PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 104(118) (2018), 69–88 DOI: https://doi.org/10.2298/PIM1818069O

APPROXIMATIONS IN BIVARIATE RENEWAL THEORY

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Abstract. We construct approximations to the renewal function for a bivariate renewal process. Suppose (X, Y), (X_1, Y_1), $(X_2, Y_2), \dots$ denote i.i.d. positive random vectors with common distribution function $F(x, y) = \mathrm{P}(X \leqslant x, Y \leqslant y)$. Let $S_n^{(1)} = X_1 + X_2 + \cdots + X_n$ and $S_n^{(2)} = Y_1 + Y_2 + \cdots + Y_n$ denote the partial sums where we set $S_0^1 = S_0^2 = 0$. Associated with $\{(X_i, Y_i)\}$, we define, respectively, the univariate and bivariate renewal counting processes: $N_i(x) = \min\{n \geqslant 1 : S_n^{(i)} > x\}$ $(i = 1, 2)$ and $N(x, y) = \min\{N_1(x), N_2(y)\}$. The bivariate renewal function is given by $U(x, y) = \mathrm{E}\, N(x, y) = \sum_0^\infty F^{*n}(x, y)$. From the practical point of view it is hard to find explicit expressions for the renewal function $U(x, y)$. Recently, Mitov and Omey (2014) introduced a new method to obtain approximations for univariate renewal functions based on expansions of Laplace–Stieltjes transforms. In this paper we generalize these approximations to the bivariate case and apply them to regularly varying increments. We show that the approximation along the diagonal is different from off the diagonal.

1. Introduction

In this paper we consider bivariate renewal processes and approximations to the bivariate renewal function. Let (X, Y), (X_1, Y_1), $(X_2, Y_2), \dots$ denote i.i.d. positive random vectors with common distribution function (d.f.) $F(x, y) = \mathrm{P}(X \leqslant x, Y \leqslant y)$ and marginal d.fs $F_1(x) = \mathrm{P}(X \leqslant x)$ and $F_2(x) = \mathrm{P}(Y \leqslant x)$. We assume that $F_i(x) < 1$ for all x and that $F_i(0+) = 0$. We also assume that X and Y are non-lattice with finite means $EX = \mu < \infty$ and $EY = \nu < \infty$, respectively.

Let $S_n^{(1)} = X_1 + X_2 + \ldots + X_n$ and $S_n^{(2)} = Y_1 + Y_2 + \ldots + Y_n$ denote the partial sums and as usual we set $S_0^1 = S_0^2 = 0$. Associated with $\{(X_i, Y_i)\}$ we define the following univariate and bivariate renewal processes:

$$N_i(x) = \min\{n \geqslant 1 : S_n^{(i)} > x\}, \quad i = 1, 2$$
$$N(x, y) = \min\{N_1(x), N_2(y)\}.$$

2010 *Mathematics Subject Classification:* 60K05, 60G50, 62E20, 26A12.

Key words and phrases: bivariate renewal process, renewal function, approximation, regular variation, equilibrium distribution, asymptotic distribution.

Communicated by Slobodanka Janković.

Note that

$$N(x,y) > n \Leftrightarrow N_1(x) > n \text{ and } N_2(y) > n$$
$$\Leftrightarrow S_n^{(1)} \leqslant x \text{ and } S_n^{(2)} \leqslant y.$$

It follows that $\mathrm{P}(N(x,y) > n) = F^{*n}(x,y)$, where F^{*n} denotes the n-fold convolution of F with itself and $F^{*0} = 1$.

The bivariate renewal function is given by $U(x,y)$, where

$$U(x,y) = \mathrm{E}\, N(x,y) = \sum_0^\infty F^{*n}(x,y).$$

For the marginals, we have $U_i(x) = \mathrm{E}\, N_i(x) = \sum_0^\infty F_i^{*n}(x)$, $i = 1,2$. Note that $U_1(x) = U(x,\infty)$ and $U_2(x) = U(\infty,x)$. Also, we have $U(x,y) \leqslant U_1(x)$ and $U(x,y) \leqslant U_2(y)$ so that $U(x,y) \leqslant \min(U_1(x), U_2(y))$.

From the practical point of view it is hard to find explicit expressions for the renewal function $U(x,y)$. In the univariate case there are many papers in which different approximations for $U_1(x)$ are obtained. We refer to [**2**,**8**,**18**,**21**] for results of this type. In bivariate renewal theory, such results are almost absent in the literature. We have Proposition 1.1 below that has been proved by [**10**] and proved by [**7**] and also by [**1**].

PROPOSITION 1.1. *$t^{-1}U(tx,ty) \to M(x,y)$, where $M(x,y) \equiv \min(x/\mu, y/\nu)$.*

PROOF. From univariate renewal theory, we have

$$\frac{1}{t}U_1(tx) \to \frac{x}{\mu} \quad \text{and} \quad \frac{1}{t}U_2(ty) \to \frac{y}{\nu}.$$

Since $t^{-1}U(tx,ty) \leqslant \min(t^{-1}U_1(tx), t^{-1}U_2(ty))$, it follows that

$$\limsup \frac{1}{t}U(tx,ty) \leqslant \min\Big(\frac{x}{\mu}, \frac{y}{\nu}\Big).$$

On the other hand, we have

$$\frac{N_1(x)}{x} \overset{a.s.}{\to} \frac{1}{\mu} \quad \text{and} \quad \frac{N_2(x)}{x} \overset{a.s.}{\to} \frac{1}{\nu}.$$

Using Fatou's lemma, we obtain that

$$\liminf \frac{1}{t}U(tx,ty) \geqslant E \liminf \frac{1}{t}\min(N_1(tx), N_2(ty) = \min\Big(\frac{x}{\mu}, \frac{y}{\nu}\Big). \qquad \square$$

Paper [**3**] was the first to study and extend the classical univariate renewal theorem to higher dimensions. Papers [**9**, **10**] studied a renewal process in two dimensions and developed further results. The author discussed the bivariate generating functions and bivariate Laplace transforms as the basic tools to generalize the classical theory of univariate renewal processes. He presented an example of a bivariate exponential distribution to illustrate the general theory and developed explicit expressions of the two dimensional renewal function, as well as the correlation between the marginal univariate renewal counting processes. The renewal theorem for the multi-dimensional case can be found in [**14**,**19**,**20**]. Papers [**22**,**23**] used bivariate renewal processes in maintenance policies. The infinite means case has been considered by [**12**] and more recently by [**11**].

More recently [**13**] introduced a new method to obtain approximations for univariate renewal functions and these approximations cover the known results. In this paper we generalize these approximations to bivariate renewal processes. Although we use a similar approach key differences appear, especially along the diagonal.

2. Approximations to the bivariate renewal function

The equilibrium distribution associated with X is given by

$$F_{1,e}(x) = \frac{1}{\mu}\int_0^x \bar{F}_1(z)dz,$$

where $\bar{F}_1(x) = 1 - F_1(x)$ is the tail of F_1. In a similar way we define $F_{2,e}(x)$. The generating function of (X, Y) or the Laplace–Stieltjes transform (LST) of F is given by

$$\hat{F}(s,t) = \mathrm{E}(e^{-sX-tY}) = \int_0^\infty\int_0^\infty e^{-sx-ty}dF(x,y).$$

For the marginals, we write $\hat{F}_i(s)$. The LST of $F_{1,e}(x)$ is given by $\hat{F}_{1,e}(s) = (1-\hat{F}_1(s))/\mu s$.

