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WAVE-FRONT SETS RELATED TO QUASI-ANALYTIC GEVREY SEQUENCES

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Abstract. Quasi-analytic wave-front sets of distributions which correspond to the Gevrey sequence $p!^s$, $s \in [1/2, 1)$ are defined and investigated. The propagation of singularities are deduced by considering sequences of Gaussian windowed short-time Fourier transforms of distributions which are modifications of the original distributions by suitable restriction-extension techniques. Basic micro-local properties of the new wave-fronts are thereafter established.

1. Introduction

In the literature it seems to be no (local) wave-front sets which detect heavier singularities than singularities involved in the analytic wave-front set, while there are different kinds of wave-front sets detecting milder singularities. For example, if $WF_A(f)$, $WF_t(f)$, $t > 1$ and $WF(f)$ are the wave-front sets of a suitable (ultra-)distribution f with respect to analyticity, Gevrey class $\mathcal{E}^t$ and smoothness, respectively, it is well-known that

$$WF(f) \subseteq WF_t(f) \subseteq WF_A(f).$$

Here $\mathcal{E}^t(X)$, X is open in $\mathbf{R}^d$, $t > 1$, is the Roumieu space of ultra-differentiable functions which correspond to the Gevrey sequence $p!^t$. (See also Section 2 for notations.) We refer to [**1**, **3**, **9**, **10**, **12**, **13**, **20**, **23**, **27**] for the spaces of non-quasi-analytic and quasi-analytic ultra-differentiable functions. Note that $WF_t(f)$ agrees to wave-front sets $WF_L(f)$ in [**10**, Section 8.4], with $L_p = p^t$ when $t \geqslant 1$. In particular, if $t = 1$, then $WF_t(f) = WF_A(f)$.

Let us mention that the analysis of various wave-fronts local and global, both defined by Hörmander, and their applications for distributions and ultra-distributons, has been given in many papers [**4**–**7**, **11**, **14**, **17**, **23**, **27**, **29**]. The homogeneous wave-front set, used and studied in [**15**–**19**, **22**], is equivalent to the Gabor wave front as well as to the global one of Hörmander, recently was studied in [**24**] and after that by [**2**, **8**, **25**, **26**]. We also refer to our references [**7**, **20**, **21**, **28**]. Actually,

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we will not compare wave-fronts or consider some specific application as it is done in many of cited papers, especially for the Schrödinger equations. Moreover, we will not recall the definitions of basic spaces of ultra-differentiable functions and Gelfand Shilov type spaces.

In this paper we define the wave-front set $WF_s(f)$, $s \in [1/2, 1)$, for $f \in \mathscr{D}'(\mathbf{R}^d)$. This is done by restricting f to a ball with radius r around x_0 ($f_{\rm res} = f_{|B(x_0,r)}$), and then, by the appropriate estimate of the sequence of short-time Fourier transforms $(V_{\phi_N} f^{\rm rex})(x_0, \xi)$, $N \in \mathbf{N}$, for ξ belonging to a cone Γ around ξ_0. Here $\phi_N = e^{-|\,\cdot\,|^2/(4N)}$ and $f^{\rm rex}$ denotes an appropriate extension of $f_{\rm res}$. (This notation is only used in the introduction.) Our definition extends the notion of ultra-distribution wave-fronts for $s = t > 1$ and can be accommodated in order to extend the notion of the analytic wave-front in the case $s = 1$ (see Remark 2.4).

We establish basic properties for the wave-front sets for $s \in [1/2, 1)$. Moreover, we introduce a subspace $\mathcal{E}^s_{0,\infty}(\mathbf{R}^d)$ of the space of global Gevrey ultra-differentiable functions $\mathcal{E}^s_\infty(\mathbf{R}^d) \subset \mathcal{E}^s(\mathbf{R}^d)$, (cf. Definition 1.14) and analyze the local regularity of an $f \in \mathscr{D}'(\mathbf{R}^d)$ with respect to $\mathcal{E}^s_{0,\infty}(\mathbf{R}^d)$ and $\mathcal{E}^s$. We have, with respect to $\mathcal{E}^s$,

$$\operatorname{sing\,supp}_s f \subset \pi_1(WF_s(f)), \tag{1.1}$$

where π_1 is the projection $\pi_1(x, \xi) = x$ from $\mathbf{R}^{2d}$ to $\mathbf{R}^d$. Considering the local singularities with respect to $\mathcal{E}^s_{0,\infty}(\mathbf{R}^d)$, we have

$$\pi_1(WF_s(f)) \subset \operatorname{sing\,supp}_{\infty,s} f. \tag{1.2}$$

We also show that the wave-front set of $f \in \mathscr{D}'(\mathbf{R}^d)$ decreases with the differentiation as well as with the multiplication by a function from $\mathcal{E}^s_{0,\infty}(\mathbf{R}^d)$, $s \in [1/2, 1)$. For the former property we assume additionally that the Fourier transform of f is a polynomially bounded locally integrable function. Consequently, the wave-front sets here can be applied on problems involving partial differential equations.

We prove the basic estimate of the propagation of the wave-front, $s \in [1/2, 1)$ related to a distribution f and a differential operator with constant coefficients $P(D)$:

$$WF_s(P(D)f) \subseteq WF_s(f) \subseteq WF(s, P, f) \cup \operatorname{Char}(P),$$

where, $WF(s, P, f)$ is a suitable set determined by the regularity of $P(D)(f^{\rm rex})$ and the polynomial growth of the Fourier transform of f^{rex}.

2. Gevrey wave-fronts

In general it is a difficult task to examine wave-front properties of Gevrey regularity of order s, when $s < 1$, since the presence of suitable compactly supported functions of such regularity are absent. In this section we introduce a new approach in this case, based on a suitable restriction-extension technique, roughly explained in the introduction, for the involved distributions.

Before the definition of the wave-front sets, we introduce some notations. In what follows we let $\mathcal{F}$ be the Fourier transform on $\mathscr{S}'(\mathbf{R}^d)$ which takes the form

$$\hat{f}(\xi) = (\mathcal{F}f)(\xi) = (2\pi)^{-d/2} \int_{\mathbf{R}^d} f(x) e^{-i\langle x, \xi \rangle} dx$$

when $f \in \mathscr{S}(\mathbf{R}^d)$. In particular, if

$$E_{x_0,N}(x) = e^{-|x-x_0|^2/(4N)}, \quad x \in \mathbf{R}^d,\ N \in \mathbf{Z}_+,$$

then

$$(\mathcal{F}^{-1}E_{-x_0,N})(\xi) = (\mathcal{F}E_{x_0,N})(\xi) = (2N)^{d/2}e^{-i\sqrt{2N}\langle x_0,\xi\rangle}e^{-N|\xi|^2}, \quad \xi \in \mathbf{R}^d, \tag{2.1}$$

and note that

$$(2\pi)^{-d/2}(\mathcal{F}E_{x_0,N})(\xi) = \frac{N^{d/2}e^{-i\sqrt{2N}\langle x_0,\xi\rangle}e^{-N|\xi|^2}}{\pi^{d/2}} \to e^{-|x_0|^2/2}\delta(\xi) \quad \text{as} \quad N \to \infty,$$

with convergence in $\mathscr{S}'(\mathbf{R}^d)$. Here and in what follows, $\mathbf{Z}_+$ denotes the positive integers, and $\mathbf{N} = \mathbf{Z}_+ \cup \{0\}$. For convenience we set $E_N = E_{0,N}$.

