to prove that the sequence $\{Sx_n^{(3)}\}_{n=0}^{\infty}$ is (T, S)-stable, it suffices to prove that $\lim_{n\to\infty} Sq_n = p$. Following the same lines as in the proof of Theorem 1.1, it follows from (1.1), (1.8), and (1.9) that

$$\begin{aligned} d(Sq_{n+1}, p) &\leqslant d(Sq_{n+1}, Tu_n) + d(Tu_n, p) \leqslant \epsilon_n + \delta d(Su_n, p) \\ &= \epsilon_n + \delta d(p, W(Tq_n, Sq_n, \beta_n)) \end{aligned} \tag{2.27}$$

Table 3. Comparison of the rate of convergence among various iterations for Example 2.2.

# of Iter.	Jungck-S Iter.		Jungck–Ishikawa Iter.		Jungck–Mann Iter.	
	x_{n+1}	Sx_{n+1}	x_{n+1}	Sx_{n+1}	x_{n+1}	Sx_{n+1}
0	0.50000000	0.50000000	0.50000000	0.50000000	0.50000000	0.50000000
1	0.6587176	0.29815662	0.5822722	0.25532828	0.5771537	0.25256502
2	0.7031548	0.32439537	0.6463351	0.29102104	0.6422389	0.28867742
3	0.7192609	0.33414928	0.6806469	0.31098177	0.6779774	0.30940769
4	0.7254767	0.33794829	0.7002339	0.32264036	0.6985682	0.32164143
5	0.7279307	0.33945346	0.7117700	0.32959663	0.7107408	0.32897332
⋮	⋮	⋮	⋮	⋮	⋮	⋮
15	0.7295605	0.34045475	0.7294097	0.34036205	0.7294007	0.34035652
16	0.7295607	0.34045488	0.7294659	0.34039660	0.7294602	0.34039309
⋮	⋮	⋮	⋮	⋮	⋮	⋮
30			0.7295606	0.34045482	0.7295605	0.34045475
31			0.7295607	0.34045488	0.7295606	0.34045482
32			⋮	⋮	0.7295607	0.34045488
⋮					⋮	⋮

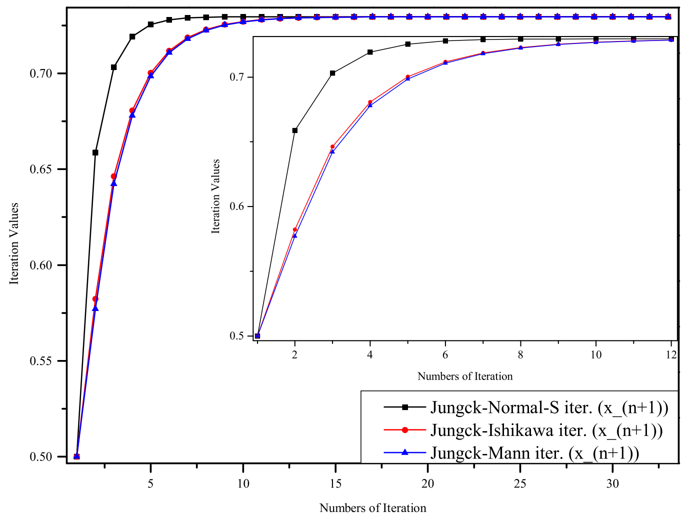

Figure 3. Comparison (rate of convergence among various iteration methods for Example 2.2 with initial approximation $x_0 = 0.5$).

$$\leqslant \epsilon_n + \delta\{\beta_n d(Sq_n, p) + (1-\beta_n)d(Tq_n, p)\}$$
$$\leqslant \epsilon_n + \delta[\beta_n + (1-\beta_n)\delta]d(Sq_n, p)$$