The LSTs of U and of U_i are given respectively by

$$\hat{U}(s,t) = \frac{1}{1-\hat{F}(s,t)}, \qquad \hat{U}_i(s) = \frac{1}{1-\hat{F}_i(s)}.$$

For $\hat{U}(s,t)$, note that as $x \to \infty$, we have

$$\frac{1}{x}\hat{U}\Big(\frac{s}{x},\frac{t}{x}\Big) = \frac{1}{x(1-\hat{F}(s/x,t/x))} \quad \to \quad \frac{1}{\mu s + \nu t}.$$

From Proposition 1.1, we get that (cf. [**16**], or the lemma below)

$$\hat{M}(s,t) = 1/(\mu s + \nu t).$$

As in the univariate case (cf. [**13**]) we define

$$\hat{F}_e(s,t) = \frac{1-\hat{F}(s,t)}{\mu s + \nu t}$$

and write

$$\hat{U}(s,t) = \frac{1}{\mu s+\nu t}\frac{1}{\hat{F}_e(s,t)} = \frac{1}{\mu s+\nu t}\frac{1}{1-(1-\hat{F}_e(s,t))}.$$

Since $1-\hat{F}(s,t) = \mathrm{E}(1-e^{-sX-tY}) \leqslant \mathrm{E}(sX+tY) = \mu s+\nu t$ and $\hat{F}(s,t) \geqslant 0$, we have $0 \leqslant 1-\hat{F}_e(s,t) \leqslant 1$. Also note that $\hat{F}_e(s,0) = \hat{F}_{1,e}(s)$. Using a Taylor expansion, we obtain that

$$\hat{U}(s,t) = \frac{1}{\mu s+\nu t}\frac{1}{1-(1-\hat{F}_e(s,t))} = \frac{1}{\mu s+\nu t}\sum_{n=0}^{\infty}(1-\hat{F}_e(s,t))^n.$$

Taking $\hat{T}_n(s,t) = (\mu s + \nu t)^{-1}(1 - \hat{F}_e(s,t))^n$, for $k \geqslant 0$, we consider

$$\hat{U}_k(s,t) = \sum_{n=0}^{k} \hat{T}_n(s,t).$$

As in the univariate case, we use $\hat{U}_k(s,t)$ as an approximation for $\hat{U}(s,t)$. For $k = 0, 1, 2$ we obtain

$$\hat{U}_0(s,t) = \frac{1}{\mu s + \nu t}, \quad \hat{U}_1(s,t) = \hat{U}_0(s,t) + \hat{T}_1(s,t), \quad \hat{U}_2(s,t) = \hat{U}_1(s,t) + \hat{T}_2(s,t).$$

In the next lemma we identify these LSTs transforms. Recall that we use $M(x,y) = \min(x/\mu, y/\nu)$.

LEMMA 2.1. *Suppose that $B(x,y)$ is a nonnegative function with $B(x,y) = 0$ whenever $x \leqslant 0$ or $y \leqslant 0$. Let $A(x,y)$ be defined as*

$$A(x,y) = \int_0^{M(x,y)} B(x - \mu z, y - \nu z)dz = \int_0^{\infty} B(x - \mu z, y - \nu z)dz.$$

Then

$$\hat{A}(s,t) = \frac{\hat{B}(s,t)}{\mu s + \nu t}.$$

As a special case ($B(x,y) \equiv 1$) we have $\hat{M}(s,t) = 1/(\mu s + \nu t)$.

PROOF. We have

$$\hat{A}(s,t) = \int_0^{\infty}\int_0^{\infty} e^{-sx-ty}dA(x,y) = st\int_0^{\infty}\int_0^{\infty}\int_{u=x}^{\infty}\int_{v=y}^{\infty} e^{-su-tv}du\,dv\,dA(x,y).$$

It follows that

$$\begin{aligned}\hat{A}(s,t) &= st\int_{u=0}^{\infty}\int_{v=0}^{\infty} e^{-su-tv}\int_{x=0}^{u}\int_{y=0}^{v} dA(x,y)du\,dv\\ &= st\int_{u=0}^{\infty}\int_{v=0}^{\infty} e^{-su-tv}A(u,v)du\,dv.\end{aligned}$$

From the definition of $A(.,.)$, we find that

$$\begin{aligned}&\hat{A}(s,t) = st\int_{u=0}^{\infty}\int_{v=0}^{\infty}\int_{z=0}^{M(u,v)} B(u - \mu z, v - \nu z)dze^{-su-tv}du\,dv\\ &= st\Big(\int_{u=0}^{\infty}\int_{v=\nu u/\mu}^{\infty}\int_{z=0}^{u/\mu} + \int_{v=0}^{\infty}\int_{u=\mu v/\nu}^{\infty}\int_{z=0}^{v/\nu}\Big)B(u-\mu z, v-nuz)dze^{-su-tv}du\,dv\\ &= I + II.\end{aligned}$$

First consider I. We have

$$\begin{aligned}I &= st\int_{u=0}^{\infty}\int_{v=\nu u/\mu}^{\infty}\int_{z=0}^{u/\mu} B(u - \mu z, v - \nu z)e^{-su-tv}dz\,du\,dv\\ &= st\int_{z=0}^{\infty}\int_{u=\mu z}^{\infty}\int_{v=\nu u/\mu}^{\infty} B(u - \mu z, v - \nu z)e^{-su-tv}du\,dv\,dz\end{aligned}$$

$$= st \int_{z=0}^{\infty} \int_{p=0}^{\infty} \int_{q=\nu p/\mu}^{\infty} B(p,q) e^{-s(p+\mu z)-t(q+\nu z)} dp\, dq\, dz$$
$$= st \int_{p=0}^{\infty} \int_{q=\nu p/\mu}^{\infty} B(p,q) e^{-sp-tq} dp dq \int_{z=0}^{\infty} e^{-(\mu s+\nu t)z} dz$$
$$= \frac{st}{\mu s+\nu t} \int_{p=0}^{\infty} \int_{q=\nu p/\mu}^{\infty} B(p,q) e^{-sp-tq} dp\, dq.$$

In a similar way, we have

$$II = \frac{st}{\mu s+\nu t} \int_{q=0}^{\infty} \int_{p=\mu q/\nu}^{\infty} B(p,q) e^{-sp-tq} dp\, dq.$$

Combining the results, we have

$$I + II = \frac{st}{\mu s+\nu t} \int_{q=0}^{\infty} \int_{p=0}^{\infty} B(p,q) e^{-sp-tq} dp\, dq = \frac{\hat{B}(s,t)}{\mu s+\nu t}. \qquad \square$$

Since $\hat{M}(s,t) = \hat{U}_0(s,t)$, then Lemma 2.1 gives $U_0(x,y) = M(x,y)$. Using $\hat{F}_e(s,t) = \frac{1}{\mu s+\nu t} - \frac{\hat{F}(s,t)}{\mu s+\nu t}$, Lemma 2.1 gives

$$F_e(x,y) = M(x,y) - \int_0^{M(x,y)} F(x-\mu z, y-\nu z) dz \tag{2.1}$$
$$= \int_0^{M(x,y)} \bar{F}(x-\mu z, y-\nu z) dz,$$

where $\bar{F}(x,y) = 1 - F(x,y)$.

For $T_1(x,y)$, we have $\hat{T}_1(s,t) = \hat{M}(s,t)(1-\hat{F}_e(s,t))$ and we find that

$$T_1(x,y) = M(x,y) - M * F_e(x,y) = M * \bar{F}_e(x,y),$$

where $\bar{F}_e(x,y) = 1 - F_e(x,y)$. Using Lemma 2.1 again, we obtain that

$$T_1(x,y) = \int_0^{M(x,y)} \bar{F}_e(x-\mu z, y-\nu z) dz.$$

For $T_2(x,y)$, we find that

$$T_2(x,y) = -\int_0^{M(x,y)} R_{e,2}(\mu x - z, \nu y - z) dz,$$

where $R_{e,2}(x,y) = 1 - F_e^{*2}(x,y) - 2(1-F_e(x,y))$.

Our approximations for $U(x,y)$ therefore are given by

$$U_0(x,y) = M(x,y),$$
$$U_1(x,y) = M(x,y) + T_1(x,y),$$
$$U_2(x,y) = M(x,y) + T_1(x,y) + T_2(x,y).$$

In the sections below we study the asymptotic behaviour of $F_e(x,y)$ and $T_1(x,y)$.

3. Definitions

In Sections 4 and 5 we study the asymptotic behaviour of $F_e(x,y)$ and $T_1(x,y)$. More precisely, we will study regularly varying behaviour of the functions involved. In this section, we present some classes of functions that will be needed.

3.1. Regular variation and related classes. In what follows we will often use univariate and multivariate regularly varying tails as follows.