Remark 2.1. Recall that the d-dimensinal Hermite polynomial of order $\alpha \in \mathbf{N}^d$ is given by

$$H_\alpha(x) = (-1)^{|\alpha|}e^{|x|^2}\partial^\alpha(e^{-|x|^2}), \quad x \in \mathbf{R}^d.$$

We have

$$e^{-|x|^2/2}|H_\alpha(x)| \lesssim \Big(\frac{2}{e}\Big)^{|\alpha|/2}\alpha^{\alpha/2}, \quad x \in \mathbf{R}^d,\ \alpha \in \mathbf{N}^d.$$

This implies

$$e^{|x|^2/8N}|E_N^{(\alpha)}(x)| \lesssim (e\sqrt{N})^{-|\alpha|}\alpha^{\alpha/2}, \quad x \in \mathbf{R}^d,\ |\alpha| \leqslant N \in \mathbf{Z}_+. \tag{2.2}$$

Especially, we have

$$|E_N^{(\alpha)}(x)| \lesssim (e\sqrt{N})^{-|\alpha|}\alpha^{\alpha/2}, \quad x \in \mathbf{R}^d,\ |\alpha| \leqslant N \in \mathbf{Z}_+.$$

Remark 2.2. For future references, we note that

$$\|\langle\xi\rangle^l\hat{E}_N(\xi)\|_{L^1} < c_{l,N}, \quad l, N \in \mathbf{Z}_+.$$

Moreover, $c_{l,N} \leqslant c$ for $l \leqslant N \in \mathbf{Z}_+$, where c does not depend on l and N.

Definition 2.1. Let $X, Y \subseteq \mathbf{R}^d$ be open, $f \in \mathscr{D}'(X)$ and $g \in \mathscr{D}'(Y)$. Then g is called *f-related* at $x_0 \in X \cap Y$, if $f = g$ in an open neighborhood of x_0. The notation $f \underset{x_0}{\sim} g$ is used when g is f-related at x_0.

Evidently, $\underset{x_0}{\sim}$ in the previous definition is an equivalence relation.

2.1. The definition of the wave-front. We now give the definition of regular points and wave-front sets with respect to the Gevrey class $s \in [1/2, 1)$. Here and in what follows we let $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$.

Definition 2.2. Let $s \in [1/2, 1)$, $X \subseteq \mathbf{R}^d$ be open, $f \in \mathscr{D}'(X)$, $x_0 \in X$ and $\xi_0 \in \mathbf{R}^d \setminus \{0\}$. Then (x_0, ξ_0) is called a *Gevrey regular point of order s for f*, if for some $g \in \mathscr{S}'(\mathbf{R}^d)$ such that $f \underset{x_0}{\sim} g$, some open cone Γ of ξ_0, $C > 0$ and $N_0 \in \mathbf{Z}_+$, there holds

$$|(\mathcal{F}(gE_{x_0,N}))(\xi)| \leqslant \frac{C^{n+1}n^{sn}}{\langle\xi\rangle^n} \quad \text{when} \quad \xi \in \Gamma,\ n \leqslant N, \tag{2.3}$$

for every integer $N \geqslant N_0$.

The complement of the set of Gevrey regular points in $\mathbf{R}^d \times (\mathbf{R} \setminus \{0\})$ is denoted by $WF_s(f)$ and is called the *s-wave-front set of f*.

REMARK 2.3. 1. With the same assumptions, (2.3) implies that for every $k \in \mathbf{N}$,

$$|(\mathcal{F}(gE_{x_0,kN}))(\xi)| \leqslant \frac{C^{n+1}n^{sn}}{\langle \xi \rangle^n} \quad \text{when} \quad \xi \in \Gamma,\ n \leqslant N, \tag{2.4}$$

2. If $f \underset{x_0}{\sim} g$, then $(x_0,\xi_0) \notin WF_s(f)$ if and only if $(x_0,\xi_0) \notin WF_s(g)$.

The following result shows that the condition $n \leqslant N$ in (2.3) can be replaced by $n \leqslant N + N_1$ for any fixed integer $N_0 \geqslant 0$.

LEMMA 2.1. *Let $s \in [1/2,1)$, $x_0 \in \mathbf{R}^d$, $g \in \mathscr{S}'(\mathbf{R}^d)$, $\Gamma \subseteq \mathbf{R}^d \setminus 0$ be an open cone, $N_1 \geqslant 0$ be an integer and let $N_0 \in \mathbf{Z}_+$. Then the following conditions are equivalent:*

(1) *there is a constant $C > 0$ such that (2.3) holds for every integer $N \geqslant N_0$;*
(2) *there is a constant $C > 0$ such that*

$$|(\mathcal{F}(gE_{x_0,N}))(\xi)| \leqslant \frac{C^{n+1}n^{sn}}{\langle \xi \rangle^n} \quad \textit{when} \quad \xi \in \Gamma,\ n \leqslant N + N_1, \tag{2.3$'$}$$

holds for every integer $N \geqslant N_0$.

PROOF. It is clear that (2) implies (1). In order to prove the reversed inclusion we only consider the case when $x_0 = 0$ and $N_1 = 1$. The general case follows by similar arguments and is left for the reader.

We need to prove that if (1) holds, then (2) holds in the case $n = N + 1$. If (1) holds, then

$$\begin{aligned}
|(\mathcal{F}(gE_N))(\xi)| &= |(\mathcal{F}(gE_{N+1}E_{N(N+1)}))(\xi)| \\
&\leqslant (N(N+1))^{d/2} \int |(\mathcal{F}(gE_{N+1}))(\xi-\eta)| e^{-N(N+1)|\eta|^2}\, d\eta \\
&\leqslant C_1^N (N+1)^{s(N+1)} \left(1 + \int_{|\eta|\geqslant 1} \langle \xi - \eta \rangle^{-N-1} e^{-N(N+1)|\eta|^2}\, d\eta\right) \\
&\leqslant \frac{2^{N+1}C_1^N (N+1)^{s(N+1)}}{\langle \xi \rangle^{N+1}} \left(1 + \int_{|\eta|\geqslant 1} |\eta|^{N+1} e^{-N(N+1)|\eta|^2}\, d\eta\right) \\
&\leqslant \frac{C_3^N N^{sN}}{\langle \xi \rangle^{N+1}} \left(1 + \int_{|\eta|\geqslant 1} |\eta|^{N+1} e^{-N(N+1)|\eta|^2}\, d\eta\right) \\
&\leqslant \frac{C_4^N N^{sN}}{\langle \xi \rangle^{N+1}} (1 + \Gamma((N+d+1)/2)(N(N+1))^{-(N+d+1)/2}) \\
&\leqslant \frac{C_5^N N^{sN}}{\langle \xi \rangle^{N+1}},
\end{aligned}$$

for some positive constants $C_1, \dots, C_5 > 0$. Hence (2) follows. □

In several results later on, we need that, additionally g in Definition 2.2 is chosen such that

$$\|\hat{g}(\xi)\langle \xi \rangle^{-l}\|_{L^\infty} < \infty \quad \text{for some } l > 0. \tag{2.5}$$

EXAMPLE 2.1. Let $g(x) = e^{-a|x|^2}$, $x \in \mathbf{R}^d$. Then

$$\mathcal{F}(g(x)E_N(x))(\xi) = \Big(\frac{2N}{4aN+1}\Big)^{d/2} e^{-\frac{N}{4Na+1}|\xi|^2}, \quad \xi \in \mathbf{R}^d.$$

One can simply show that (2.2) holds in any cone. We have the similar conclusion for $x_0 \neq 0$.

