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ALGEBRAIC DISTANCES IN VARIOUS ALGEBRAIC CONE METRIC SPACES

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Abstract. We define several variants of algebraic distance in an algebraic cone metric space. Then we obtain some new results about their properties and compare these definitions.

1. Introduction and preliminaries

Ordered normed spaces and cones have many applications in applied mathematics. Thus, fixed point theory in K-metric and K-normed spaces was developed in the mid-20th century (see [**4, 19**]). In 2007, Huang and Zhang [**6**] reintroduced such spaces under the name of cone metric spaces by substituting the set of real numbers by an ordered normed space as a codomain of a metric and obtained some fixed point results (see also [**8, 12, 14, 17**] and references contained therein).

In 1996, Kada et al. [**9**] defined the concept of w-distance in metric spaces. Afterwards, many researchers proved some fixed point theorems under w-distance in complete metric spaces. In the sequel, Cho et al. [**2**] defined the concept of c-distance in a cone metric space, which is a cone version of w-distance. Then, some fixed point results under c-distance in cone metric spaces and TVS-cone metric spaces were obtained in [**3, 5, 11, 15, 18**] (see also the references therein).

On the other hand, recently, Niknam et al. [**13**] defined the concept of algebraic cone metric space and studied some of its elementary properties. Consistent with the content of [**13**], the following definitions and lemma will be needed in the sequel.

Let Y be a real vector space and P be a convex subset of Y. A point $x \in P$ is said to be an algebraic interior point of P if for each $y \in Y$ there exists $\epsilon > 0$ such that $x + ty \in P$, for all $t \in [0, \epsilon]$. This means that a point x is called an algebraic interior point of a convex set $P \subseteq Y$ if $x \in P$ and for each $y \in Y$ there exists $\epsilon > 0$ such that $[x, x+\epsilon y] \subset P$, where $[x, x+\epsilon y] = \{\lambda x+(1-\lambda)(x+\epsilon y) : \text{ for all } \lambda \in [0,1]\}$

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(see [**13**]). The set of all algebraic interior points of P is called the algebraic interior of P and is denoted by $\operatorname{aint} P$. Also, P is called algebraically open if $P = \operatorname{aint} P$.

Definition 1.1. [**13**] Let Y be a vector space with the zero vector θ. A proper nonempty and convex subset P of Y is called an algebraic cone if:

(i) $P + P \subseteq P$;
(ii) $\lambda P \subseteq P$, for each $\lambda > 0$;
(iii) if $x \in P$ and $x \in -P$, then $x = \theta$.

Given an algebraic cone $P \subseteq Y$, we define a partial ordering $\preceq_a$ with respect to P by $x \preceq_a y$ if and only if $y - x \in P$. We shall write $x \prec_a y$ to mean that $x \preceq_a y$ and $x \neq y$. Also, we write $x \ll_a y$ if and only if $y - x \in \operatorname{aint} P$. The algebraic cone P is said to be Archimedean if for each $x, y \in P$ there exists $n \in \mathbb{N}$ such that $x \preceq_a ny$. For example, $P = \{(x, y) \in \mathbb{R}^2 : x, y \geqslant 0\}$ is an algebraic cone with the Archimedean property in the real vector space $\mathbb{R}^2$. In the sequel we assume that (Y, P) has the Archimedean property.

Lemma 1.1. [**13**] *Let Y be a real vector space and P be an algebraic cone in Y with nonempty algebraic interior. Then we have*

(i) $P + \operatorname{aint} P \subseteq \operatorname{aint} P$; (ii) $\lambda \operatorname{aint} P \subseteq \operatorname{aint} P$, *for each* $\lambda > 0$.

Definition 1.2. [**13**] Let X be a nonempty set and (Y, P) be an algebraic cone space with $\operatorname{aint} P \neq \emptyset$. Suppose that a vector valued function $d_a : X \times X \to Y$ satisfies the conditions:

(ACM1) $\theta \preceq_a d_a(x, y)$ for all $x, y \in X$ with $x \neq y$ and $d_a(x, y) = \theta$ if and only if $x = y$;
(ACM2) $d_a(x, y) = d_a(y, x)$ for all $x, y \in X$;
(ACM3) $d_a(x, z) \preceq_a d_a(x, y) + d_a(y, z)$ for all $x, y, z \in X$.

Then d_a is called an algebraic cone metric and (X, d_a) is called an algebraic cone metric space.

Definition 1.3. [**13**] Let (X, d_a) be an algebraic cone metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then the following statements hold:

(i) $\{x_n\}$ converges to x if for every $c \in Y$ with $c \in \operatorname{aint} P$ there exists an $n_0 \in \mathbb{N}$ such that $d_a(x_n, x) \ll_a c$ for all $n > n_0$. We denote this by $d_a\text{-}\lim_{n\to\infty} x_n = x$ or $x_n \xrightarrow{d_a} x$ as $n \to \infty$;
(ii) $\{x_n\}$ is called a Cauchy sequence if for every $c \in Y$ with $c \in \operatorname{aint} P$ there exists an $n_0 \in \mathbb{N}$ such that $d_a(x_n, x_m) \ll_a c$ for all $m, n > n_0$;
(iii) (X, d_a) is a complete algebraic cone metric space if every Cauchy sequence in X is convergent.