DEFINITION 3.1. A positive and measurable function h is regularly varying with index $\beta \in \mathbb{R}$ if it satisfies $\lim_{t\to\infty} \frac{h(tx)}{h(t)} = x^{\beta}$, $\forall x > 0$. Notation: $h \in \mathrm{RV}(\beta)$.

The d.f. $F(x,y)$ is regularly varying if for some function $h(x) \in \mathrm{RV}(\beta)$ we have

$$\lim_{t\to\infty} h(t)\,\mathrm{P}(X > xt, Y > ty) = \lambda(x,y) \tag{3.1}$$

for all $(x,y) \geqslant (0,0)$ with $x+y>0$. Equivalently, we have

$$\lim h(t)\bar{F}(tx,ty) = \lim h(t)(1-F(tx,ty)) = \theta(x,y)$$

for all (x,y) with $\min(x,y) < \infty$. Notation: $F \in RVF(h,\theta)$.

The relation between θ and λ is given by

$$\theta(x,y) = \theta_1(x) + \theta_2(y) - \lambda(x,y) \leqslant \theta_1(x) + \theta_2(y).$$

Note that $\lim h(t)(1-F(txz,tyz)) = \theta(xz,yz)$ and that

$$h(t)(1-F(txz,tyz)) = \frac{h(t)}{h(tz)} h(tz)(1-F(txz,tyz)) \to z^{-\beta}\theta(x,y).$$

It follows that $\theta(xz,yz) = z^{-\beta}\theta(x,y)$.

REMARK 3.1. Note that $F \in RVF(h,\theta)$ implies that

$$h(t)\,\mathrm{P}(X > xt) = h(t)\bar{F}_1(tx) \to \theta_1(x) = \theta(x,\infty).$$

It follows that $\theta_1(x) = c_1 x^{-\beta}$ for some constant $c_1 \geqslant 0$. If $c_1 \neq 0$, we have $\bar{F}_1 \in \mathrm{RV}(-\beta)$. Similarly, we have

$$h(t)\,\mathrm{P}(Y > xt) = h(t)\bar{F}_2(tx) \to \theta_2(x) = c_2 x^{-\beta}.$$

REMARK 3.2. Also note that the defining property of $F \in RVF(h,\theta)$ holds locally uniformly in (x,y) and that the limit function θ is continuous.

REMARK 3.3. Regularly varying distributions have proved to be useful in studying domains of attractions in extreme value theory and in studying sums of random vectors. See [**4**–**6**]. See also [**15**].

Now consider the equilibrium distribution $F_{1,e}$. If $\beta \geqslant 1$ and $\mu < \infty$, we have

$$\bar{F}_{1,e}(x) = \frac{1}{\mu}\int_x^{\infty} \bar{F}_1(z)dz.$$

For $\beta > 1$, we have

$$\frac{1}{t} h(t)\bar{F}_{1,e}(tx) = h(t)\frac{1}{\mu}\int_x^{\infty} \bar{F}_1(tu)du \to \frac{1}{\mu}\int_x^{\infty} \theta_1(u)du = \frac{c_1}{\mu(\beta-1)} x^{1-\beta}.$$

If $c_1 \neq 0$, it follows that $\bar{F}_{1,e}(x) \in \mathrm{RV}(1-\beta)$.

If $\beta = 1$ and $\mu < \infty$, then we have

$$F_{1,e}(xt) - F_{1,e}(t) = \frac{1}{\mu}\int_t^{tx} \bar{F}_1(z)dz = \frac{t}{\mu}\int_1^x \bar{F}_1(tz)dz$$

and it follows that

$$\frac{h(t)}{t}(\bar{F}_{1,e}(xt) - \bar{F}_{1,e}(t)) \to -\frac{1}{\mu}\int_1^x \theta_1(z)dz = -\frac{c_1}{\mu}\log(x),$$

and we have $\bar{F}_{1,e}(x) \in \Pi_{-c_1/\mu}(t/h(t))$, cf. Definition 3.2 below. Note that in this case, we have

$$\frac{t}{h(t)\bar{F}_{1,e}(t)} \to 0.$$

For further use, we define the univariate and bivariate classes Π.

Definition 3.2. A positive and measurable function $f : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\Pi_c(L)$ if $L \in \mathrm{RV}(0)$ and if

$$\frac{f(tx) - f(t)}{L(t)} \to c\log x, \quad \forall x > 0.$$

A positive and measurable function $f\colon \mathbb{R}_+^2 \to \mathbb{R}_+$ belongs to the class $\Pi(L,\psi)$ if $L \in \mathrm{RV}(0)$ and if

$$\frac{f(tx,ty) - f(t,t)}{L(t)} \to \psi(x,y) < \infty, \quad \forall x,y > 0.$$

Clearly we have

$$\frac{f(txz,tyz) - f(t,t)}{L(t)} = \frac{f(txz,tyz) - f(tz,tz)}{L(tz)}\frac{L(tz)}{L(t)} + \frac{f(tz,tz) - f(t,t)}{L(t)}.$$

If $f(x,y) \in \Pi(L,\psi)$, it follows that $\psi(zx,zy) = \psi(x,y) + \psi(z,z)$ and we have $\psi(x,x) = c\log x$ for some constant c. It follows that $f(x,x) \in \Pi_c(L)$. If $f(x,y) \in \Pi(L,\psi)$ is a monotone function, then the defining property holds locally uniformly in $(x,y) > (0,0)$.

4. The function $F_e(x,y)$

4.1. Asymptotics of $F_e(\mu x, \nu y)$.

We have

$$F_e(\mu x,\nu y) = \int_0^{\min(x,y)} \bar{F}(\mu(x-z),\nu(y-z))dz.$$

Note that $F_e(\mu x,\infty) = F_{e,1}(x\mu)$, where $F_{e,1}(x)$ is the equilibrium distribution corresponding to the first marginal.

In the univariate case the equilibrium distribution $F_{1,e}(x)$ is a real distribution function. In the bivariate case it is not the case, i.e., $F_e(x,y)$ is, in general, not a distribution function. Moreover, as $(x,y) \to (\infty,\infty)$, $F_e(x,y)$ doesn't always tend to 1. We have the following limit results for F_e.

Lemma 4.1. *Suppose that* $\min(x,y) \to \infty$.

(i) *If* $x = y \to \infty$, *then* $F_e(\mu x, \nu y) \to \mathrm{E}\max\left(\frac{X}{\mu}, \frac{Y}{\nu}\right) \geqslant 1$.

(ii) *Suppose that* $x - y$ *converges to limit (finite or not finite). Then*

$$\begin{aligned} F_e(\mu x, \nu y) &\to \mathrm{E}\max\Big(\frac{X}{\mu}, \frac{Y}{\nu} - a\Big), \quad \textit{if } x \leqslant y \ \&\ y - x \to a, \quad 0 \leqslant a \leqslant \infty, \\ &\to \mathrm{E}\max\Big(\frac{X}{\mu} - b, \frac{Y}{\nu}\Big), \quad \textit{if } x \geqslant y \ \&\ x - y \to b, \quad 0 \leqslant b \leqslant \infty. \end{aligned}$$

(iii) *Suppose that* $x - y$ *converges to* $\pm\infty$, *then* $F_e(\mu x, \nu y) \to 1$.