EXAMPLE 2.2. Let f_n be a sequence of entire functions over $\mathbf{C}$ and $\{s_n\}_{n\in\mathbf{Z}_+}$ be a strictly decreasing sequence in $[1/2, 1)$ tending to $1/2$ as $n \to \infty$. Let the sequence of restriction of f_n on $\mathbf{R}$ satisfy $f_n \in \mathcal{S}^{s_n}_{s_n}(\mathbf{R}) \smallsetminus \mathcal{S}^{s_{n+1}}_{s_{n+1}}(\mathbf{R}^d)$, where $\mathcal{S}^{s_n}_{s_n}(\mathbf{R})$ are Gelfand–Shilov spaces. Denote by χ_n the characteristic function of the set $(-n,-n+1)\cup(n-1,n), n \in \mathbf{Z}_+$. Put $f = \sum_{n=1}^{\infty} \chi_n f_n$ and $g_n = f_n, n \in \mathbf{Z}_+$. Then, $f \underset{x_0}{\sim} g_n$ for every $x_0 \in (-n,-n+1) \cup (n-1,n)$. Since

$$g_n \in \mathcal{S}^{s_n}_{s_n}(\mathbf{R} \smallsetminus ([-n+1, n-1] \cup I_n)),$$

where $I_n = \{n, -n, n+1, -n-1, \dots, n+k, -n-k, \dots\}$, we obtain

$$WF_{s_n}(f) \subset ([-n+1, n-1] \cup I_n) \times \mathbf{R}^d \smallsetminus \{0\}, \quad n \in \mathbf{Z}_+.$$

EXAMPLE 2.3. Let f be a distribution on $\mathbf{R}$ such that

$$\hat{f}(\xi) = \begin{cases} e^{-\xi^2/2}, & \xi \geqslant 0, \\ 1, & \xi < 0. \end{cases}$$

Then,

$$f(x) = \sqrt{\frac{\pi}{2}}\delta(x) + \frac{1}{2i\pi}\ \mathrm{vp}\ \frac{1}{x} + \left(\sqrt{\frac{\pi}{2}}\delta(x) - \frac{1}{2i\pi}\ \mathrm{vp}\ \frac{1}{x}\right) * e^{-x^2/2}$$

Put $g(x) = f(x), x \in \mathbf{R}$. Clearly, $f \underset{x_0}{\sim} g$ for every $x_0 \in \mathbf{R}$. Moreover,

$$\hat{g} * \widehat{E_{x_0,N}}(\xi) = 2N^{1/2} e^{-i(2N)^{1/2}x_0\xi} \left(\int_{-\infty}^{0} e^{-N(\xi-\eta)^2} + \int_0^\infty e^{-\eta^2/4} e^{-N(\xi-\eta)^2} \right)$$

and, one can see that for every $x_0 \in \mathbf{R}$, $(x_0, \xi) \in WF_s(f)$ when $\xi < 0$, while $(x_0, \xi) \notin WF_s(f)$ when $\xi > 0$, for every $s \geqslant 1/2$.

Consider $x_0 \neq 0$. Then we can also take $f \underset{x_0}{\sim} g_0$, where

$$g_0(x) = \frac{1}{2i\pi}\ \mathrm{vp}\ \frac{1}{x} + \left(\sqrt{\frac{\pi}{2}}\delta(x) - \frac{1}{2i\pi}\ \mathrm{vp}\ \frac{1}{x}\right) * e^{-x^2/2} \tag{2.6}$$

since it is equal to f in every neighbourhood of x_0 not containing zero. The "bad" part $vp\frac{1}{x}$, has the Fourier transform

$$\mathcal{F}\Big(vp\frac{1}{x}\Big)(\xi) = -i\frac{\sqrt{\pi}}{2}\ \mathrm{sgn}\xi,$$

which, in convolution with $e^{-N\xi^2}$ can not be estimated as in (2.3), neither for $\xi < 0$ nor for $\xi > 0$. The convolution part of g_0 in (2.6) may not compensate the growth of the "bad" part for $\xi < 0$ or $\xi > 0$, as well.

Remark 2.4. In the case $t > 1, f \in \mathscr{D}(\mathbf{R}^d)$, the product of f and any cut-of function κ, with a sufficiently small support, belonging to the space of ultra-differentiable functions $\mathscr{D}^t(\mathbf{R}^d)$, equals one in a neighborhood of x_0, is a suitable extension leading to the same definition of $WF_t(f)$.

Remark 2.5. In the case $s = 1$, for the analytic wave-front one has to use a suitable sequence of $g_N \in \mathscr{S}'(\mathbf{R}^d), n \in \mathbf{N}_+$, such that (2.3) is changed into

$$|\mathcal{F}(g_N)(\xi)| = |\langle g_N, e^{-i\langle\,\cdot\,,\xi\rangle}\rangle| \leqslant \frac{C^{n+1}n^{sn}}{\langle\xi\rangle^n}, \quad \xi \in \Gamma,\ n \leqslant N,\ N_0 < N \in \mathbf{Z}_+,$$

where $g_N = f\kappa_N$, and κ_N is a sequence of compactly supported smooth functions equals one in a neighborhood of x_0 such that for some $C > 0$,

$$|\kappa_N^{(\alpha)}(x)| \leqslant (CN)^{|\alpha|}, \quad x \in \mathbf{R}^d,\ |\alpha| \leqslant N,$$

see (8.4.5) in [**10**, Section 8.4].

Remark 2.6. Let $(x_0, \xi_0) \notin WF_s(f)$. If $y \in B(x_0, r)$ and $\eta \in \Gamma$, then $(y, \eta) \notin WF_s(f)$. Thus, $WF_s(f)$ is a closed set of $\mathbf{R}^d \times (\mathbf{R}^d \setminus \{0\})$.

In the sequel, we will always assume, without mentioning this explicitly, that s is a parameter so that $s \in [1/2, 1)$.

2.2. Basic properties. The next result links the s-wave-front sets to Gevrey regularity of order s.

Proposition 2.1. *Let $X \subseteq \mathbf{R}^d$ be open, $f \in \mathscr{D}'(X)$ and $x_0 \in X$. Assume that there are $N_0 \in \mathbf{Z}_+$, $C > 0$ and $g \in \mathscr{S}'(\mathbf{R}^d)$ such that $f \underset{x_0}{\sim} g$ and*

$$|(\mathcal{F}(gE_{x_0,N}))(\xi)| \leqslant \frac{C^{n+1}n^{sn}}{\langle\xi\rangle^n}, \quad \xi \in \mathbf{R}^d,\ n \leqslant N, \tag{2.7}$$

for every $N \geqslant N_0$. Then

$$\sup_{x \in U} |D^{(\alpha)} f(x)| \leqslant C^{|\alpha|+1}\alpha!^s, \quad \alpha \in \mathbf{N}^d, \tag{2.8}$$

for some open neighborhood U of x_0.

Proof. We only prove the result in the case when $N_0 = 1$, $x_0 = 0$. The general case follows by similar arguments.