In this paper, we define various algebraic distances in various algebraic cone metric spaces, investigate their properties and compare our results. Also, as an application, we prove some fixed point results.

2. Algebraic distances

In this section, we suppose that (X, d_a) is an algebraic cone metric space. Then the following properties are easy to prove and these properties are often useful.

Lemma 2.1. *Let X be a nonempty set, (Y, P) be an algebraic cone space with* $\operatorname{aint} P \neq \emptyset$ *and (X, d_a) be an algebraic cone metric space. Then, for all $u, v, w, c \in Y$, the following assertions hold:*

(p_1) *If $u \preceq_a v$ and $v \ll_a w$, then $u \ll_a w$.*
(p_2) *If $u \ll_a v$ and $v \preceq_a w$, then $u \ll_a w$.*
(p_3) *If $u \preceq_a v$ and $v \preceq_a w$, then $u \preceq_a w$. Also, if $u \ll_a v$ and $v \ll_a w$, then $u \ll_a w$.*
(p_4) *If $\theta \preceq_a u \ll_a c$ for each $c \in \operatorname{aint} P$, then $u = \theta$.*
(p_5) *If $u \preceq_a \lambda u$, where $u \in P$ and $0 < \lambda < 1$, then $u = \theta$.*
(p_6) *Let $\{b_n\}$ be a sequence in Y, algebraic convergent to θ (i.e., $b_n \stackrel{a}{\to} \theta$), $\theta \preceq_a b_n$ and $c \in \operatorname{aint} P$. Then there exists a positive integer n_0 such that $b_n \ll_a c$ for each $n > n_0$.*
(p_7) *If $\theta \preceq_a u \preceq_a v$ and k is a nonnegative real number, then $\theta \preceq_a ku \preceq_a kv$.*
(p_8) *If $\theta \preceq_a u_n \preceq_a v_n$ for all $n \in \mathbb{N}$ and $u_n \stackrel{a}{\to} u$, $v_n \stackrel{a}{\to} v$ as $n \to \infty$, then $\theta \preceq_a u \preceq_a v$.*
(p_9) *$x_n \xrightarrow{d_a} x$ and $x_n \xrightarrow{d_a} y$ (in the algebraic cone metric) imply that $x = y$.*
(p_{10}) *Let $\theta \ll_a c$. If $\theta \preceq_a d_a(x_n, x) \preceq_a b_n$ and $b_n \stackrel{a}{\to} \theta$, then eventually $d_a(x_n, x) \ll_a c$, where x_n, x are a sequence and a given point in X.*
(p_{11}) *If $u \preceq_a v + c$ for every $c \in \operatorname{aint} P$, then $u \preceq_a v$.*
(p_{12}) *The family $\{N_a(x, c) : x \in X, \theta \ll_a c\}$, where $N_a(x, c) = \{y \in X : d_a(y, x) \ll_a c\}$, is a subbasis for topology on X (see [**13**]). We denote this algebraic cone topology by τ_a, and note that τ_a is a Hausdorff topology.*

Remark 2.1. Huang and Zhang [**6**] proved that if P is a normal cone then $x_n \in X$ converges to $x \in X$ if and only if $d(x_n, x) \to \theta$, as $n \to \infty$, and that $x_n \in X$ is a Cauchy sequence if and only if $d(x_n, x_m) \to \theta$, as $n, m \to \infty$. It follows from (p_6) and (p_{10}) that the sequence $\{x_n\}$ converges to $x \in X$ in an algebraic cone metric space if $d_a(x_n, x) \stackrel{a}{\to} \theta$, as $n \to \infty$ and $\{x_n\}$ is a Cauchy sequence if $d_a(x_n, x_m) \stackrel{a}{\to} \theta$, as $n \to \infty$. In our case, we have only one half of the statements of Lemmas 1 and 4 from [**6**]. Also, in this case, the fact that $d_a(x_n, y_n) \stackrel{a}{\to} d_a(x, y)$ if $x_n \xrightarrow{d_a} x$ and $y_n \xrightarrow{d_a} y$ is not applicable (for more details, one can see [**5, 8, 10, 13**]).

Now, we define the first version of an algebraic distance and introduce some of its properties.