Proof. (i) First suppose that $x = y$. We have

$$F_e(\mu x, \nu x) = \int_0^x \bar{F}(\mu z, \nu z)dz \to \int_0^\infty \bar{F}(\mu z, \nu z)dz.$$

The limit equals

$$\int_0^\infty \bar{F}(\mu z, \nu z)dz = \int_0^\infty \mathrm{P}\Big(\max\Big(\frac{X}{\mu}, \frac{Y}{\nu}\Big) > z\Big)dz = \mathrm{E}\max\Big(\frac{X}{\mu}, \frac{Y}{\nu}\Big) < \infty.$$

(ii) Next suppose that $x < y$. We have

$$F_e(\mu x, \nu y) = \int_0^x \bar{F}(\mu(x - z), \nu(y - z))dz = \int_0^x \bar{F}(\mu z, \nu(y - x + z))dz.$$

If $y - x \to a \leqslant \infty$, we have $\lim_{x\to\infty} \bar{F}(\mu z, \nu(y - x + z)) = \bar{F}(\mu z, \nu(a + z))$ and also we have $\bar{F}(\mu z, \nu(y - x + z)) \leqslant \bar{F}(\mu z, \nu z)$. Since

$$\int_0^\infty \bar{F}(\mu z, \nu z)dz = \mathrm{E}\max\Big(\frac{X}{\mu}, \frac{Y}{\nu}\Big) < \infty,$$

applying Lebesgue's dominated convergence theorem we obtain that

$$F_e(\mu x, \nu y) = \int_0^x \bar{F}(\mu z, \nu(y - x + z))dz \to \int_0^\infty \bar{F}(\mu z, \nu(a + z))dz.$$

This limit equals:

$$\begin{aligned} \int_0^\infty \bar{F}(\mu z, \nu(a + z))dz &= \int_0^\infty \Big(1 - \mathrm{P}\Big(\frac{X}{\mu} \leqslant z, \frac{Y}{\nu} - a \leqslant z\Big)\Big)dz \\ &= \int_0^\infty \mathrm{P}\Big(\max\Big(\frac{X}{\mu}, \frac{Y}{\nu} - a\Big) > z\Big)dz. \end{aligned}$$

It follows that $F_e(\mu x, \nu y) \to \mathrm{E}\max\left(\frac{X}{\mu}, \frac{Y}{\nu} - a\right)$.

If $x > y$, in a similar way we have the following result: if $y \to \infty$ and $x - y \to b \leqslant \infty$, then

$$F_e(\mu x, \nu y) \to \int_0^\infty \bar{F}(\mu(b + z), \nu z)dz = \mathrm{E}\max\Big(\frac{X}{\mu} - b, \frac{Y}{\nu}\Big).$$

(iii) This follows immediately from (ii). □

When we consider convergence along the lines of the form $(\mu tx, \nu ty)$, we obtain:

Corollary 4.1. *Let us consider* $F_e(\mu tx, \nu ty)$.

(i) *If* $x = y$, *then* $F_e(\mu tx, \nu tx) \to \mathrm{E}\max\left(\frac{X}{\mu}, \frac{Y}{\nu}\right) = 1 + \frac{1}{2}\mathrm{E}\left|\frac{X}{\mu} - \frac{Y}{\nu}\right|$

(ii) *If* $x < y$, *then* $t(y-x) \to \infty$, *and* $F_e(\mu tx, \nu ty) \to 1$;
(iii) *If* $x > y$, *then* $t(x-y) \to \infty$, *and* $F_e(\mu tx, \nu ty) \to 1$.

The corollary shows that the asymptotic behaviour of $F_e(\mu tx, \nu ty)$ is different on or off the line $y = x$. In what follows we restrict attention to the case where $x \neq y$.

4.2. Limit for $\bar{F}_e(\mu x, \nu y) = 1 - F_e(\mu x, \nu y)$ ($x \neq y$). If $x \neq y$, Corollary 4.1 shows that $F_e(\mu tx, \nu ty) \to 1$. We discuss $\bar{F}_e(\mu tx, \nu ty) = 1 - F_e(\mu tx, \nu ty)$ $x \neq y$ in this section. We reconsider the case where $x < y$. Recall that for $x < y$ we have (cf. (2.1))

$$F_e(\mu x, \nu y) = \int_0^x \bar{F}(\mu z, \nu(y-x+z))dz.$$

If $y - x \to \infty$ and $x \to \infty$, we have $\bar{F}(\mu z, \nu(y-x+z) \to \bar{F}_1(\mu z)$ and it makes sense to introduce $\bar{F}(\mu z, \nu(y-x+z)) - \bar{F}_1(\mu z)$ in the formula. We find

$$\begin{aligned} F_e(\mu x, \nu y) - F_{1,e}(\mu x) &= \int_0^x (\bar{F}(\mu z, \nu(y-x+z)) - \bar{F}_1(\mu z))dz \\ &= \int_0^x \mathrm{P}(X \leqslant \mu z, Y > \nu(y-x+z))dz \\ &= \int_0^x (\mathrm{P}(Y > \nu(y-x+z)) \\ &\qquad - \mathrm{P}(X > \mu z, Y > \nu(y-x+z)))dz. \end{aligned} \tag{4.1}$$

To study the asymptotics in (4.1), we assume $F \in RVF(h, \theta)$ with $h \in \mathrm{RV}(\beta)$. If $x < y$, from (4.1) it follows that

$$\begin{aligned} F_e(\mu tx, \nu ty) - F_{1,e}(\mu tx) &= \int_0^{tx} (\bar{F}(\mu z, \nu(ty - tx + z)) - \bar{F}_1(\mu z))dz \\ &= t\int_0^x (\bar{F}(\mu tu, \nu t(y-x+u)) - \bar{F}_1(\mu tu))dz. \end{aligned} \tag{4.2}$$

We obtain the following result.

Theorem 4.1. *Suppose that* $F \in RVF(h, \theta)$ *with* $h \in \mathrm{RV}(\beta)$ *and let* $\beta \geqslant 1$. *For* $x < y$ *we have*

$$\frac{h(t)}{t}(\bar{F}_{1,e}(\mu tx) - \bar{F}_e(\mu tx, \nu ty)) \to \phi_1(x, y), \tag{4.3}$$

where $\phi_1(x, y) = \int_0^x (\theta(\mu u, \nu(y-x+u)) - \theta_1(\mu u))du$.

For $x > y$, *we have*

$$\frac{h(t)}{t}(\bar{F}_{2,e}(\nu ty) - \bar{F}_e(\mu tx, \nu ty)) \to \phi_2(x, y),$$

where $\phi_2(x, y) = \int_0^y (\theta(\mu(x-y+z), \nu z) - \theta_2(vz))dz$.

Proof. From (4.2) we find that

$$\frac{h(t)}{t}(\bar{F}_{1,e}(\mu tx) - \bar{F}_e(\mu tx, \nu ty)) = h(t)\int_0^x (\bar{F}(\mu tu, \nu t(y-x+u)) - \bar{F}_1(tu\mu))dz.$$

To prove (4.3), note the following facts:

1) By the regular variation of F, we have
$$h(t)(\bar{F}(\mu tu, \nu t(y-x+u)) - \bar{F}_1(\mu tu)) \to \theta(\mu u, \nu(y-x+z)) - \theta_1(\mu u).$$
2) We have
$$h(t)(\bar{F}(\mu tu, \nu t(y-x+u)) - \bar{F}_1(\mu tu)) \leqslant h(t)\,\mathrm{P}(Y > \nu t(y-x+u)).$$
3) By regular variation of the marginals, we have
$$h(t)\,\mathrm{P}(Y > \nu t(y-x+u)) \to \theta_2(\nu(y-x+u)).$$
4) Since $y-x \leqslant y-x+u \leqslant y$, we have
$$h(t)\int_0^x \mathrm{P}(Y > \nu t(y-x+u))du \to \int_0^x \theta_2(\nu(y-x+u))du < \infty,$$

because of local uniform convergence. The Lemma of Pratt [**17**] now shows that (4.3) holds with
$$\phi_1(x,y) = \int_0^x (\theta(\mu u, \nu(y-x+u)) - \theta_1(\mu u))du.$$

For $y < x$, the result follows in a similar way. □

Remark 4.1. Note that $0 \leqslant \theta(\mu u, \nu(y-x+u) - \theta_1(\mu u)) \leqslant \theta_2(\nu(y-x+u))$ and that
$$\int_0^x \theta_2(\nu(y-x+u))du = c_2\nu^{-\beta}\int_0^x (y-x+u)^{-\beta}du$$
$$= c_2\nu^{-\beta}\int_{y-x}^y u^{-\beta}du < \infty.$$

It follows that $\phi_1(x,y) < \infty$. Also we have
$$\phi_1(x,y) = \int_0^x (\theta_2(\nu(y-x+u)) - \lambda(\mu u, \nu(y-x+u)))du.$$

Now we reconsider the case where $\beta > 1$ for which
$$\frac{h(t)}{t}\bar{F}_{1,e}(\mu tx) \to \int_x^\infty \theta_1(\mu u)du < \infty.$$

We have the following theorem.