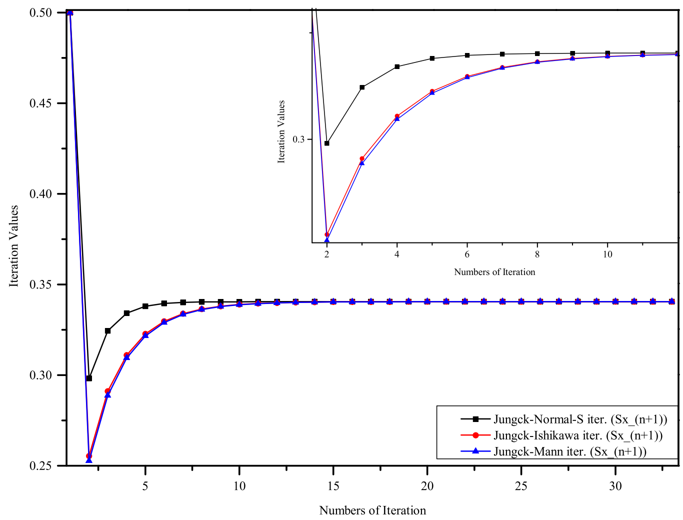

FIGURE 4. Comparison (rate of convergence among various iteration methods for Example 2.2 with initial approximation $x_0 = 0.5$).

$$\leqslant \epsilon_n + \delta(\beta_n + 1 - \beta_n)d(Sq_n, p) = \delta d(Sq_n, p) + \epsilon_n.$$

It is easy to see that (2.27) satisfies all the requirements of Lemma 1.1. So by its conclusion, we have $\lim_{n\to\infty} Sq_n = p$.

Conversely, let $\lim_{n\to\infty} d(Sq_n, p) = 0$. Using again (1.1), (1.8), and (1.9), we obtain

$$\begin{aligned} 0 &\leqslant \epsilon_n = d(Sq_{n+1}, Tu_n) \\ &\leqslant d(Sq_{n+1}, p) + d(p, Tu_n) \leqslant d(Sq_{n+1}, p) + \delta d(Su_n, p) \\ &\leqslant d(Sq_{n+1}, p) + \delta[\beta_n + (1-\beta_n)\delta]d(Sq_n, p). \end{aligned} \tag{2.28}$$

Taking the limits of both sides of inequality (2.28) and then using the assumption $\lim_{n\to\infty} d(Sq_n, p) = 0$, we have $\lim_{n\to\infty} \epsilon_n = 0$. □

THEOREM 2.5. *Let (X, d, W) be an HS, $(\tilde{T}, \tilde{S})\colon Y \to X$ an approximate operator of the pair $(T, S)\colon Y \to X$ such that $\tilde{T}(Y) \subseteq \tilde{S}(Y)$ and $\tilde{S}(Y)$ is a complete subset of X, and p and $\tilde{p}$ be the points of coincidence of the pairs $(\tilde{T}, \tilde{S})$ and (T, S), respectively. Let $\{Sx_n^{(3)}\}_{n=0}^{\infty}$ be the iterative sequence (1.9) and $\{\tilde{S}\tilde{x}_n^{(3)}\}_{n=0}^{\infty}$ be an iteration method generated by*

$$\begin{aligned} &\tilde{x}_0^{(3)} \in Y, \quad \tilde{S}\tilde{x}_{n+1}^{(3)} = \tilde{T}\tilde{y}_n^{(3)}, \\ &\tilde{S}\tilde{y}_n^{(3)} = W\big(\tilde{T}\tilde{x}_n^{(3)}, \tilde{S}\tilde{x}_n^{(3)}, \beta_n\big) \text{ for all } n \in \mathbb{N}, \end{aligned} \tag{2.29}$$

where $\{\beta_n\}_{n=0}^{\infty}$ is a real sequence in $[0,1]$ satisfying $\lim_{n\to\infty} \beta_n = 0$. If $\{\tilde{S}\tilde{x}_n^{(3)}\}_{n=0}^{\infty}$ converges to $\tilde{p}$, then we have $d(p, \tilde{p}) \leqslant \frac{\delta\varepsilon_2 + \varepsilon_1}{1-\delta}$.

Proof. From (1.4), (1.8), (1.9), and (2.29), we have

$$d\big(Sx_{n+1}^{(3)}, \tilde{S}\tilde{x}_{n+1}^{(3)}\big) \leqslant d\big(Ty_n^{(3)}, T\tilde{y}_n^{(3)}\big) + d\big(T\tilde{y}_n^{(3)}, \tilde{T}\tilde{y}_n^{(3)}\big) \\ \leqslant \delta d\big(Sy_n^{(3)}, S\tilde{y}_n^{(3)}\big) + \varepsilon_1 + Lu_4(x,y), \tag{2.30}$$