Definition 2.1. Let (X, d_a) be an algebraic cone metric space. A function $q_a : X \times X \to Y$ is called a c-algebraic distance (or briefly, algebraic distance) on X if the following properties are satisfied:

(q_1) $\theta \preceq_a q_a(x, y)$ for all $x, y \in X$;
(q_2) $q_a(x, z) \preceq_a q_a(x, y) + q_a(y, z)$ for all $x, y, z \in X$;
(q_3) for $x \in X$, if $q_a(x, y_n) \preceq_a u$ for some $u = u_x$ and all $n \geqslant 1$, then $q_a(x, y) \preceq_a u$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$;
(q_4) for all $c \in Y$ with $\theta \ll_a c$, there exists $e \in Y$ with $\theta \ll_a e$ such that $q_a(z, x) \ll_a e$ and $q_a(z, y) \ll_a e$ imply $d_a(x, y) \ll_a c$.

EXAMPLE 2.1. Let $Y = \mathbb{R}$, $P = \{x \in Y : x \geqslant 0\}$, $\preceq_a$ be an algebraic cone and partial ordering with respect to P be defined by $x \preceq_a y$ if and only if $y - x \in P$. Let $X = [0, \infty)$ and define a mapping $d : X \times X \to P$ by $d_a(x, y) = |x - y|_a$ for all $x, y \in X$. Then (X, d_a) is an algebraic cone metric space. Define a mapping $q_a : X \times X \to Y$ by $q_a(x, y) = y$ for all $x, y \in X$. Then, q is an algebraic distance. In fact, (q_1)–(q_3) are immediate. From $d_a(x, y) = |x-y|_a \leqslant_a x+y = q_a(z, x)+q_a(z, y)$, it follows that (q_4) holds. Hence q_a is an algebraic distance.

EXAMPLE 2.2. Let (Y, P) be an algebraic cone space with $\operatorname{aint} P \neq \emptyset$ and (X, d_a) be an algebraic cone metric space such that the metric $d_a(\cdot, \cdot)$ is a continuous function in second variable. Then, $q_a(x, y) = d_a(x, y)$ is an algebraic distance. In fact, (q_1) and (q_2) are immediate. But, property (q_3) is nontrivial and it follows from $q_a(x, y_n) = d_a(x, y_n) \preceq u$, passing to the limit when $n \to \infty$ and using continuity of d_a. Let $c \in Y$ with $c \in \operatorname{aint} P$ be given and put $e = \frac{c}{2}$. Suppose that $q_a(z, x) \ll_a e$ and $q_a(z, y) \ll_a e$. Then $d_a(x, y) = q_a(x, y) \preceq q_a(x, z) + q_a(z, y) \ll_a e + e = c$. Using (p_1), this shows that $d_a(x, y) \ll_a c$ and thus q_a satisfies (q_4). Hence, q_a is an algebraic distance.

In Examples 2.1 and 2.2, we have introduced two known algebraic distances in an algebraic cone metric space. There exist other examples of distances in [**2, 15**] which the reader can consider in algebraic version. Also, similar to Example 3 of Djordjević [**5**], one can consider algebraic distances which are not c-distances in cone metric spaces of [**2, 15**].

REMARK 2.2. Based on Examples 2.1 and 2.2, we have two important notes:

(i) For an algebraic distance q_a, $q_a(x, y) = \theta$ is not necessarily equivalent to $x = y$ for all $x, y \in X$.
(ii) For an algebraic distance q_a, $q_a(x, y) = q_a(y, x)$ does not necessarily hold for all $x, y \in X$.

We recall that a sequence $\{u_n\}$ in an algebraic cone P is a c-sequence if for every $c \in \operatorname{aint} P$ there exists $n_0 \in \mathbb{N}$ such that $u_n \ll_a c$ for $n \geqslant n_0$. It is easy to prove that if $\{u_n\}$ and $\{v_n\}$ are c-sequences in Y and $\alpha, \beta > 0$, then $\{\alpha u_n + \beta v_n\}$ is a c-sequence. Note that in the case that the cone P is normal, a sequence in Y is a c-sequence if and only if it is a θ-sequence (see property (p_6)). However, a c-sequence need not be a θ-sequence in an arbitrary algebraic cone metric space.

LEMMA 2.2. *Let (X, d_a) be an algebraic cone metric space and q_a be an algebraic distance on X. Also, let $\{x_n\}$ and $\{y_n\}$ be sequences in X, $x, y, z \in X$, and $\{u_n\}$ and $\{v_n\}$ be two c-sequences in the respective algebraic cone P. Then the following properties hold:*

(qp_1) *If $q_a(x_n, y) \preceq_a u_n$ and $q_a(x_n, z) \preceq_a v_n$ for $n \in \mathbb{N}$, then $y = z$. In particular, if $q_a(x, y) = \theta$ and $q_a(x, z) = \theta$, then $y = z$.*
(qp_2) *If $q_a(x_n, y_n) \preceq_a u_n$ and $q_a(x_n, z) \preceq_a v_n$ for $n \in \mathbb{N}$, then $\{y_n\}$ converges to z.*
(qp_3) *If $q_a(x_n, x_m) \preceq_a u_n$ for $m > n$, then $\{x_n\}$ is a Cauchy sequence in X.*
(qp_4) *If $q_a(y, x_n) \preceq_a u_n$ for $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence in X.*