Theorem 4.2. *Suppose that $F \in RVF(h,\theta)$ with $h \in \mathrm{RV}(\beta)$ and suppose that $\beta > 1$. If $x \neq y$, we have*
$$\frac{h(t)}{t}\bar{F}_e(\mu tx, \nu ty) \to \Psi(x,y),$$
where the limit function $\Psi(x,y)$ is given by
$$\Psi_1(x,y) = \int_x^\infty \theta_1(\mu u)du - \phi_1(x,y), \quad x < y,$$
$$\Psi_2(x,y) = \int_y^\infty \theta_2(\nu u)du - \phi_2(x,y), \quad x > y, \tag{4.4}$$

$$\phi_1(x,y) = \int_0^x (\theta(\mu u, \nu(y-x+u) - \theta_1(\mu u))du, \quad x<y,$$
$$\phi_2(x,y) = \int_0^y (\theta(\mu(x-y+u, \nu u) - \theta_2(\nu u))du, \quad x>y. \tag{4.5}$$

PROOF. It follows from Theorem 4.1 that for $x<y$,

$$\frac{h(t)}{t}\bar{F}_e(\mu tx, \nu ty) \to \Psi_1(x,y),$$

where $\Psi_1(x,y) = \int_x^\infty \theta_1(\mu u)du - \phi_1(x,y)$.

If $x>y$ and $\beta>1$, in a similar way we find that

$$\frac{h(t)}{t}\bar{F}_e(\mu tx, \nu ty) \to \Psi_2(x,y),$$

where $\Psi_2(x,y) = \int_x^\infty \theta_2(\nu z)dz - \phi_2(x,y), x>y$. □

Now we consider the case where $\beta = 1$. We obtain the following theorem.

THEOREM 4.3. *We have*

$$\frac{h(t)}{t}(\bar{F}_{1,e}(t) - \bar{F}_e(\mu tx, \nu ty)) \to \frac{c_1}{\mu}\log(\mu x) + \phi_1(x,y).$$

PROOF. Since $\bar{F}_{1,e}(x) \in \Pi_{-c_1/\mu}(t/h(t))$, for $x<y$, (4.3) implies that

$$\begin{aligned}\frac{h(t)}{t}(\bar{F}_{1,e}(t) - \bar{F}_e(\mu tx, \nu ty)) = & -\frac{h(t)}{t}(\bar{F}_{1,e}(\mu tx) - \bar{F}_{1,e}(t)) \\ & + \frac{h(t)}{t}(\bar{F}_{1,e}(\mu tx) - \bar{F}_e(\mu tx, \nu ty)) \\ \to & \frac{c_1}{\mu}\log(\mu x) + \phi_1(x,y).\end{aligned}$$

□

A similar statement can be obtained when $x>y$.

REMARK 4.2. If $h(t)t^{-1}(F_{1,e}(t) - F_{2,e}(t)) \to \delta$, then it follows that

$$\lim_{t\to\infty}\frac{h(t)}{t}(\bar{F}_{1,e}(t) - \bar{F}_e(\mu tx, \nu ty)) = \Psi(x,y)$$

exists for all $x \neq y$ and for some limit function $\Psi(x,y)$.

5. Limit results for $T_1(x,y)$

In this section we consider the asymptotic behaviour of $T_1(x,y)$ under various circumstances. As in the case of F_e we treat only the case where $x \neq y$.

5.1. Finite variance.

In this section we assume that $EX^2 + EY^2 < \infty$ and we consider $T_1(x,y)$ for $x \neq y$. We require the bounds given in the following lemma.

LEMMA 5.1. (i) *For $x<y$ we have*

$$\bar{F}_{1,e}(\mu x) - x\bar{F}_2(\nu(y-x)) \leqslant \bar{F}_e(\mu x, \nu y) \leqslant \bar{F}_{1,e}(\mu x).$$

(ii) *For $x>y$ we have*

$$\bar{F}_{2,e}(\nu x) - y\bar{F}_1(\mu(x-y)) \leqslant \bar{F}_e(\mu x, \nu y) \leqslant \bar{F}_{2,e}(\nu y).$$

Proof. (i) For $x < y$, we have proved earlier (cf. (4.1)) that

$$F_e(\mu x, \nu y) - F_{1,e}(\mu x) = \int_0^x \mathrm{P}(X \leqslant z\mu, Y > \nu(y - x + z))dz.$$

It follows that

$$0 \leqslant F_e(\mu x, \nu y) - F_{1,e}(\mu x) \leqslant \int_0^x \mathrm{P}(Y > \nu(y - x + z))dz$$

and then we obtain

$$0 \leqslant F_e(\mu x, \nu y) - F_{1,e}(\mu x) \leqslant \frac{1}{\nu} \int_{\nu(y-x)}^{\nu y} \mathrm{P}(Y > z)dz.$$

It follows that $0 \leqslant F_e(\mu x, \nu y) - F_{1,e}(\mu x) \leqslant x\,\mathrm{P}(Y > \nu(y - x))$ and this gives the first result.

The second result (ii) follows in a similar way. □

Now we consider $T_1(\mu x, \nu y), x \neq y$ and we prove the following result.

Theorem 5.1. *Assume that* $EX^2 + EY^2 < \infty$.

(i) *If* $x < y$, $y - x \to \infty$ *and* $\liminf(y - x)/x > 0$, *then*

$$U_1(\mu x, \nu y) - x = T_1(\mu x, \nu y) \to \int_0^\infty \bar{F}_{1,e}(\mu z)dz.$$

(ii) *If* $x > y$, $x - y \to 0$ *and* $\liminf(x - y)/y > 0$, *then*

$$U_1(\mu x, \nu y) - y = T_1(\mu x, \nu y) \to \int_0^\infty \bar{F}_{2,e}(\nu z)dz.$$

Proof. (i) If $x < y$, then $T_1(\mu x, \nu y) = \int_0^x \bar{F}_e(\mu z, \nu(y - x + z))dz$. Using Lemma 5.1, we have $T_1(\mu x, \nu y) \leqslant \int_0^x \bar{F}_{1,e}(\mu z)dz$ and

$$\begin{aligned} T_1(\mu x, \nu y) &\geqslant \int_0^x \bar{F}_{1,e}(\mu z)dz - \int_0^x z\bar{F}_2(\nu(y - x))dz \\ &= \int_0^x \bar{F}_{1,e}(\mu z)dz - \frac{1}{2}x^2\bar{F}_2(\nu(y - x)). \end{aligned}$$

If $EX^2 < \infty$, we have as $x \to \infty$,

$$\int_0^x \bar{F}_{1,e}(\mu z)dz \uparrow \int_0^\infty \bar{F}_{1,e}(\mu z)dz < \infty.$$

If $EY^2 < \infty$, then $t^2\bar{F}_2(t) \to 0$ $(t \to \infty)$. If $x \to \infty$ and $y - x \to \infty$, then we obtain that

$$\frac{1}{2}x^2\bar{F}_2(\nu(y - x)) = o(1)\frac{x^2}{(y - x)^2}.$$

If also $\liminf(y - x)/x > 0$, it follows that $x^2\bar{F}_2(\nu(y - x)) \to 0$, and then we obtain that $T_1(\mu x, \nu y) \to \int_0^\infty \bar{F}_{1,e}(\mu z)dz$.

The second result (ii) follows in a similar way. □

REMARK 5.1. If $y = \infty$, then we obtain $U_1(\mu x, \infty) - x \to \int_0^\infty \bar{F}_{1,e}(\mu z)dz = EX^2/2\mu$, which gives the well-known univariate result for the univariate renewal function.