We have $f = g$ on $U = B(0, r)$ for some choice of $r > 0$. Let $\alpha \in \mathbf{N}^d$, $x \in U$ and let $C_1 > C$. Then

$$\sup_{\alpha \in \mathbf{N}^d} \Big(\frac{C^{|\alpha|+d+1}(|\alpha| + d + 1)!^s}{C_1^{|\alpha|}|\alpha|!^s} \Big) < \infty, \tag{2.9}$$

$$|D^\alpha(f(x)E_N(x))| = |D^\alpha(g(x)E_N(x))| \lesssim I_1 + I_2,$$

where

$$I_1 = \Big| \int_{|\xi| \leqslant 1} \xi^\alpha (\mathcal{F}(gE_N))(\xi) e^{i\langle x,\xi\rangle} d\xi \Big|,$$

$$I_2 = \Big| \int_{|\xi| \geqslant 1} \xi^\alpha (\mathcal{F}(gE_N))(\xi) e^{i\langle x,\xi\rangle} d\xi \Big|.$$

By (2.7) we get

$$I_1 \leqslant C, \quad \alpha \in \mathbf{N}^d.$$

In order to estimate I_2 we let $n = |\alpha| + d + 1$ and let $N > n$. Then (2.9) gives

$$I_2 \leqslant C^{n+1} n!^s \int_{|\xi|\geqslant 1} |\xi|^{|\alpha|-n} d\xi,$$

which implies that

$$\|D^\alpha (gE_N)\|_{L^\infty(U)} \leqslant C_1^{|\alpha|+d+1} (|\alpha| + d + 1)!^s \leqslant C_2^{|\alpha|+1} |\alpha|^{s|\alpha|}, \quad |\alpha| \leqslant N,$$

for some positive constants C_1 and C_2. Letting $N \to \infty$, the left-hand side uniformly converges to $D^\alpha g$ on U; since $\|D^\alpha g\|_{L^\infty(U)} = \|D^\alpha f\|_{L^\infty(U)}$, (2.8) follows. □

We also consider spaces as in the following definition.

Definition 2.3. (1) $\mathcal{E}^s_{0,\infty}(\mathbf{R}^d)$ consists of all $\varphi \in C^\infty(\mathbf{R}^d)$ such that $\hat{\varphi} \in L^\infty(\mathbf{R}^d)$, $\alpha \in \mathbf{N}^d$, and

$$\|\varphi\|_{\mathcal{E}^s_{0,\infty,h}} \equiv \sup_{\alpha \in \mathbf{N}^d} \frac{h^{|\alpha|} \|\langle\xi\rangle^{|\alpha|} \hat{\varphi}(\xi)\|_{L^\infty(\mathbf{R}^d)}}{\alpha!^s} < \infty, \tag{2.10}$$

for some $h > 0$;

(2) $\mathcal{E}^s_\infty(\mathbf{R}^d)$ consists of all $\varphi \in C^\infty(\mathbf{R}^d)$ such that

$$\|\varphi\|_{\mathcal{E}^s_{\infty,h}} \equiv \sup_{\alpha \in \mathbf{N}^d} \frac{h^{|\alpha|} \|\varphi^{(\alpha)}\|_{L^\infty(\mathbf{R}^d)}}{\alpha!^s} < \infty.$$

for some $h > 0$;

If A and B are topological spaces, then $A \to B$ means that $A \subseteq B$ and that the injection map from A to B is continuous, while $A \hookrightarrow B$ additionally means that A is dense in B.

Proposition 2.2. $\mathcal{E}^s_{0,\infty}(\mathbf{R}^d) \hookrightarrow \mathcal{E}^s_\infty(\mathbf{R}^d) \to \mathcal{E}^s(\mathbf{R}^d)$.

Proof. The second embedding is an immediate consequence of the definition. Let $\varphi \in \mathcal{E}^s_{0,\infty}(\mathbf{R}^d)$. By (2.10) and the fact that $|\alpha|! \leqslant d^{|\alpha|}\alpha!$ we have, with suitable $h_1 > 0$,

$$\frac{h^\alpha |\varphi^{(\alpha)}(x)|}{\alpha!^s} \leqslant \int \frac{h_1^{|\alpha|+d+1} |\hat{\varphi}(\xi)| \langle\xi\rangle^{|\alpha|+d+1}}{|\alpha|!^{s+d+1}} \frac{d\xi}{\langle\xi\rangle^{d+1}} < \infty.$$

Let $\delta_N = (\pi^{-1}N)^{d/2} e^{-N|\xi|^2}, N \in \mathbf{N}$ and $\theta \in \mathcal{E}^s_\infty(\mathbf{R}^d)$. Then $\theta_N = \delta_N * \theta$ is a sequence in $\mathcal{E}^s_{0,\infty}(\mathbf{R}^d)$ which converges to θ in $\mathcal{E}^s_\infty(\mathbf{R}^d)$ as $N \to \infty$. For the proof we have to use the fact that $||\delta_N||_{L^1(\mathbf{R}^d)} = 1, N \in \mathbf{N}$ and

$$\|(\delta_N * \theta)^{(\alpha)}||_{L^1(\mathbf{R}^d)} \leqslant \|\delta_N\|_{L^1(\mathbf{R}^d)} \|\theta^{(\alpha)}\|_{L^\infty(\mathbf{R}^d)},$$

$$\|\langle\xi\rangle^{|\alpha|} \widehat{\delta_N * \theta}\|_{L^\infty(\mathbf{R}^d)} = \|\mathcal{F}^{-1}(\delta_N * \theta)^{(\alpha)}\|_{L^\infty(\mathbf{R}^d)} \leqslant c\|(\delta_N * \theta)^{(\alpha)}\|_{L^1(\mathbf{R}^d)}.$$ □

We have now the following wave-front result.

Proposition 2.3. *Let $f \in \mathscr{D}'(\mathbf{R}^d)$, P be a polynomial on $\mathbf{R}^d$, and let $\varphi \in \mathcal{E}^s_{0,\infty}(\mathbf{R}^d)$. Then the following is true:*

(1) *if* $(x_0,\xi_0)\notin WF_s(f)$ *and* $f \underset{x_0}{\sim} g$ *for some* $g\in \mathscr S'(\mathbf R^d)$ *such that* (2.5) *holds, then* $(x_0,\xi_0)\notin WF_s(\varphi f)$.

(2) $WF_s(P(D)f)\subseteq WF_s(f)$.

Proof. Assume that f is Gevrey s-regular at $(x_0,\xi_0)\in \mathbf R^d\times (\mathbf R^d\setminus \{0\})$, and choose $g\in \mathscr S'(\mathbf R^d)$ such that $f \underset{x_0}{\sim} g$ and (2.5) hold. We shall prove that $(x_0,\xi_0)\notin WF_s(\varphi f)$ and $(x_0,\xi_0)\notin WF_s(P(D)f)$. We only prove these relations in the case $x_0=0$ and $N_0=1$ in Definition 2.2. The general case follows by similar arguments and is left for the reader.