PROOF. (qp_1) It is enough to prove that $d_a(y,z) \ll_a c$ for each $c \in \operatorname{aint} P$ in order to prove that $y = z$. For the given c, choose $e \in \operatorname{aint} P$ such that property (q_4) is satisfied. Since $\{u_n\}$ and $\{v_n\}$ are c-sequences, there exists $n_0 \in \mathbb{N}$ such that $u_n \ll_a e$ and $v_n \ll_a e$ for each $n \geqslant n_0$. By (p_1), since $u_n \ll_a e$ and $q_a(x_n, y) \preceq_a u_n$, we have $q_a(x_n, y) \ll_a e$. Similarly, we have $q_a(x_n, z) \ll_a e$. Now, using (q_4), we get $d_a(y, z) \ll_a c$.

(qp_2) Let again $c \in \operatorname{aint} P$ be arbitrary and choose a corresponding $e \in \operatorname{aint} P$ satisfying property (q_4). If $n_0 \in \mathbb{N}$ such that $u_n \ll_a e$ and $v_n \ll_a e$ for each $n \geqslant n_0$, then (p_1) implies that $q_a(x_n, y_n) \ll_a e$ and $q(x_n, z) \ll_a e$ for $n \geqslant n_0$. Hence, by (q_4), we have $d(y_n, z) \ll c$ for all $n \geqslant n_0$; that is, $y_n \xrightarrow{d_a} z$ as $n \to \infty$.

(qp_3) Let $c \in Y$ with $c \in \operatorname{aint} P$. As in the proof of ($qp_1$), choose $e \in Y$ with $e \in \operatorname{aint} P$. Then there exists a positive integer $n_0 \in \mathbb{N}$ such that $q_a(x_n, x_{n+1}) \ll_a e$ and $q_a(x_n, x_m) \ll_a e$ for any $m > n \geqslant n_0$ and hence $d_a(x_{n+1}, x_m) \ll_a c$ (by (q_4)). This implies that $\{x_n\}$ is a Cauchy sequence in X.

As in the proof of (qp_3), we can easily prove (qp_4). □

Our main result in this section is the following theorem of Chatterjea type (see [**1**]) under an algebraic distance in an algebraic cone metric space.

THEOREM 2.1. *Let (X, d_a) be a complete algebraic cone metric space, q_a be an algebraic distance on X and $T\colon X \to X$ be a continuous mapping. Suppose that there exist nonnegative constants α, β, γ such that*

$$q_a(Tx, Ty) \preceq_a \alpha q_a(x, y) + \beta q_a(x, Ty) + \gamma q_a(y, Tx), \tag{2.1}$$

$$q_a(Ty, Tx) \preceq_a \alpha q_a(y, x) + \beta q_a(Ty, x) + \gamma q_a(Tx, y) \tag{2.2}$$

for all $x, y \in X$, where $\alpha + 2\beta + 2\gamma < 1$. Then T has a fixed point in X. If $Tu = u$, then $q_a(u, u) = \theta$.

PROOF. Let x_0 be an arbitrary point of X. If $Tx_0 = x_0$, then x_0 is a fixed point of T and the proof is finished. Suppose that $Tx_0 \neq x_0$. Then we construct a sequence $\{x_n\}$ in X such that $x_n = T^n x_0 = Tx_{n-1}$. In order to prove that it is a Cauchy sequence, put $x = x_n$ and $y = x_{n-1}$ in (2.1). We have

$$\begin{aligned} q_a(x_{n+1}, x_n) &= q_a(Tx_n, Tx_{n-1}) \\ &\preceq_a \alpha q_a(x_n, x_{n-1}) + \beta q_a(x_n Tx_{n-1}) + \gamma q_a(x_{n-1}, Tx_n) \\ &= \alpha q_a(x_n, x_{n-1}) + \beta q_a(x_n, x_n) + \gamma q_a(x_{n-1}, x_{n+1}) \\ &\preceq_a \alpha q_a(x_n, x_{n-1}) + \beta[q_a(x_n, x_{n+1}) + q_a(x_{n+1}, x_n)] \\ &\quad + \gamma[q_a(x_{n-1}, x_n) + q_a(x_n, x_{n+1})] \\ &= \alpha q_a(x_n, x_{n-1}) + \beta q_a(x_{n+1}, x_n) + (\beta+\gamma) q_a(x_n, x_{n+1}) \\ &\quad + \gamma q_a(x_{n-1}, x_n). \end{aligned} \tag{2.3}$$

Similarly, putting $x = x_n$ and $y = x_{n-1}$ in (2.2), we have

$$\begin{aligned} q_a(x_n, x_{n+1}) \preceq_a{}& \alpha q_a(x_{n-1}, x_n) + \beta q_a(x_n, x_{n+1}) \\ &+ (\beta+\gamma) q_a(x_{n+1}, x_n) + \gamma q_a(x_n, x_{n-1}). \end{aligned} \tag{2.4}$$