Replacing (x, y) by (tx, ty) in the previous result, we obtain the following

THEOREM 5.2. *Assume that $EX^2 + EY^2 < \infty$.*

(i) *If $x < y$, then as $t \to \infty$,*

$$T_1(t\mu x, t\nu y) = U_1(\mu tx, \nu ty) - tx \to \int_0^\infty \bar{F}_{1,e}(\mu z)dz.$$

(ii) *If $x > y$, then as $t \to \infty$,*

$$T_1(t\mu x, t\nu y) = U_1(\mu tx, \nu ty) - ty \to \int_0^\infty \bar{F}_{2,e}(\nu z)dz.$$

5.2. Infinite variance. In the case where the variance is infinite, we assume that $F \in RVF(h, \theta)$ with $h \in \mathrm{RV}(\beta)$ and we distinguish several cases. Note that if $\beta > 2$, then the variances are finite.

5.2.1. *The case* $1 < \beta < 2$. We consider the case where $x \neq y$. For $x < y$, we have

$$T_1(\mu x, \nu y) = \int_0^x \bar{F}_e(\mu(x - z), \nu(y - z))dz,$$

and we find

$$T_1(\mu tx, \nu ty) = t \int_0^x \bar{F}_e(\mu tu, \nu t(y - x + u))du$$

so that

$$\frac{h(t)}{t^2} T_1(\mu tx, \nu ty) = \frac{h(t)}{t} \int_0^x \bar{F}_e(\mu tu, \nu t(y - x + u))du \to \int_0^x \Psi(\mu u, \nu(y - x + u)du,$$

where Ψ is given in Theorem 4.2.

To prove this, note that Lemma 5.1 gives

$$\frac{h(t)}{t} \bar{F}_{1,e}(\mu tu) - h(t)\mu u \bar{F}_2(\nu t(y - x)) \leqslant \frac{h(t)}{t} \bar{F}_e(\mu tu, \nu t(y - x + u)) \leqslant \frac{h(t)}{t} \bar{F}_{1,e}(\mu tu)$$

and we have squeezed the function between two other functions for which we can apply Lebesgue's dominated convergence theorem; noting that for $1 < \beta < 2$, we have $\frac{1}{t} h(t) \bar{F}_{1,e}(tx) \to C_1 x^{1-\beta}$ and

$$\frac{h(t)}{t} \int_0^x \bar{F}_{1,e}(\mu tu)du \to \int_0^x C_1(\mu u)^{1-\beta} du < \infty.$$

A similar result holds when $x > y$. So we have the following theorem

THEOREM 5.3. *If $x \neq y$ and $1 < \beta < 2$, then we have*

$$\frac{h(t)}{t^2} T_1(\mu tx, \nu ty) = \frac{h(t)}{t^2} \big(U_1(t\mu x, t\nu y) - t \min(x, y)\big) \to \Theta(x, y)$$

where

$$\Theta(x, y) = \begin{cases} \int_0^x \Psi(\mu u, \nu(y - x + u)du, & x < y, \\ \int_0^y \Psi(\mu(x - y + u), \nu u)du, & x > y. \end{cases} \tag{5.1}$$

5.2.2. *The case* $\beta = 1$. This subsection and the following contain specific limit results for $\beta = 1$ and $\beta = 2$, respectively.

THEOREM 5.4. *Suppose* $\beta = 1$ *and* $x < y$. *Then*

$$\frac{h(t)}{t^2}\left(t\int_0^x \bar{F}_{1,e}(\mu t u)du - T_1(\mu t x, \nu t y)\right) \to \int_0^x \phi_1(u, y - x + u)du.$$

PROOF. We have proved (cf. Theorem 4.1) that

$$\frac{h(t)}{t}(\bar{F}_{1,e}(\mu t x) - \bar{F}_e(\mu t x, \nu t y)) \to \phi_1(x, y)$$

and (cf. Lemma 5.1)

$$0 \leqslant \bar{F}_{1,e}(\mu t x) - \bar{F}_e(\mu t x, \nu t y) \leqslant t x \bar{F}_2(\nu t(y - x)).$$

We have

$$\frac{h(t)}{t^2}\left(t\int_0^x \bar{F}_{1,e}(\mu t u)du - T_1(\mu t x, \nu t y)\right) = \frac{h(t)}{t}\int_0^x (\bar{F}_{1,e}(\mu t u) \\ - \bar{F}_e(\mu t u, \nu t(y - x + u)))du.$$

Now note that

$$0 \leqslant \bar{F}_{1,e}(\mu t u) - \bar{F}_e(\mu t u, \nu t(y - x + u)) \leqslant t u \bar{F}_2(\nu t(y - x)),$$
$$\frac{h(t)}{t} t u \bar{F}_2(\nu t(y - x)) = h(t) u \bar{F}_2(\nu t(y - x)) \to u\theta_2(y - x).$$

The result follows from an application of Lebesgue's theorem. □

A similar result holds for $x > y$.

5.2.3. *The case* $\beta = 2$.

THEOREM 5.5. *Suppose* $\beta = 2$ *and* $x < y$. *Then*

$$\frac{h(t)}{t^2}(T_1(\mu t x, \nu t y) - T_1(\mu t x, \infty)) \to -\int_0^x \phi_1(u, y - x + u)du,$$

converges to a finite limit.

PROOF. Recall that for $x < y$ we have $T_1(\mu x, \nu y) = \int_0^x \bar{F}_e(\mu u, \nu(y - x + u))du$. Note that we have $\bar{F}_e(\mu u, \infty) = \bar{F}_{1,e}(\mu u)$ and

$$T_1(\mu x, \nu y) - T_1(\mu x, \infty) = \int_0^x (\bar{F}_e(\mu u, \nu(y - x + u)) - \bar{F}_{1,e}(\mu u))du,$$

where $T_1(\mu x, \infty) = \int_0^x \bar{F}_{1,e}(\mu u))du$. Using Lemma 5.1 we have

$$-u\bar{F}_2(\nu(y - x)) \leqslant \bar{F}_e(\mu u, \nu(y - x + u)) - \bar{F}_{1,e}(\mu u) \leqslant 0.$$

Replacing (x, y) by (tx, ty), we have

$$\frac{1}{t}(T_1(\mu t x, \nu t y) - T_1(\mu t x, \infty)) = \int_0^x (\bar{F}_e(\mu t u, \nu t(y - x + u)) - \bar{F}_{1,e}(\mu t u))du,$$
$$-t u \bar{F}_2(\nu t(y - x)) \leqslant \bar{F}_e(\mu t u, \nu t(y - x + u)) - \bar{F}_{1,e}(\mu t u) \leqslant 0.$$

Since
$$\frac{h(t)}{t}\int_0^x tu\bar{F}_2(\nu t(y-x))du \to h(t)\frac{x^2}{2}\theta_2(\nu(y-x)),$$
using Theorem 4.1, we obtain the result. □

In a similar way, for $x > y$ we have that
$$\frac{h(t)}{t^2}(T_1(\mu tx, \nu ty) - T_2(\infty, \nu ty)) \to -\int_0^x \phi_2(x-y+u, u)du.$$

6. Examples

6.1. Example 1. Let $X \stackrel{d}{=} Y \stackrel{d}{=} Z$ and $\mathrm{P}(X > x, Y > y) = \mathrm{P}(Z > x + y)$. Suppose that $\bar{F}_Z(x)$ is such that $h(t)\bar{F}_Z(tx) \to x^{-\beta}$ where $h \in \mathrm{RV}(\beta)$, $\beta > 0$. We have $\bar{F}(x,y) = \bar{F}_Z(x)+\bar{F}_Z(y)-\bar{F}_Z(x+y)$ and it follows that $h(t)\bar{F}(tx,ty) \to \theta(x,y)$ where $\theta(x,y) = x^{-\beta} + y^{-\beta} - (x+y)^{-\beta}$. We also have $\theta_1(x) = \theta_2(x) = x^{-\beta}$.

Note that for $\beta > 1$, the mean $\mu = \mathrm{E}\,Z < \infty$ and for $\beta > 2$, $\mathrm{var}(Z) < \infty$.

As a special case we can consider a Pareto distribution where $\mathrm{P}(Z > z) = (1+z)^{-\beta}$, $z \geqslant 0$.

As another special case, we can consider the following distribution:
$$\mathrm{P}(Z > z) = \frac{C}{(2+z)^{\beta}(\log(2+z))^4}, \quad z \geqslant 0.$$
If $\beta \geqslant 1$, then $\mu = \mathrm{E}\,Z < \infty$ and if $\beta \geqslant 2$, then $\mathrm{E}\,Z^2 < \infty$.