$$\begin{aligned} d\big(Sy_n^{(3)}, S\tilde{y}_n^{(3)}\big) &\leqslant d\big(Sy_n^{(3)}, \tilde{S}\tilde{y}_n^{(3)}\big) + d\big(S\tilde{y}_n^{(3)}, \tilde{S}\tilde{y}_n^{(3)}\big) \\ &\leqslant d\big(W\big(Tx_n^{(3)}, Sx_n^{(3)}, \beta_n\big), W\big(\tilde{T}\tilde{x}_n^{(3)}, \tilde{S}\tilde{x}_n^{(3)}, \beta_n\big)\big) + \varepsilon_2 \\ &\leqslant \beta_n d\big(Tx_n^{(3)}, \tilde{T}\tilde{x}_n^{(3)}\big) + (1-\beta_n) d\big(Sx_n^{(3)}, \tilde{S}\tilde{x}_n^{(3)}\big) + \varepsilon_2 \\ &\leqslant \beta_n d\big(Tx_n^{(3)}, T\tilde{x}_n^{(3)}\big) + \beta_n d\big(T\tilde{x}_n^{(3)}, \tilde{T}\tilde{x}_n^{(3)}\big) \\ &\quad + (1-\beta_n) d\big(Sx_n^{(3)}, \tilde{S}\tilde{x}_n^{(3)}\big) + \varepsilon_2 \\ &\leqslant \beta_n \delta d\big(Sx_n^{(3)}, S\tilde{x}_n^{(3)}\big) + (1-\beta_n) d\big(Sx_n^{(3)}, \tilde{S}\tilde{x}_n^{(3)}\big) \\ &\quad + \beta_n Lu_3(x,y) + \beta_n \varepsilon_1 + \varepsilon_2 \\ &\leqslant [1-\beta_n(1-\delta)] d\big(\tilde{S}\tilde{x}_n^{(3)}, Sx_n^{(3)}\big) + \beta_n \varepsilon_1 + \beta_n \delta \varepsilon_2 \\ &\quad + \varepsilon_2 + \beta_n Lu_3(x,y), \end{aligned} \tag{2.31}$$

where

$$u_3(x,y) = \min\{d_{31}, d_{32}, d_{33}, d_{34}, d_{35}, d_{36}\}, \tag{2.32}$$

$$u_4(x,y) = \min\{d_{41}, d_{42}, d_{43}, d_{44}, d_{45}, d_{46}\}, \tag{2.33}$$

$$\begin{aligned} d_{31} &= d\big(Sx_n^{(3)}, Tx_n^{(3)}\big), \qquad d_{32} = d\big(S\tilde{x}_n^{(3)}, T\tilde{x}_n^{(3)}\big), \\ d_{33} &= d\big(Sx_n^{(3)}, T\tilde{x}_n^{(3)}\big), \qquad d_{34} = d\big(S\tilde{x}_n^{(3)}, Tx_n^{(3)}\big), \\ d_{35} &= \tfrac{1}{2}\big[d\big(Sx_n^{(3)}, Tx_n^{(3)}\big) + d\big(S\tilde{x}_n^{(3)}, T\tilde{x}_n^{(3)}\big)\big], \\ d_{36} &= \tfrac{1}{2}\big[d\big(Sx_n^{(3)}, T\tilde{x}_n^{(3)}\big) + d\big(S\tilde{x}_n^{(3)}, Tx_n^{(3)}\big)\big], \\ d_{41} &= d\big(Sy_n^{(3)}, Ty_n^{(3)}\big), \qquad d_{42} = d\big(S\tilde{y}_n^{(3)}, T\tilde{y}_n^{(3)}\big), \\ d_{43} &= d\big(Sy_n^{(3)}, T\tilde{y}_n^{(3)}\big), \qquad d_{44} = d\big(S\tilde{y}_n^{(3)}, Ty_n^{(3)}\big), \\ d_{45} &= \tfrac{1}{2}\big[d\big(Sy_n^{(3)}, Ty_n^{(3)}\big) + d\big(S\tilde{y}_n^{(3)}, T\tilde{y}_n^{(3)}\big)\big], \\ d_{46} &= \tfrac{1}{2}\big[d\big(Sy_n^{(3)}, T\tilde{y}_n^{(3)}\big) + d\big(S\tilde{y}_n^{(3)}, Ty_n^{(3)}\big)\big]. \end{aligned}$$