(1) We have $(0,\xi_0)\notin WF_s(f)$. We shall apply the standard technique as in [**10**, Lemma 8.1.1]. Let Γ be an open cone such that $\xi_0\in\Gamma$ and that (2.3) holds, and let $\Gamma_1\subseteq \Gamma\cup\{0\}$ be a closed cone with ξ_0 as an interior point. Then with a suitable $c\in(0,1)$,

$$
\begin{gathered}
\xi\in\Gamma_1, |\xi|>1 \text{ and } |\xi-\eta|\leqslant c|\xi| \quad\Rightarrow\quad \eta\in\Gamma,\\
|\xi-\eta|\leqslant c|\xi| \quad\Rightarrow\quad |\xi|\leqslant (1-c)^{-1}|\eta|.
\end{gathered}\tag{2.11}
$$

We have

$$
(\mathcal F(\varphi g E_N))(\xi) = I_1(\xi)+I_2(\xi),
$$

where

$$
\begin{aligned}
I_1(\xi) &= \int_{|\xi-\eta|\leqslant c|\xi|} \hat\varphi(\xi-\eta)(\mathcal F(gE_N))(\eta)\,d\eta,\\
I_2(\xi) &= \int_{|\eta|\geqslant c|\xi|} \hat\varphi(\eta)(\mathcal F(gE_N))(\xi-\eta)\,d\eta, \quad \xi\in\Gamma_1\subset\Gamma.
\end{aligned}
$$

We need to estimate $|I_1(\xi)|$ and $|I_2(\xi)|$. We have

$$
\begin{aligned}
\sup_{\xi\in\Gamma_1}|\xi|^nI_1(\xi) &\leqslant \sup_{\xi\in\Gamma_1}|\xi|^n \sup_{|\xi-\eta|\leqslant c|\xi|} |(\mathcal F(gE_N))(\eta)| \int |\hat\varphi(\xi-\eta)|\,d\eta\\
&\leqslant C_1(1-c)^{-n}\sup_{\eta\in\Gamma}|\eta|^n|(\mathcal F(gE_N))(\eta)| \leqslant C^{n+1}n^{sn}, \quad n\leqslant N.
\end{aligned}\tag{2.12}
$$

Here the second inequality follows from the fact that $|\xi|\leqslant (1-c)^{-1}|\eta|$ when $|\xi-\eta|\leqslant c|\xi|$.

Next, we estimate $|I_2(\xi)|$. By (2.5) we get

$$
\begin{aligned}
\|\langle\xi\rangle^{-l}\mathcal F(gE_N)(\xi)\|_{L^\infty} &\leqslant \|(\langle\xi\rangle^{-l}(\mathcal Fg)(\xi))*(\langle\xi\rangle^{l}\hat E_N(\xi))\|_{L^\infty}\\
&\leqslant \|\langle\xi\rangle^{-l}(\mathcal Fg)(\xi)\|_{L^\infty}\|\langle\xi\rangle^{l}\hat E_N(\xi)\|_{L^1}<\infty.
\end{aligned}\tag{2.13}
$$

Let $n\leqslant N$. It follows from (2.7), (2.13) and the assumptions on φ that if $C>0$ is chosen large enough, then

$$
\begin{aligned}
\Big|\frac{|\xi|^n}{C^{n+1}n!^s}I_2(\xi)\Big| &\leqslant \int_{|\eta|\geqslant c|\xi|}\frac{|\eta^n\hat\varphi(\eta)|}{C^{n+1}n!^s}\langle\xi-\eta\rangle^{l}\langle\xi-\eta\rangle^{-l}|(\mathcal F(gE_N))(\xi-\eta)|\,d\eta\\
&\leqslant C_1\int_{|\eta|\geqslant c|\xi|}\frac{|\eta^n\hat\varphi(\eta)|}{C^{n+1}n!^s}\langle\eta\rangle^{l+d+1}\frac{d\eta}{\langle\eta\rangle^{d+1}}\\
&\leqslant C_2\sup_{|\eta|>c|\xi|}\Big(\frac{|\eta^{n+r}\hat\varphi(\eta)|}{C^{n+1}n!^s}\Big)<\infty,
\end{aligned}
$$

where $r > l + d + 1$, for some constants C_1 and C_2. This gives

$$|I_2(\xi)| \leqslant \frac{C^{n+1} n^{sn}}{|\xi|^n}, \quad \xi \in \Gamma_1, \ n \leqslant N \tag{2.14}$$

for some constant $C > 0$. The assertion now follows by combining (2.12) and (2.14).

(2) The assertion follows if we prove $(0, \xi) \notin WF_s(\partial_{x_k} f)$, $1 \leqslant k \leqslant d$. Let $\xi \in \mathbf{R} \setminus \{0\}$. We have

$$\mathcal{F}((\partial_{x_k} g) E_N)(\xi) = i\xi_k \mathcal{F}(g E_N)(\xi) - \frac{1}{2N} \mathcal{F}(x_k g E_N)(\xi). \tag{2.15}$$

We estimate the terms on the right-hand side separately.

In view of Lemma 2.1, the first term in the right-hand side of (2.15) can be estimated as

$$|i\xi_k \mathcal{F}(g E_N)(\xi)| \leqslant \frac{C^{n+1} n^{sn}}{\langle \xi \rangle^{n-1}} \leqslant \frac{C_1^n (n-1)^{s(n-1)}}{\langle \xi \rangle^{n-1}}, \quad \xi \in \Gamma, \ n \leqslant N+1,$$

for some constants C and C_1. Hence (with a new C)

$$|i\xi_k \mathcal{F}(g E_N)(\xi)| \leqslant \frac{C^{n+1} n^{sn}}{\langle \xi \rangle^n}, \quad \xi \in \Gamma, \ n \leqslant N,$$

for some constant C.

Differentiating (2.1), using that

$$|\mathcal{F}(x_k E_N)(\xi)| = |\partial_{\xi_k} \mathcal{F}(E_N)(\xi)|, \quad \xi \in \mathbf{R}^d,$$

and taking $\sqrt{N}\xi$ as new variables of integration we obtain

$$\frac{1}{2N} \int |\mathcal{F}(x_k E_N)(\xi)| d\xi = C_1 \int \xi_k e^{-N|\xi|^2} N^{d/2} d\xi \leqslant C_2 N^{-1/2} \leqslant C_2,$$

for some constants C_1 and C_2. This also gives

$$\frac{1}{2N} \int |\mathcal{F}(x_k E_{2N})(\xi)| d\xi < C,$$

where C is independent of N. Thus, if Γ_1 and Γ are the same as in the first part of the proof, it follows from that part that for the second term in (2.15) we have, using in the end (2.4),

$$\sup_{\xi \in \Gamma_1} \Big(\frac{\langle \xi \rangle^n}{C^{n+1} n!^s} |\mathcal{F}(g(\partial_{x_k} E_N))(\xi)| \Big) \frac{1}{2N} \sup_{\xi \in \Gamma_1} \Big(\frac{\langle \xi \rangle^n}{C^{n+1} n!^s} \big| |\mathcal{F}(g E_{2N})| * |\mathcal{F}(\partial_{x_k} E_{2N})|(\xi) \big| \Big)$$
$$\leqslant \sup_{\xi \in \Gamma} \Big(\frac{\langle \xi \rangle^n}{C^{n+1} n!^s} |\mathcal{F}(g E_{2N})(\xi)| \Big) \frac{1}{2N} \int |\mathcal{F}(x_k E_{2N}))(\xi)| d\xi < C, \quad n \leqslant N.$$

where $C > 0$ is a suitable constant not depending on n and N, and the assertion follows. □

Remark 2.7. For the later use, with the notation of Definition 2.2, we note that $(x_0, \xi_0) \notin WF_s(P(D)f)$ if and only if $(x_0, \xi_0) \notin WF_s(P(D)g)$, where $f \underset{x_0}{\sim} g$ and $P(D)$ is a differential operator with constant coefficients.