6.1.1. *Function F_e.* We suppose without loss of generality that $\mu = EZ = 1$. For $x < y$ we have (recall $\nu = \mu = 1$)
$$\begin{aligned} F_e(x,y) &= \int_0^x \bar{F}(z,(y-x+z))dz \\ &= \int_0^x (\bar{F}_Z(z) + \bar{F}_Z(y-x+z) - \bar{F}_Z(y-x+2z))dz \\ &= \int_0^x \bar{F}_Z(z)dz + \int_{y-x}^{y} \bar{F}_Z(z)dz - \frac{1}{2}\int_{y-x}^{y+x} \bar{F}_Z(z)dz. \end{aligned}$$
Now we have $\int_0^x \bar{F}_Z(z)dz = F_{e,Z}(x)$. It follows that
$$F_e(x,y) = F_{e,Z}(x) + F_{e,Z}(y) - \tfrac{1}{2}F_{e,Z}(y-x) - \tfrac{1}{2}F_{e,Z}(x+y),$$
$$\begin{aligned} \bar{F}_e(x,y) &= \bar{F}_{e,Z}(x) + \bar{F}_{e,Z}(y) - \bar{F}_{e,Z}(x+y) + \tfrac{1}{2}(F_{e,Z}(y-x) - F_{e,Z}(x+y)) \\ &= \bar{H}(x,y) + \tfrac{1}{2}(F_{e,Z}(y-x) - F_{e,Z}(x+y)), \end{aligned}$$
where $\bar{H}(x,y) = \bar{F}_{e,Z}(x)+\bar{F}_{e,Z}(y)-\bar{F}_{e,Z}(x+y)$ is of the same form as (X,Y) with Z replaced by Z_e.

If $x > y$, in a similar way we find that
$$\bar{F}_e(x,y) = \bar{H}(x,y) + \tfrac{1}{2}(\bar{F}_{e,Z}(x+y) - \bar{F}_{e,Z}(x-y)).$$
We conclude that for $x \neq y$ we have:
$$\bar{F}_e(x,y) = \bar{H}(x,y) + \tfrac{1}{2}(\bar{F}_{e,Z}(x+y) - \bar{F}_{e,Z}(|x-y|)).$$

Regularly varying asymptotics. Suppose $h(t)\bar{F}_Z(tx) \to x^{-\beta}$ and $h \in \mathrm{RV}(\beta)$. Since $\bar{F}(x,y) = \bar{F}_Z(x) + \bar{F}_Z(y) - \bar{F}_Z(x+y)$ we have $h(t)\bar{F}(tx,ty) \to \theta(x,y) = x^{-\beta} + y^{-\beta} - (x+y)^{-\beta}$ and $\theta_1(x) = \theta_2(x) = x^{-\beta}$.

(i) Now suppose that $\beta > 1$. In this case we have the following as an example of Theorem 4.2:

$$\frac{1}{t}h(t)\bar{F}_{e,Z}(tx) = \int_x^\infty h(t)\bar{F}_Z(tz)dz \to \frac{x^{1-\beta}}{(\beta-1)}$$

and it follows that $\frac{h(t)}{t}\bar{H}(tx,ty) \to \frac{1}{\beta-1}(x^{1-\beta} + y^{1-\beta} - (x+y)^{1-\beta})$ and

$$\frac{h(t)}{t}\bar{F}_e(tx,ty) \to \frac{1}{\beta-1}\big[x^{1-\beta} + y^{1-\beta} - (x+y)^{1-\beta} + \tfrac{1}{2}(|x-y|^{1-\beta} - (x+y)^{1-\beta})\big]$$
$$= \Psi(x,y) \qquad \text{(Theorem 4.2)}.$$

(ii) If $\beta = 1$, we have the following as an example of Theorem 4.3:

$$\frac{1}{t}h(t)(\bar{F}_{e,Z}(tx) - \bar{F}_{e,Z}(t)) = \int_x^1 h(t)\bar{F}_Z(tz)dz \to -\log x,$$

$$\bar{H}(tx,ty) - \bar{F}_{e,Z}(t) = \bar{F}_{e,Z}(tx) - \bar{F}_{e,Z}(t) + \bar{F}_{e,Z}(ty) - \bar{F}_{e,Z}(t) - \bar{F}_{e,Z}(t(x+y)) + \bar{F}_{e,Z}(t),$$

so we find that $\frac{h(t)}{t}(\bar{H}(tx,ty) - \bar{F}_{e,Z}(t)) \to -\log x - \log y + \log(x+y)$.

For the other term we find

$$\bar{F}_{e,Z}(t(x+y)) - \bar{F}_{e,Z}(t|x-y|) = \bar{F}_{e,Z}(t(x+y)) - \bar{F}_{e,Z}(t) - \bar{F}_{e,Z}(t|x-y|) + \bar{F}_{e,Z}(t),$$
$$\frac{h(t)}{t}(\bar{F}_{e,Z}(t(x+y)) - \bar{F}_{e,Z}(t|x-y|)) \to -\log(x+y) + \log(|x-y|).$$

It follows that

$$\frac{h(t)}{t}(\bar{F}_e(tx,ty) - \bar{F}_{e,Z}(t)) \to \log\frac{x+y}{xy} + \frac{1}{2}\log\frac{|x-y|}{x+y}$$
$$= -c_1\log x - \phi_1(x,y) \qquad \text{(Theorem 4.3)}.$$

6.1.2. *Function* T_1. We still assume $\mathrm{E}\,Z = 1$. Here we assume that $h(t)\bar{F}_Z(tx) \to \theta(x) = x^{-\beta}$, where $h(x) \in \mathrm{RV}(\beta), 1 < \beta < 2$.

Consider $x < y$. Recall that we have

$$F_e(x,y) = \begin{cases} F_{e,Z}(x) + F_{e,Z}(y) - \frac{1}{2}F_{e,Z}(|y-x|) - \frac{1}{2}F_{e,Z}(x+y), & x \neq y, \\ 2F_{e,Z}(x) - \frac{1}{2}F_{e,Z}(2x), & x = y. \end{cases}$$

It follows that

$$\bar{F}_e(x,y) = \bar{F}_{e,Z}(x) + \bar{F}_{e,Z}(y) - \tfrac{1}{2}\bar{F}_{e,Z}(|y-x|) - \tfrac{1}{2}\bar{F}_{e,Z}(x+y), \quad x \neq y,$$
$$\tfrac{3}{2} - F_e(x,y) = 2(1 - F_{e,Z}(x)) - \tfrac{1}{2}(1 - F_{e,Z}(2x)), \quad x = y.$$

Using $h(t)\bar{F}_Z(tx) \to x^{-\beta}$, we have $t^{-1}h(t)\bar{F}_{e,Z}(tx) \to x^{1-\beta}/(\beta-1)$. We find that

$$(\beta-1)t^{-1}h(t)\bar{F}_e(tx,ty) \to x^{1-\beta} + y^{1-\beta} - \tfrac{1}{2}|y-x|^{1-\beta} - \tfrac{1}{2}(x+y)^{1-\beta}, \quad x \neq y,$$

$$(\beta-1)t^{-1}h(t)(\tfrac{3}{2}-F_e(tx,ty)) \to 2x^{1-\beta}-\tfrac{1}{2}(2x)^{1-\beta}, \quad x=y.$$

For $x<y$ we have

$$\begin{aligned}
T_1(x,y) &= \int_0^x \bar{F}_e(x-z,y-z)dz \\
&= \int_0^x (\bar{F}_{e,Z}(x-z)+\bar{F}_{e,Z}(y-z)-\tfrac{1}{2}\bar{F}_{e,Z}(y-x)-\tfrac{1}{2}\bar{F}_{e,Z}(x+y-2z))dz \\
&= \int_0^x \bar{F}_{e,Z}(z)dz + \int_{y-x}^y \bar{F}_{e,Z}(z)dz - \tfrac{1}{2}x\bar{F}_{e,Z}(y-x) - \frac{1}{4}\int_{y-x}^{x+y}\bar{F}_{e,Z}(z)dz \\
&= H(x)+H(y)-\tfrac{3}{4}H(y-x)-\tfrac{1}{4}H(x+y)-\tfrac{1}{2}x\bar{F}_{e,Z}(y-x),
\end{aligned}$$

where $H(x)=\int_0^x \bar{F}_{e,Z}(z)dz$. For $x>y$ we have a similar result and as a final result we find

$$T_1(x,y)=H(x)+H(y)-\tfrac{3}{4}H(|y-x|)-\tfrac{1}{4}H(x+y)-\tfrac{1}{2}\min(x,y)\bar{F}_{e,Z}(|y-x|).$$