Inserting (2.31) into (2.30) and then using the fact $1-\beta_n(1-\delta) < 1$ for all $n \in \mathbb{N}$, we get that

$$d\big(Sx_{n+1}^{(3)}, \tilde{S}\tilde{x}_{n+1}^{(3)}\big) \leqslant \delta d\big(\tilde{S}\tilde{x}_n^{(3)}, Sx_n^{(3)}\big) + \delta\beta_n Lu_3(x,y) + Lu_4(x,y) \\ + \beta_n \delta \varepsilon_1 + \beta_n \delta^2 \varepsilon_2 + \delta \varepsilon_2 + \varepsilon_1. \tag{2.34}$$

By (1.8), we have the following estimates:

$$\begin{cases} d_{31} \leqslant (1+\delta)d\big(Sx_n^{(3)},p\big) = d'_{31}, \\ d_{32} \leqslant (1+\delta)d\big(Sx_n^{(3)},p\big) + (1+\delta)d\big(Sx_n^{(3)},S\tilde{x}_n^{(3)}\big) = d'_{32}, \\ d_{33} \leqslant (1+\delta)d\big(Sx_n^{(3)},p\big) + \delta d\big(Sx_n^{(3)},S\tilde{x}_n^{(3)}\big) = d'_{33}, \\ d_{34} \leqslant (1+\delta)d\big(Sx_n^{(3)},p\big) + d\big(Sx_n^{(3)},S\tilde{x}_n^{(3)}\big) = d'_{34}, \\ d_{35} \leqslant (1+\delta)d\big(Sx_n^{(3)},p\big) + \frac{1}{2}(1+\delta)d\big(Sx_n^{(3)},S\tilde{x}_n^{(3)}\big) = d'_{35}, \\ d_{36} \leqslant (1+\delta)d\big(Sx_n^{(3)},p\big) + \frac{1}{2}(1+\delta)d\big(Sx_n^{(3)},S\tilde{x}_n^{(3)}\big) = d'_{36}, \end{cases} \tag{2.35}$$

$$\begin{cases} d_{41} \leqslant (1+\delta)d\big(Sy_n^{(3)},p\big) = d'_{41}, \\ d_{42} \leqslant (1+\delta)d\big(Sy_n^{(3)},p\big) + (1+\delta)d\big(Sy_n^{(3)},S\tilde{y}_n^{(3)}\big) = d'_{42}, \\ d_{43} \leqslant (1+\delta)d\big(Sy_n^{(3)},p\big) + \delta d\big(Sy_n^{(3)},S\tilde{y}_n^{(3)}\big) = d'_{43}, \\ d_{44} \leqslant (1+\delta)d\big(Sy_n^{(3)},p\big) + d\big(Sy_n^{(3)},S\tilde{y}_n^{(3)}\big) = d'_{44}, \\ d_{45} \leqslant (1+\delta)d\big(Sy_n^{(3)},p\big) + \frac{1}{2}(1+\delta)d\big(Sy_n^{(3)},S\tilde{y}_n^{(3)}\big) = d'_{45}, \\ d_{46} \leqslant (1+\delta)d\big(Sy_n^{(3)},p\big) + \frac{1}{2}(1+\delta)d\big(Sy_n^{(3)},S\tilde{y}_n^{(3)}\big) = d'_{46}. \end{cases} \tag{2.36}$$

Set

$$u'_3(x,y) = \min\{d'_{31},d'_{32},d'_{33},d'_{34},d'_{35},d'_{36}\}, \tag{2.37}$$

$$u'_4(x,y) = \min\{d'_{41},d'_{42},d'_{43},d'_{44},d'_{45},d'_{46}\}. \tag{2.38}$$

It follows from (2.32)–(2.38) that

$$u_3(x,y) \leqslant u'_3(x,y) = d'_{31} = (1+\delta)d\big(Sx_n^{(3)},p\big), \tag{2.39}$$

$$u_4(x,y) \leqslant u'_4(x,y) = d'_{41} = (1+\delta)d\big(Sy_n^{(3)},p\big). \tag{2.40}$$

Also, by (1.1), (1.8), and (1.9), we have

$$d\big(Sy_n^{(3)},p\big) \leqslant [1-\beta_n(1-\delta)]d\big(Sx_n^{(3)},p\big). \tag{2.41}$$