3. Local regularity

Proposition 3.1. *Let $U \subseteq \mathbf{R}^d$ be open, $x_0 \in U$, $f \in \mathscr{D}'(U)$. Assume that $g \in \mathcal{E}^s_{0,\infty}(\mathbf{R}^d)$ be such that $f \underset{x_0}{\sim} g$. Then there exists $C > 0$ such that* (2.7) *holds for $\mathcal{F}(gE_N)$.*

Proof. Let $n \leqslant N \in \mathbf{R}^d$. Then,

$$\sup_{\xi \in \mathbf{R}^d} |\langle \xi \rangle^n (\mathcal{F}(gE_N))(\xi)| \leqslant \sup_{\xi \in \mathbf{R}^d} |\langle \xi \rangle^n \hat{g}(\xi)| \| \langle \xi \rangle^n \hat{E}_N(\xi)\|_{L^1(\mathbf{R}^d)},$$

and the result follows from the fact that $\| \langle \xi \rangle^n \hat{E}_N(\xi)\|_{L^1(\mathbf{R}^d)} < c$, for some c, see Remark 2.2. □

As a consequence we have the following. Here $\operatorname{sing\,supp}_{\infty,s} f$ is the set of points $x \in \mathbf{R}^d$ such that there exists no $g \in \mathcal{E}^s_{0,\infty}(\mathbf{R}^d)$ such that $f \underset{x_0}{\sim} g$.

Corollary 3.1. *Let U be open, $x_0 \in U$, $f \in \mathscr{D}'(U)$ and $g \in \mathcal{E}^s_{0,\infty}(\mathbf{R}^d)$ be such that $f \underset{x_0}{\sim} g$. Then $(x_0, \xi) \notin WF_s(f)$, for any $\xi \in \mathbf{R}^d \setminus \{0\}$. In particular,* (1.2) *holds.*

Now, we compare the projection of $WF_s(f)$ with the singular support with respect to $\mathcal{E}^s$. Definition 2.2 and the compactness of the sphere $\mathbf{S}^{d-1}$ imply the following proposition.

Proposition 3.2. *Let $f \in \mathscr{D}'(\mathbf{R}^d)$, $K \subseteq \mathbf{R}^d$ be compact, and let F be a closed cone. If $WF_s(f) \cap (K \times F) = \emptyset$, then there exist an open set U, an open cone Γ and $g \in \mathscr{S}'$ such that $f = g$ on U, $K \times F \subset U \times \Gamma$ and, for some $C > 0$,*

$$|(\mathcal{F}(gE_N))(\xi)| \leqslant C^{n+1} \frac{n!^s}{\langle \xi \rangle^n}, \quad \xi \in \Gamma, \ n \leqslant N \in \mathbf{N}. \tag{3.1}$$

Proof. Let $K = \{x_0\}$, $x_0 = 0$ and $\xi_0 \in F = \Gamma_{\xi_0}$ be a closed conic neighbourhood of ξ_0 contained in an open cone Γ such that (2.3) holds in Γ for $g = f$ in an open set U, say an open ball with center at x_0. Then (3.1) for $U \times \Gamma$. In the case $K = \{x_0\}$ and F being a closed cone of $\mathbf{R}^d \setminus \{0\}$, the intersection of F with the unit sphere in $\mathbf{R}^d$ is compact. Hence, we may choose a finite number of balls, $B(x_0, r_{x_0,\xi_j})$, closed cones Γ_{ξ_j} compactly included in open cones Γ_j, $j = 1, \dots, k$ ((2.3) holds in Γ_j), then take for U the intersection of open balls, and for Γ, $\Gamma \equiv \bigcup_{j=1}^k \Gamma_j$.

In a general case, for any $x \in K$ we can repeat the proceeding procedure. We cover K by finite number of open balls $B(x_l, r_{x_l,\xi_{l,j}})$, $l = 1, \dots m, j = 1, \dots, k_l$, make intersections of balls with respect to j and the union of corresponding cones, obtain $B(x_l, r_l) \times \Gamma_l$, make intersection of cones, $\Gamma = \bigcap_{l=1}^m \Gamma_l$, and obtain $U \times \Gamma = \bigcup_{l=1}^m U_l \times \Gamma$. □

The following result links the $\operatorname{sing\,supp}_s$ with th s-wave-front set.

Corollary 3.2. *Let $f \in \mathscr{D}'(\mathbf{R}^d)$. Then* (1.1) *holds.*

PROOF. Assume that $(x_0,\xi_0) \notin WF_s(f)$ for all $\xi_0 \in \mathbf{R}^d \setminus 0$. Then there is a neighborhood U of x_0 such that $WF_s(f) \cap (U \times \mathbf{R}^d) = \emptyset$ and $g \in \mathscr{S}'$ equal to f on U so that with suitable C (and $x_0 = 0$, $N_0 = 1$)

$$|(\mathcal{F}(gE_N))(\xi)| \leqslant \frac{C^{n+1}n!^s}{\langle\xi\rangle^n}, \quad \xi \in \mathbf{R}^d,\ n \leqslant N,\ N \in \mathbf{Z}_+.$$

By Proposition 2.1, we conclude that $g \in \mathcal{E}^s(U)$. That is, $0 \notin \operatorname{sing\,supp}_s f$. □

The next statement is a straightforward consequence of the definition and previous results.

PROPOSITION 3.3. *Let $f \in \mathscr{D}'(\mathbf{R}^d)$ and $1/2 \leqslant s_1 < s_2 < 1$. Then $WF_{s_2}(f) \subset WF_{s_1}(f)$.*

4. Wave-front for $P(D)f = h$

Let $D^\alpha = (-i)^{|\alpha|}\partial^{\alpha_1+\cdots+\alpha_d}/(\partial_{x_1}^{\alpha_1}\ldots\partial_{x_d}^{\alpha_d})$, $P(D) = \sum_{|\alpha|\leqslant m} a_\alpha D^\alpha$ be a differential operator with constant coefficients, $P_m(\xi) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha$ its principal symbol, and let $f \in \mathscr{D}'(\mathbf{R}^d)$. Recall, $\operatorname{Char}(P)$ is defined by $\operatorname{Char}(P) = \{\xi \in \mathbf{R}^d \setminus 0, P_m(\xi) \neq 0\}$.

DEFINITION 4.1. The set $\operatorname{Reg}(s,P,f)$ consists of all points $(x_0,\xi_0) \in \mathbf{R}^d \times (\mathbf{R}^d \setminus \{0\})$ such that for some $g \in \mathscr{S}'(\mathbf{R}^d)$, the following conditions are satisfied:

(1) $f \underset{x_0}{\sim} g$ and (2.5) holds true;
(2) for some open conical neighborhood Γ of ξ_0, some $N_0 \in \mathbf{Z}_+$ and $C > 0$, (2.3) holds with $P(D)g$ in place of g, for every $N \geqslant N_0$.

The complement of $\operatorname{Reg}(s,P,f)$ is denoted by $WF(s,P,f)$.

Evidently, $WF(s,P,f)$ is a closed set.

REMARK 4.1. The second assumption (2) is needed in the proof of the next theorem and it is an open problem whether this condition can be given in a weaker form.

THEOREM 4.1. *Let $P(D)$ be a differential operator with constant coefficients and $f \in \mathscr{D}'(\mathbf{R}^d)$. Then,*

$$WF_s(P(D)f) \subset WF_s(f) \subset WF(s,P,f) \cup \operatorname{Char}(P). \tag{4.1}$$

REMARK 4.2. Let $A >$ and $P_A(D) = \frac{1}{A}P(D)$. We can simply conclude that (4.1) holds for $P(D)$ if and only if it holds for $P_A(D)$ since $\operatorname{Char}(P_A(D)) = \operatorname{Char} P(D)$ and $P_A(D)Af = h$. This remark will be important in the proof which is to follow when we need to have that $r_0 = \sum_{|\alpha|\leqslant m} |a_\alpha|$ is small enough, more precisely, $r_0 < e/4$. This will be explained in the proof.