Adding up (2.3) and (2.4), we have

$$q_a(x_{n+1}, x_n) + q_a(x_n, x_{n+1}) \preceq_a (\alpha+\gamma)[q_a(x_n, x_{n-1}) + q_a(x_{n-1}, x_n)] \\ + (2\beta+\gamma)[q_a(x_{n+1}, x_n) + q_a(x_n, x_{n+1})].$$

Set $u_n = q_a(x_{n+1}, x_n) + q_a(x_n, x_{n+1})$. We get $u_n \preceq_a (\alpha+\gamma)u_{n-1} + (2\beta+\gamma)u_n$. Thus, we have $u_n \preceq_a \lambda u_{n-1}$, where $\lambda = \frac{(\alpha+\gamma)}{1-(2\beta+\gamma)} < 1$ (by $\alpha + 2\beta + 2\gamma < 1$). By repeating the procedure, we get $u_n \preceq_a \lambda^n u_0$ (by (p_3)) for all $n \in \mathbb{N}$. Thus,

$$q_a(x_n, x_{n+1}) \preceq_a v_n \preceq_a \lambda^n[q_a(x_1, x_0) + q_a(x_0, x_1)]. \tag{2.5}$$

Let $m > n$. In the usual way, it follows from (2.5) and $\lambda \in [0, 1)$ that

$$q_a(x_n, x_m) \preceq_a q_a(x_n, x_{n+1}) + \cdots + q_a(x_{m-1}, x_m) \\ \preceq_a \frac{\lambda^n}{1-\lambda}[q_a(x_1, x_0) + q_a(x_0, x_1)] = v_n,$$

where $\{v_n\}$ is a c-sequence. Lemma 2.2 (qp_3) implies that $\{x_n\}$ is a Cauchy sequence in the algebraic cone metric space X and, since X is complete, $x_n \xrightarrow{d_a} x^* \in X$ as $n \to \infty$. Continuity of T implies that $x_{n+1} = Tx_n \xrightarrow{d_a} Tx^*$, and since the limit of a sequence in an algebraic cone metric space is unique, we obtain $Tx^* = x^*$; that is, x^* is a fixed point of T.

Now, we suppose that $Tu = u$. It follows from (2.1) that

$$q_a(u, u) = q_a(Tu, Tu) \preceq_a \alpha q_a(u, u) + \beta q_a(u, Tu) + \gamma q_a(u, Tu) = (\alpha+\beta+\gamma)q_a(u, u).$$

Since $\alpha + \beta + \gamma < \alpha + 2\beta + 2\gamma < 1$, so by Lemma ($p_5$), we have $q_a(u, u) = \theta$. □

Now, we define another version of algebraic distance according to Ćirić et al. [**3**]. Let (X, d_a) be an algebraic cone metric space. Then $G : X \to P$ is lower semicontinuous at $x \in X$ if for each $c \in Y$ with $c \in \operatorname{aint} P$, there is $n_0 \in \mathbb{N}$ such that $G(x) \preceq_a G(x_n) + c$ for all $n \geqslant n_0$, whenever $\{x_n\}$ is a sequence in X and $x_n \xrightarrow{d_a} x$.

Definition 2.2. Let (X, d_a) be an algebraic cone metric space. A function $p_a \colon X \times X \to P$ is called an algebraic w-cone distance on X if it satisfies:

(w_1) $p_a(x, z) \preceq_a p_a(x, y) + p_a(y, z)$ for all $x, y, z \in X$;
(w_2) for any $x \in X$, $p_a(x, \cdot) \colon X \to P$ is lower semicontinuous;
(w_3) for all $c \in Y$ with $\theta \ll_a c$, there exists $e \in Y$ with $\theta \ll_a e$ such that $p_a(z, x) \ll_a e$ and $p_a(z, y) \ll_a e$ imply $d_a(x, y) \ll_a c$.

Example 2.3. Let (X, d_a) be an algebraic cone metric space. Clearly, d_a satisfies (w_1) and (w_3). So we have only to prove that d_a satisfies (w_2). Suppose that $\{y_n\}$ is a sequence in X such that $y_n \xrightarrow{d_a} y$ and $c \in Y$ with $\theta \ll_a c$ is arbitrary. Since $y_n \xrightarrow{d_a} y$, then there exists $N_0 \in \mathbb{N}$ such that $d_a(y_n, y) \ll_a c$ for all $n \geqslant N_0$. Define $G \colon X \to P$ by $G(y) = d(x, y)$ where $x \in X$. Then for all $n \geqslant N_0$ we have

$$G(y) = d_a(x, y) \preceq_a d_a(x, y_n) + d_a(y_n, y) \preceq_a d_a(x, y_n) + c = G(y_n) + c.$$

Therefore $p_a(x, \cdot) = d_a(x, \cdot)$ is lower semicontinuous. Hence, d_a is an algebraic w-cone distance on X.