Using (2.39)–(2.41) in (2.34), we obtain

$$\begin{aligned} d\big(Sx_{n+1}^{(3)},\tilde{S}\tilde{x}_{n+1}^{(3)}\big) \leqslant{}& (1-\sigma)d\big(Sx_n^{(3)},\tilde{S}\tilde{x}_n^{(3)}\big) \\ &+ L(1+\delta)[1-\beta_n(1-2\delta)]d\big(Sx_n^{(3)},p\big) \\ &+ \beta_n\delta\varepsilon_1 + \beta_n\delta^2\varepsilon_2 + \delta\varepsilon_2 + \varepsilon_1, \end{aligned} \tag{2.42}$$

where $\sigma = 1-\delta \in (0,1)$.

Set

$$\varphi_n^{(1)} = d\big(Sx_n^{(3)},\tilde{S}\tilde{x}_n^{(3)}\big) \geqslant 0, \quad \varphi_n^{(3)} = \sigma \in (0,1),$$

$$\varphi_n^{(2)} = \frac{L(1+\delta)[1-\beta_n(1-2\delta)]d\big(Sx_n^{(3)},p\big) + \beta_n\delta\varepsilon_1 + \beta_n\delta^2\varepsilon_2 + \delta\varepsilon_2 + \varepsilon_1}{\sigma}.$$

It is now easy to check that (2.42) satisfies the requirements in Lemma 1.2 and so by its conclusion, we get that

$$0 \leqslant d(p,\tilde{p}) \leqslant \frac{\delta\varepsilon_2 + \varepsilon_1}{1-\delta}. \qquad \square$$

Example 2.3. Let $Y = [1.08, 1.35]$ be endowed with the usual metric. Define operators T, $S\colon [1.08, 1.35] \to [0.541644, 0.709534]$ by

$$Tx = \tfrac{1}{8}\big(5 - 0.5\sqrt{x}\big), \quad Sx = \tfrac{1}{2}\exp(x-1), \tag{2.43}$$

with the unique point of coincidence $p = 0.55910110 = T(1.111722) = S(1.111722)$. It is easy to check that $T([1.08, 1.35]) = [0.552382, 0.560048] \subset S([1.08, 1.35]) = [0.541644, 0.709534]$, $S([1.08, 1.35])$ is a complete subspace of $[0.541644, 0.709534]$, and the operator S is invertible. Now we show that the pair (T, S) satisfies condition (1.8). By some simple calculations, we see that

$$|Tx - Ty| = \frac{1}{16}\big|\sqrt{x} - \sqrt{y}\big| = \frac{|x-y|}{16\big|\sqrt{x} + \sqrt{y}\big|} \leqslant \frac{5}{96\sqrt{3}}|x-y|, \tag{2.44}$$

for all $x, y \in [1.08, 1.35]$. On the other hand, without loss of generality, suppose that $x < y$. Applying the Lagrange mean value theorem to $Sx = \frac{1}{2}\exp(x-1)$ on $[x, y] \subseteq [1.08, 1.35]$, we get that $|Sx - Sy| = \frac{1}{2}e^{c-1}|x-y|$, which implies

$$\frac{5}{96\sqrt{3}}|x-y| = \frac{5}{48\sqrt{3}e^{c-1}}|Sx - Sy| \tag{2.45}$$

for some $c \in (x, y) \subseteq [1.08, 1.35]$. Using (2.44) and (2.45), we have

$$|Tx - Ty| \leqslant \frac{5}{96\sqrt{3}}|x-y| \leqslant \delta|Sx - Sy| + Lu(x, y),$$

where $\delta = 0.0555168 \in \big[\max_{x\in[1.08,1.35]}\big\{\frac{5}{48\sqrt{3}e^{x-1}}\big\}, 1\big] \subset (0, 1]$, $L \geqslant 0$, and

$$u(x, y) = \min\{|Sx - Tx|, |Sy - Ty|, |Sx - Ty|, |Sy - Tx|, \tfrac{1}{2}[|Sx - Tx| + |Sy - Ty|], \tfrac{1}{2}[|Sx - Ty| + |Sy - Tx|]\}.$$