PROOF. Assume that (x_0,ξ_0) does not belong to the right-hand side of (4.1) i.e., there exist a neighbourhood U of x_0 and an open conic neighbourhood Γ of ξ_0 in $\mathbf{R}^n \setminus \{0\}$ such that $P_m(\xi) \neq 0$ in Γ, $(U \times \Gamma) \cap WF(s,P,f) = \emptyset$. We assume that $x_0 = 0$. We use the notation $P(D)g = h$ such that g satisfies (2.5) and consequently h satisfies (2.5), with another exponent and with h in the place of g. Moreover, h satisfies (2.3). (See also Remark 2.7).

We will follow the proof in [**10**, Theorem 8.6.1]. However, we make several important modifications which makes this proof different from that of quoted theorem in [**10**].

We consider equation

$$({}^tP(D)\varphi)(x,\xi) = E_N(x)e^{-i\langle x,\xi\rangle}, \quad x,\xi \in \mathbf{R}^d,\ N \in \mathbf{Z}_+.$$

With $\varphi(x,\xi) = w(x)e^{-i\langle x,\xi\rangle}/P_m(\xi)$, $x,\xi \in \mathbf{R}^d$, as in [**10**], one pass to an equation of the form

$$w - Rw = E_N, \quad R = R_1 + \cdots + R_m,$$

where $|\xi|^j R_j$ is a differential operator of order less than or equal to j and homogeneous of degree zero with respect to ξ when $\xi \in \Gamma, j = 1,\dots,m$. Formally, a solution should have a form $w = \sum_{j=0}^{\infty} R^j E_N$.

Let $x,\xi \in \mathbf{R}^d$ and

$$w_N(x,\xi) = \sum_{p=0}^{2N-m-1} \sum_{j_1+\cdots+j_k=p} (R_{j_1}\cdots R_{j_k}E_N)(x,\xi), \quad N \in \mathbf{Z}_+, \tag{4.2}$$

where the composition $R_{j_1}\cdots R_{j_k}$, with $j_1+\cdots+j_k = p$, has the form

$$R_{j_1}\cdots R_{j_k} = |\xi|^{-p} \sum_{|\alpha|\leqslant p} b_\alpha \partial_x^\alpha \tag{4.3}$$

and, by Remark 4.2, coefficients b_α satisfy the estimate $|b_\alpha| \leqslant r_0^p, |\alpha| \leqslant p$.

For the indices $j_1,\dots,j_k$, we introduce the set

$$J_N = \bigcup_{k\geqslant 1} \big\{(j_1,\dots,j_k) \in \mathbf{N}^k;\ j_2+\cdots+j_k < N \leqslant j_1+j_2+\cdots+j_k\big\}.$$

Then,

$$w_N - Rw_N = E_N - \sum_{(j_1,\dots,j_k)\in J_{2N-m}} R_{j_1}\cdots R_{j_k}E_N. \tag{4.4}$$

By (4.4) we have

$${}^tP(D)(e^{-i\langle x,\xi\rangle}w_N(x,\xi)/P_m(\xi)) = e^{-i\langle x,\xi\rangle}(E_N(x) - e_N(x,\xi)),$$

where

$$e_N(x,\xi) = \sum_{(j_1,\dots,j_k)\in J_{2N-m}} (R_{j_1}\cdots R_{j_k}E_N)(x,\xi). \tag{4.5}$$

Next,

$$\mathcal{F}(gE_N)(\xi) = \mathcal{F}(g\,\cdot\,e_N(\,\cdot\,,\xi))(\xi) + \langle he^{-i\langle\,\cdot\,,\xi\rangle}, w_N(\,\cdot\,,\xi)/P_m(\xi)\rangle, \quad \xi \in \mathbf{R}^d.$$

We need to estimate e_N and begin with estimating σ_p, the number of operators $R_{j_1}\cdots R_{j_k}$, $j_1+\cdots+j_k = p$ of the form (4.3). More precisely, we have to find out the number of presentations of p,

$$p = j_1+\cdots+j_k, \quad j_i \in \{1,\dots,m\},\ i = 1,\dots,k$$

with $k \leqslant p$. Here $k = p$ when $j_i = 1, i = 1, \dots, p$. One can find (with symbol $\asymp$ for the asymptotic equality) suitable $c > 0$ such that

$$\sigma_p \leqslant \binom{2p-1}{p} - \binom{2p-2m-3}{p-m-1} \asymp \frac{1}{2}\left(\frac{4^p}{\sqrt{\pi p}} - \frac{4^{p-m}}{\sqrt{\pi(p-m)}}\right) \leqslant c4^p. \tag{4.6}$$

Let us explain this rough estimate. The number of p units can be divided into p boxes in $\binom{2p-1}{p}$ ways but if one of the boxes, at least, has $m+1$ units this possibility should be subtracted. One has $\binom{2p-2m-3}{p-m-1}$ such possibilities.

The summation over the set of indices in (4.5), can be estimated by the number of terms in (4.5) multiplied by the maximal one.

Next we estimate the number s of terms in (4.5). If $p = 2N - m - i$, $i = 1, \dots, m-1$, with application of R_{j_1} on $R_{j_2} \cdots R_{j_k}$, one can reach one of the members of the sum in (4.5). The choice of j_1 depends on i, but the number of such j_1 is less than $m(m-1)/2$. Thus, by (4.6), and with another constant c, we have

$$s \leqslant c4^{2N-m}. \tag{4.7}$$

With the similar argument we estimate S, the number of terms in w_N:

$$S \leqslant c4^{2N-m}, \tag{4.8}$$

with another constant c.

With the notation of Remark 4.2, we have

$$|\xi|^p |R_{j_1} \cdots R_{j_k} E_N(x)| \leqslant c r_0^p \sup_{|\alpha| \leqslant p} |\partial_x^\alpha E_N(x)|, \quad x \in \mathbf{R}^d.$$

Thus, (4.7) and (2.2) imply

$$|\xi|^{2N-m} |e_N(x,\xi)| \leqslant c(4r_0)^{2N-m} \left(\frac{1}{e\sqrt{N}}\right)^{2N-m} (2N-m)^{(2N-m)/2} e^{-|x|^2/(8N)}.$$

We return to Remark 4.2. From the early beginning we should assume that r_0 is so small so that $4r_0/e < 1$. With this, we have

$$|e_N(x,\xi)| \leqslant c|\xi|^{-2N+m} \left(\frac{4r_0}{e}\right)^{2N-m} e^{-|x|^2/(8N)} \leqslant c|\xi|^{-2N+m} e^{-|x|^2/(8N)}.$$

Differentiating $e_N(x,\xi)$ with respect to x and taking the Fourier transform with respect to x, it follows, with $t = d+1$ if d is odd or $t = d+2$ if t is even, that there exists $C > 0$, not depending on N, such that

$$\sup_{\eta \in \mathbf{R}^d} \left|(1+|\eta|^2)^{t/2} (\mathcal{F} e_{N,\xi})(\eta)\right| \leqslant C\langle \xi \rangle^{-2N+m}, \quad \xi \in \mathbf{R}^d, \tag{4.9}$$

where $e_{N,\xi}(x) = e_N(x,\xi)$ is considered as a function in x, parameterized by N and ξ.