Concerning the work using algebraic w-cone distances, note that the following properties hold:

(1) Using the notion of c-sequence, condition (w_2) of the previous definition can be formulated in the following statement: If $y_n, y \in X$, $y_n \xrightarrow{d_a} y$ as $n \to \infty$ and $g(y) = p_a(x, y)$, then $g(y) - g(y_n)$ is a c-sequence.

(2) Definition 2.1 of an algebraic distance is different from Definition 2.2 of an algebraic w-cone distance in the way that the (q_3) is used instead of (w_2). Also, it is clear that each algebraic w-cone distance is a c-algebraic distance, but the converse does not hold.

(3) There exists an extension of Lemma 2.2 for an algebraic distance (or w-distance of Kada et al. [**9**]) to an algebraic w-cone distance. Also, similarly as in the case of an algebraic distance, Lemma 2.2 and Theorem 2.1 can be proved for an algebraic w-cone distance.

3. Algebraic *b*-distance

In this section, we define an algebraic b-distance on algebraic cone metric spaces as another version of algebraic distance in an algebraic cone metric space. Very recently, Rahimi et al. [**16**] defined an algebraic cone b-metric space.

Definition 3.1. [**16**] Let X be a nonempty set, (Y, P) be an algebraic cone space with $\operatorname{aint} P \neq \emptyset$ and $s \geqslant 1$ be a given real number. Suppose that a vector-valued function $D_a \colon X \times X \to Y$ satisfies the following conditions:

(ACbM1) $\theta \preceq_a D_a(x, y)$ for all $x, y \in X$ and $D_a(x, y) = \theta$ if and only if $x = y$;

(ACbM2) $D_a(x, y) = D_a(y, x)$ for all $x, y \in X$;

(ACbM3) $D_a(x, z) \preceq_a s[D_a(x, y) + D_a(y, z)]$ for all $x, y, z \in X$.

Then D_a is called an algebraic cone b-metric with parameter $s \geqslant 1$ and (X, D_a) is called an algebraic cone b-metric space.

Definition 3.2. [**16**] Let (X, D_a) be an algebraic cone b-metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then

(i) $\{x_n\}$ converges to x if, for every $c \in Y$ with $c \in \operatorname{aint} P$ there exists an $n_0 \in \mathbb{N}$ such that $D_a(x_n, x) \ll_a c$ for all $n > n_0$. We denote this by $D_a\text{-}\lim_{n\to\infty} x_n = x$ or $x_n \xrightarrow{D_a} x$ as $n \to \infty$;

(ii) $\{x_n\}$ is called a Cauchy sequence if, for every $c \in Y$ with $c \in \operatorname{aint} P$ there exists an $n_0 \in \mathbb{N}$ such that $D_a(x_n, x_m) \ll_a c$ for all $m, n > n_0$;

(iii) (X, D_a) is a complete algebraic cone b-metric space if every Cauchy sequence in X is convergent.

Now, we define another version of an algebraic distance in algebraic cone b-metric space.

Definition 3.3. Let (X, D_a) be an algebraic cone b-metric space with parameter $s \geqslant 1$. A function $Q_a \colon X \times X \to Y$ is called a c-b-algebraic distance (or briefly, algebraic b-distance) on X if it satisfies the following properties:

(Q_1) $\theta \preceq_a Q_a(x, y)$ for all $x, y \in X$;

(Q$_2$) $Q_a(x,z) \preceq_a s[Q_a(x,y)+Q_a(y,z)]$ for all $x,y,z\in X$;

(Q$_3$) for $x \in X$, if $Q_a(x,y_n) \preceq_a u$ for some $u=u_x$ and all $n \geqslant 1$, then $Q_a(x,y) \preceq_a su$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$;

(Q$_4$) for all $c \in Y$ with $\theta \ll_a c$, there exists $e \in Y$ with $\theta \ll_a e$ such that $Q_a(z,x) \ll_a e$ and $Q_a(z,y) \ll_a e$ imply $D_a(x,y) \ll_a c$.

Example 3.1. Let $Y=\mathbb{R}$, $X=[0,1]$ and $P=\{x\in Y : x \geqslant 0\}$. Define a mapping $D_a\colon X\times X \to Y$ by $D_a(x,y)=|x-y|^2$ for all $x,y\in X$. Then (X,D_a) is an algebraic cone b-metric space with parameter $s=2$. Define a mapping $Q_a\colon X\times X\to Y$ by $Q_a(x,y)=y^2$ for all $x,y\in X$. Then Q_a is an algebraic b-distance on X.