Define operators $\tilde{T}$, $\tilde{S}\colon [1.08, 1.35] \to [0.536143, 0.715196]$ by

$$\tilde{T}x = 0.203 + 0.015x\cos(0.015\pi x)\frac{0.631}{\sqrt{2.5 + 0.9x} + \sin(0.01\pi x^2)}, \tag{2.46}$$

$$\tilde{S}x = -0.17 + 0.045\,\mathrm{arccosh}(\exp(x+1)^3 + 6), \tag{2.47}$$

with a point of coincidence $\tilde{p} = 0.55035276 = \tilde{T}(1.1040505) = \tilde{S}(1.1040505)$. Clearly,

$$\tilde{T}([1.08, 1.35]) = [0.541148, 0.551290] \subset \tilde{S}([1.08, 1.35]) = [0.536143, 0.715196],$$

$\tilde{S}([1.08, 1.35])$ is a complete subspace of $[0.536143, 0.715196]$, and $\tilde{S}$ is invertible. The Wolfram Mathematica 9 software package implies that

$$\sup_{x\in[1.08,1.35]} |T - \tilde{T}| = 0.0112336 \quad \text{and} \quad \sup_{x\in[1.08,1.35]} |S - \tilde{S}| = 0.00566221,$$

which imply that

$$|Tx - \tilde{T}x| \leqslant 0.0112336 = \varepsilon_1 \quad \text{and} \quad |Sx - \tilde{S}x| \leqslant 0.00566221 = \varepsilon_2$$

for all $x \in [1.08, 1.35]$. Thus, we can consider the operators $\tilde{T}$ and $\tilde{S}$ as approximate operators of T and S, respectively, in the sense of Definition 1.4. On the other hand, for given $x_0 = \tilde{x}_0 = 1.2 \in Y$, iterative schemes (1.9) and (2.29) of the pairs (T, S)

TABLE 4. Convergence behaviors of iteration methods (1.9) and (2.29).

# of Iter.	Iteration (1.9)		Iteration (2.29)	
	x_{n+1}	Sx_{n+1}	$\tilde{x}_{n+1}$	$S\tilde{x}_{n+1}$
0	1.20000000	1.20000000	1.20000000	1.20000000
1	1.1075370	0.55676605	1.0984528	0.54701618
2	1.1119286	0.55921650	1.1043886	0.55055485
3	1.1117113	0.55909500	1.1040285	0.55033957
4	1.1117228	0.55910145	1.1040519	0.55035356
5	1.1117222	0.55910110	1.1040504	0.55035266
6	⋮	⋮	1.1040505	0.55035276
⋮			⋮	⋮

in (2.43) and $(\tilde{T}, \tilde{S})$ in (2.46)) and (2.47) with $\beta_n = \frac{1}{10+5^{n^2}}$ for all $n \in \mathbb{N}$, converges to $p = 0.55910110$ and $\tilde{p} = 0.55035276$, respectively, as shown in Table 4 below.

Consequently, $|p - \tilde{p}| = 0.00874834$. As a matter of fact, without knowing the point of coincidence of the pair $(\tilde{T}, \tilde{S})$ and without computing it, we can find the following upper bound for the error in approximating $\tilde{p}$ by p by using the conclusion of Theorem 2.5:

$$|p - \tilde{p}| = 0.00874834 \leqslant \frac{0.0555168 \times 0.00566221 + 0.0112336}{1 - 0.0555168} = 0.0122267.$$

3. Conclusion

For the iteration methods employed in [**1**, **6**, **12**–**16**, **20**, **23**–**25**, **30**, **31**, **36**, **38**], it is the usual practice to impose some conditions on the parametric sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty} \subseteq [0,1]$, like $\sum_{n=0}^{\infty} \alpha_n = \infty$ (or $\sum_{n=0}^{\infty} \beta_n = \infty$), $\sum_{n=0}^{\infty} \alpha_n \beta_n = \infty$, $\sum_{n=0}^{\infty} (\alpha_n + \beta_n) = \infty$, $0 < \alpha \leqslant \alpha_n$, $0 < \beta \leqslant \beta_n$, $\beta_n \leqslant \alpha_n$, $\frac{1}{2} \leqslant \alpha_n$, $\frac{1}{2} \leqslant \alpha_n(1-\delta)$ for all $n \in \mathbb{N}$ to obtain convergence, stability and data dependence results. None of these conditions has been used in our corresponding results. Therefore, our results are improvement of the corresponding results established in all the above mentioned references.

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