In order to give more details we write

$$(1-\Delta)^{t/2} = \sum_{|\beta| \leqslant t} c_\beta \partial_x^\beta,$$

and let $K = \sum_{|\beta| \leqslant t} |c_\beta|$. Then, by (4.5), (4.8), and (4.9)

$$|\xi|^p|(1-\Delta)^{t/2}e_N(x,\xi)| \leqslant \sum_{|\beta|\leqslant t}|c_\beta| \sum_{j_1,j_2,\cdots,j_k\in J_{2N-m}} |(R_{j_1}\cdots R_{j_k}\partial_x^\beta E_N)(x,\xi)|$$
$$\leqslant K \sum_{j_1,j_2,\cdots,j_k\in J_{2N-m}} cr_0^p \sup_{|\alpha|\leqslant p,|\beta|\leqslant t} |\partial_x^{\alpha+\beta}E_N(x)|, \quad x\in \mathbf{R}^d$$
$$\leqslant cK(4r_0)^{2N-m}\Big(\frac{1}{e\sqrt{N}}\Big)^{2N-m+t}(2N-m+t)^{(2N-m+t)/2}e^{-|x|^2/(8N)}.$$

Now, by the determined assumption on r_0, we have

$$\frac{4r_0\sqrt{2N-m-t}}{e\sqrt{N}} \leqslant 1,$$

and obtain (4.9).

By similar arguments, it follows that (4.9) holds true also with the L^1 norm on the left-hand side, provided the constant C has been replaced by a larger one if necessary.

Since g satisfies (2.5), we have

$$|\langle g(\,\cdot\,), e^{-i\langle\,\cdot\,,\xi\rangle}e_{N,\xi}(\,\cdot\,)\rangle| = |(\hat g * \hat e_{N,\xi})(\xi)|, \quad \xi\in\mathbf{R}^d,$$

and by (4.9),

$$|\langle g, e^{-i\langle\,\cdot\,,\xi\rangle}e_{N,\xi}\rangle| \leqslant c\langle\xi\rangle^{-2N+l+m}.$$

Thus, with $N_0=l+m$, we have an even better estimate that we need:

$$|\langle g, e^{-i\langle\,\cdot\,,\xi\rangle}e_{N,\xi}\rangle| \leqslant \frac{C^{n+1}n^{sn}}{\langle\xi\rangle^n}, \quad \xi\in\mathbf{R}^d,\ n\leqslant N,\ N>N_0. \tag{4.10}$$

Similarly as for $e_{N,\xi}$, concerning the estimate of $w_N(\cdot,\xi)=w_{N,\xi}$, by (2.2), (4.2), and (4.8), we may conclude that

$$\Big|\frac{D_x^\alpha w_N(x,\xi)}{P_m(\xi)}\Big| \leqslant c\langle\xi\rangle^{-2N}e^{-\frac{|x|^2}{8N}}, \quad x,\xi\in\mathbf{R}^d,\ |\alpha|\leqslant s$$

and, for $t=d+1$ or $t=d+2$ there exists $C>0$ such that

$$|P_m(\xi)|^{-1}\|\hat w_{N,\xi}\cdot\langle\,\cdot\,\rangle^t\|_{L^1(\mathbf{R}^d)} \leqslant C\langle\xi\rangle^{-2N}, \quad \xi\in\mathbf{R}^d. \tag{4.11}$$

We shall estimate

$$\Big|\frac{\hat w_N(\,\cdot\,,\xi)}{P_m(\xi)} * \mathcal{F}(hE_N(\,\cdot\,))(\xi)\Big|$$

using similar arguments as in the proof of Proposition 2.3. More precisely, let $\Gamma_1\subset\subset\Gamma$. Then (2.11) holds. We have

$$\Big|\frac{\hat w_{N,\xi}}{P_m} * \mathcal{F}(hE_N)(\xi)\Big| \leqslant I_1(\xi)+I_2(\xi),$$

where

$$I_1(\xi) = \int_{|\eta|\leqslant c|\xi|}\Big|\frac{\hat w_N(\eta,\xi)}{P_m(\xi)}\Big||(\mathcal{F}(hE_N))(\xi-\eta)|\,d\eta,$$
$$I_2(\xi) = \int_{|\eta|> c|\xi|}\Big|\frac{\hat w_N(\eta,\xi)}{P_m(\xi)}\Big||(\mathcal{F}(hE_N))(\xi-\eta)|\,d\eta.$$

For I_1 we have

$$I_1(\xi) \leqslant \sup_{|\eta-\xi|<c|\xi|} |(\mathcal{F}(hE_N))(\eta)| \int_{|\xi-\eta|\leqslant c|\xi|} \Big|\frac{\hat{w}_N(\xi-\eta,\xi)}{P_m(\xi)}\Big| d\eta.$$

Let $n \leqslant N$. Estimate (2.3) for $\mathcal{F}(hE_N)(\xi-\eta)$, (4.11) and (2.11) imply

$$\begin{aligned} I_1(\xi)|\xi|^n &\leqslant (1-c)^{-d} \sup_{\eta\in\Gamma} |(\mathcal{F}(hE_N))(\eta)||\eta|^d \int_{|\eta|\geqslant(1-c)|\xi|} \Big|\frac{\hat{w}_N(\xi-\eta,\xi)}{P_m(\xi)}\Big| d\eta \\ &\leqslant C^{n+1} n^{sn}, \quad n \leqslant N,\ N > N_0,\ \xi \in \Gamma_1,\ |\xi| > 1. \end{aligned} \tag{4.12}$$

In order to estimate I_2 we use

$$|(\mathcal{F}(hE_N))(\xi)| \leqslant C\langle\xi\rangle^l, \quad \xi \in \mathbf{R}^d.$$

$$\begin{aligned} &\int_{|\eta|>c|\xi|} \Big|\frac{\hat{w}_N(\eta,\xi)}{P_m(\xi)}\Big| |(\mathcal{F}(hE_N))(\xi-\eta)| d\eta \\ &\qquad \leqslant C \int_{|\eta|>c|\xi|} \langle\xi\rangle^l \langle\eta\rangle^l \Big|\frac{\hat{w}_N(\eta,\xi)}{P_m(\xi)}\Big| |\langle\xi-\eta\rangle^{-l} (\mathcal{F}(hE_N))(\xi-\eta)| d\eta \\ &\qquad\qquad \leqslant C \int_{|\eta|>c|\xi|} ((1+c^{-1})^l \langle\eta\rangle^{2l} \Big|\frac{\hat{w}_N(\eta,\xi)}{P_m(\xi)}\Big| d\eta, \end{aligned}$$

where we have used the fact that $|\eta| > c|\xi|$ implies $|\xi-\eta| \leqslant (1+c^{-1})|\eta|$.

Let $n < N$. Then

$$\langle\xi\rangle^n \int_{|\eta|>c|\xi|} \Big|\frac{\hat{w}_N(\eta,\xi)}{P_m(\xi)}\Big| |(\mathcal{F}(hE_N))(\xi-\eta)| d\eta \leqslant C\langle\xi\rangle^n \int_{\mathbf{R}^d} \langle\eta\rangle^{2l} \Big|\frac{\hat{w}_N(\eta,\xi)}{P_m(\xi)}\Big| d\eta.$$

By use of (4.11) and Remark 2.2, we get

$$\begin{aligned} \langle\xi\rangle^n \int_{|\eta|>c|\xi|} \Big|\frac{\hat{w}_N(\eta,\xi)}{P_m(\xi)}\Big| |(\mathcal{F}(hE_N))(\xi-\eta)| d\eta &\leqslant C^{n+1} n^{sn} \langle\xi\rangle^{n-2N}, \\ &\xi \in \mathbf{R}^d,\ n < N,\ N > N_0. \end{aligned} \tag{4.13}$$

The result now follows from (4.10), (4.12), and (4.13). □

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