Lemma 3.1. *Let (X,D_a) be an algebraic cone b-metric space with parameter $s \geqslant 1$ and Q_a be an algebraic b-distance on X. Also, let $\{x_n\}$ and $\{y_n\}$ be sequences in X and $x,y,z\in X$, and $\{u_n\}$ and $\{v_n\}$ be two c-sequences in algebraic cone P. Then the following properties hold:*

(Qp$_1$) *If $Q_a(x_n,y) \preceq_a u_n$ and $Q_a(x_n,z) \preceq_a v_n$ for $n\in\mathbb{N}$, then $y=z$. Particularly, if $Q_a(x,y)=\theta$ and $q_a(x,z)=\theta$, then $y=z$.*

(Qp$_2$) *If $Q_a(x_n,y_n) \preceq_a u_n$ and $Q_a(x_n,z) \preceq_a v_n$ for $n\in\mathbb{N}$, then $\{y_n\}$ converges to z.*

(Qp$_3$) *If $Q_a(x_n,x_m) \preceq_a u_n$ for $m>n$, then $\{x_n\}$ is a Cauchy sequence in X.*

(Qp$_4$) *If $Q_a(y,x_n) \preceq_a u_n$ for $n\in\mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence in X.*

Proof. The proof is similar to Lemma 2.2. □

Our main result in this section is the following fixed point theorem under an algebraic b-distance in an algebraic cone b-metric space.

Theorem 3.1. *Let (X,D_a) be a complete algebraic cone b-metric space with parameter $s\geqslant 1$, Q_a be an algebraic b-distance on X and $T\colon X\to X$ be a continuous mapping. Suppose that there exist nonnegative real numbers α,β,γ such that the following conditions hold:*

(t$_1$) *$s(\alpha+\beta)+\gamma<1$;*

(t$_2$) *for all $x,y\in X$,*

$$Q_a(Tx,Ty) \preceq_a \alpha Q_a(x,y)+\beta Q_a(x,Tx)+\gamma Q_a(y,Ty). \tag{3.1}$$

Then T has a fixed point in X. If $Tu=u$, then $Q_a(u,u)=\theta$.

Proof. Let x_0 be an arbitrary point in X. If $Tx_0=x_0$, then x_0 is a fixed point of T and the proof is finished. Suppose that $Tx_0\neq x_0$. Then we construct a sequence $\{x_n\}$ in X such that $x_n=T^nx_0=Tx_{n-1}$. In order to prove that it is a Cauchy sequence, put $x=x_{n-1}$ and $y=x_n$ in (3.1) and use (t$_1$). We have

$$\begin{aligned} Q_a(x_n,x_{n+1}) &= Q_a(Tx_{n-1},Tx_n)\\ &\preceq_a \alpha Q_a(x_{n-1},x_n)+\beta Q_a(x_{n-1},Tx_{n-1})+\gamma Q_a(x_n,Tx_n)\\ &= (\alpha+\beta)Q_a(x_{n-1},x_n)+\gamma Q_a(x_n,x_{n+1}), \end{aligned}$$

which implies that $Q_a(x_n, x_{n+1}) \preceq_a \frac{\alpha+\beta}{1-\gamma} Q_a(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$. Repeating this process, we have

$$Q_a(x_n, x_{n+1}) \preceq_a \delta^n Q_a(x_0, x_1) \tag{3.2}$$

for all $n \in \mathbb{N}$, where $0 \leqslant \delta = \frac{\alpha+\beta}{1-\gamma} < \frac{1}{s}$ by (t_1). Let $m > n$. In the usual way, it follows from (3.2) that

$$\begin{aligned} Q_a(x_n, x_m) &\preceq_a sQ_a(x_n, x_{n+1}) + s^2 Q_a(x_{n+1}, x_{n+2}) + \cdots + s^{m-n} Q_a(x_{m-1}, x_m) \\ &\preceq_a (s\delta^n + \cdots + s^{m-n}\delta^{m-1}) Q_a(x_0, x_1) \\ &= s\delta^n (1 + s\delta + \cdots + s^{m-n-1}\delta^{m-n-1}) Q_a(x_0, x_1) \\ &\preceq_a \frac{s\delta^n}{1 - s\delta} Q_a(x_0, x_1) = v_n, \end{aligned}$$

where $\{v_n\}$ is a c-sequence. Lemma 3.1(Qp_3) implies that $\{x_n\}$ is a Cauchy sequence in the algebraic cone b-metric space X and, since X is complete, $x_n \xrightarrow{D_a} x^* \in X$ as $n \to \infty$. Continuity of T implies that $x_{n+1} = Tx_n \xrightarrow{D_a} Tx^*$, and since the limit of a sequence in an algebraic cone b-metric space is unique, we obtain $Tx^* = x^*$; that is, x^* is a fixed point of T.

Now, we suppose that $Tu = u$. It follows from (3.1) that

$$\begin{aligned} Q_a(u, u) = Q_a(Tu, Tu) &\preceq_a \alpha Q_a(u, u) + \beta Q_a(u, Tu) + \gamma Q_a(u, Tu) \\ &= (\alpha + \beta + \gamma) Q_a(u, u), \end{aligned}$$

which is, by property (p_5) and (t_1), possible only if $Q_a(u, u) = \theta$. □

EXAMPLE 3.2. Consider Y, P, X, D_a, s and Q_a as in Example 3.1. Also, let a mapping $T\colon X \to X$ be defined by $Tx = \frac{x^2}{4}$ for all $x \in X$. Take $\alpha = \frac{1}{16}$, $\beta = \frac{1}{7}$ and $\gamma = 0$. Now, we have

(i) $s(\alpha + \beta) + \gamma = 2(\frac{1}{16} + \frac{1}{7}) = \frac{23}{56} < 1$;

(ii) for all comparable $x, y \in X$,

$$Q_a(Tx, Ty) = (Ty)^2 = \frac{y^4}{16} \preceq_a \alpha Q_a(x, y) + \beta Q_a(x, Tx) + \gamma Q_a(y, Ty).$$

Moreover, T is continuous on X. Therefore, all the conditions of Theorem 3.1 are satisfied and hence T has a fixed point (which is $x = 0$ with $Q_a(0, 0) = 0$).

The following corollary is an algebraic b-distance version of Theorems 3 and 6 of Rahimi et al. [**16**].

COROLLARY 3.1. *Let (X, D_a) be a complete algebraic cone b-metric space with parameter $s \geqslant 1$ and Q_a be an algebraic b-distance on X. If a continuous mapping $T\colon X \to X$ satisfies the contractive condition $Q_a(Tx, Ty) \preceq_a \alpha Q_a(x, y)$ for all $x, y \in X$, where $\alpha \in [0, \frac{1}{s})$, then T has a fixed point in X. If $Tu = u$, then $Q_a(u, u) = \theta$.*

Now, we define another version of an algebraic b-distance according to Ćirić et al.'s work [**3**]. Let us recall that a real-valued function $F\colon X \to P$ defined on an algebraic cone b-metric space X is said to be lower b-semicontinuous at a point $x \in X$ if for each $c \in Y$ with $c \in \operatorname{aint} P$, there is $n_0 \in \mathbb{N}$ such that $F(x) \preceq_a sF(x_n)+c$ for all $n \geqslant n_0$, whenever $\{x_n\}$ is a sequence in X and $x_n \xrightarrow{D_a} x$.

Definition 3.4. Let (X, D_a) be an algebraic cone b-metric space. A function $P_a\colon X \times X \to P$ is called an algebraic W-b-cone distance on X if the following conditions are satisfied:

- (W_1) $P_a(x,z) \preceq_a s[P_a(x,y) + P_a(y,z)]$ for all $x, y, z \in X$;
- (W_2) for any $x \in X$, $P_a(x,\cdot)\colon X \to P$ is lower b-semicontinuous;
- (W_3) for all $c \in Y$ with $\theta \ll_a c$, there exists $e \in Y$ with $\theta \ll_a e$ such that $P_a(z,x) \ll_a e$ and $P_a(z,y) \ll_a e$ imply $D_a(x,y) \ll_a c$.

Example 3.3. Let (X, D_a) be an algebraic cone b-metric space. Then D_a is an algebraic W-b-cone distance on X.

Remark 3.1. Let $s = 1$. Then an algebraic b-distance of Definition 3.3 is an algebraic distance of Definition 2.1 and an algebraic W-b-cone distance of Definition 3.4 is an algebraic w-cone distance of Definition 2.2.

Remark 3.2. Concerning the work using algebraic W-b-cone distances, note that the following properties hold:

(1) Definition 3.3 of an algebraic b-distance is different from Definition 3.4 of an algebraic W-b-cone distance in the way that the (Q_3) is used instead of (W_2). Also, it is clear that each algebraic W-b-cone distance is a c-b-algebraic distance, but the converse does not hold.
(2) There exists an extension of Lemma 3.1 for an algebraic distance (or wt-distance of Hussain et al. [**7**]) to an algebraic W-b-cone distance. Also, similarly as for algebraic distances, Lemma 3.1 and Theorem 3.1 can be proved for an algebraic W-b-cone distance.

In Theorem 3.1, take $s = 1$. Then we obtain the following corollary for algebraic distances.

Theorem 3.2. *Let (X, d_a) be a complete algebraic cone metric space, q_a be an algebraic distance on X and $T\colon X \to X$ be a continuous mapping. Suppose that there exist nonnegative constants α, β, γ such that the following conditions hold:*

- (t_1) $\alpha + \beta + \gamma < 1$;
- (t_2) *for all $x, y \in X$,*

$$q_a(Tx,Ty) \preceq_a \alpha q_a(x,y) + \beta q_a(x,Tx) + \gamma q_a(y,Ty).$$

Then T has a fixed point in X. If $Tu = u$, then $q_a(u,u) = \theta$.

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