For $x=y$, we have

$$\begin{aligned}
T_1(x,x) &= \int_0^x \bar{F}_e(x-z,x-z)dz = x-\int_0^x F_e(z,z)dz \\
&= x-2\int_0^x F_{e,Z}(z)dz - \frac{1}{2}\int_0^x F_{e,Z}(2z)dz = x-2H(x)-\tfrac{1}{4}H(2x).
\end{aligned}$$

Since $H(x)\in \mathrm{RV}(2-\beta)$, $H(x)\sim x\bar{F}_{e,Z}(x)/(2-\beta)$ and $\bar{F}_{e,Z}\in \mathrm{RV}(1-\beta)$, we obtain that for $x\neq y$

$$\begin{aligned}
\frac{1}{H(t)}T_1(tx,ty) \to x^{2-\beta}+y^{2-\beta}-\tfrac{3}{4}|y-x|^{2-\beta}-\tfrac{1}{4}(x+y)^{2-\beta} \\
-\tfrac{1}{2}(2-\beta)\min(x,y)|y-x|^{1-\beta}
\end{aligned}$$

and for $x=y$ we have $\frac{1}{H(t)}(tx-T_1(tx,tx)) \to 2x^{2-\beta}+\frac{1}{4}(2x)^{2-\beta}$.

6.2. Example 2. Take $F(x,y)=\min(F(x),F(y))$ or equivalently that

$$\bar{F}(x,y)=\max(\bar{F}(x),\bar{F}(y)).$$

Assume that $h\in \mathrm{RV}(\beta), \beta>1$, and $h(t)\bar{F}(tx)\to x^{-\beta}$. Using $\bar{F}(x,y)=\max(\bar{F}(x),\bar{F}(y))$, we find that $h(t)\bar{F}(tx,ty)\to\max(x^{-\beta},y^{-\beta})$.

6.2.1. *Function F_e.* Take $F(x,y)=\min(F(x),F(y))$ and $\mu=1$. We have

$$\begin{aligned}
F_e(x,y) &= \int_0^{\min(x,y)} (1-\min(F(x-z),F(y-z)))dz \\
&= \int_0^{\min(x,y)} \max(\bar{F}(x-z),\bar{F}(y-z))dz.
\end{aligned}$$

For $x<y$, we have $x-z<y-z$ and we find that $F_e(x,y)=\int_0^x \bar{F}(x-z)dz=F_e(x)$. In a similar way, we have for $x>y$ that $F_e(x,y)=F_e(y)$. For $x=y$, we have $F_e(x,y)=F_e(x)$. Summarizing, we have that $F_e(x,y)=\min(F_e(x),F_e(y))$ and

$$\bar{F}_e(x,y)=\max(\bar{F}_e(x),\bar{F}_e(y)).$$

6.2.2. *Function* T_1. For T_1, we have

$$T_1(x,y) = \int_0^{\min(x,y)} \bar{F}_e(x-z, y-z)dz = \int_0^{\min(x,y)} \max(\bar{F}_e(x-z), \bar{F}_e(y-z))$$

and we find that $T_1(x,y) = \mu_e \min(F_{e,e}(x), F_{e,e}(y))$. where $F_{e,e}(x) = \frac{1}{\mu_e}\int_0^x \bar{F}_e(z)dz$.

6.3. Example 3. In the independent case we have $F(x,y) = F_1(x)F_2(y)$ and $\mathrm{P}(X > x, Y > y) = \mathrm{P}(X > x)\,\mathrm{P}(Y > y)$. If $h(t)\bar{F_i}(tx) \to c_i x^{-\beta}$, we have $h(t)\,\mathrm{P}(X > xt, Y > ty) \to 0$.

6.3.1. *Function* F_e. Take $F(x,y) = F_1(x)F_2(y)$ and assume $\mu = \nu = 1$. For $x < y$ we have

$$\begin{aligned} F_e(x,y) &= \int_0^x (1 - F_1(x-z)F_2(y-z))dz \\ &= \int_0^x (\bar{F}_1(x-z) + \bar{F}_2(y-z) - \bar{F}_1(x-z)\bar{F}_2(y-z))dz \\ &= \int_0^x \bar{F}_1(z)dz + \int_{y-x}^y \bar{F}_2(z)dz - \int_0^x \bar{F}_1(z)\bar{F}_2(y-x+z)dz \\ &= F_{1,e}(x) + F_{2,e}(y) - F_{2,e}(y-x) - \int_0^x \bar{F}_1(z)\bar{F}_2(y-x+z)dz. \end{aligned}$$

For $x > y$, in a similar way we find

$$F_e(x,y) = F_{1,e}(x) + F_{2,e}(y) - F_{1,e}(x-y) - \int_0^x \bar{F}_2(z)\bar{F}_1(x-y+z)dz.$$

For $x = y$, we have

$$\begin{aligned} F_e(x,x) &= \int_0^x (1 - F_1(x-z)F_2(x-z))dz \\ &= \int_0^x (\bar{F}_1(x-z) + \bar{F}_2(x-z) - \bar{F}_1(x-z)\bar{F}_2(x-z))dz \\ &= \int_0^x \bar{F}_1(z)dz + \int_0^x \bar{F}_2(z)dz - \int_0^x \bar{F}_1(z)\bar{F}_2(z)dz \\ &= F_{1,e}(x) + F_{2,e}(x) - \int_0^x \bar{F}_1(z)\bar{F}_2(z)dz. \end{aligned}$$

6.3.2. *Function* T_1. Consider the regular varying case with $1 < \beta < 2$ and $x < y$. Assume $\mu = \nu = 1$. From (3.1) we obtain $\lambda(x,y) = 0$ giving $\theta(x,y) = \theta_1(x) + \theta_2(y)$. Inserting this into (4.5) gives

$$\phi_1(x,y) = \int_0^x \theta_2(y-x+u)du = \int_0^x \theta_2(y-u)du$$

from which we obtain for (4.4)

$$\Psi_1(x,y) = \int_x^\infty \theta_1(u)du - \int_0^x \theta_2(y-u)du.$$

Applying Theorem 5.3 we have that $h(t)T_1(x,y)/t^2 \to \Theta(x,y)$ where $\Theta(x,y)$ is given by (5.1).

The case $x > y$ is similar.

6.4. Example 4. Consider $F(x,y) = F_1(x)F_2(y)(1+\theta\bar{F}_1(x)\bar{F}_2(y))$. We have

$$\begin{aligned}\bar{F}(x,y) &= 1 - F_1(x)F_2(y) - \theta F_1(x)F_2(y)\bar{F}_1(x)\bar{F}_2(y)\\ &= \bar{F}_1(x) + F_1(x)\bar{F}_2(y) - \theta F_1(x)F_2(y)\bar{F}_1(x)\bar{F}_2(y)\\ &= \bar{F}_1(x) + \bar{F}_2(y) - (1+\theta F_1(x)F_2(y))\bar{F}_1(x)\bar{F}_2(y).\end{aligned}$$

Suppose that as in Example 4 we have $h(t)\bar{F}_i(tx) \to c_i x^{-\beta}$. Now we get that

$$h(t)\bar{F}(tx,ty) \to c_1 x^{-\beta} + c_2 y^{-\beta}.$$

7. Conclusions and final remarks

The next order approximation requires us to consider T_2 that involves integrals containing $R_{e,2}(x,y) = 1 - F_e^{*2}(x,y) - 2(1 - F_e(x,y))$. Other types of result are Blackwell type results involving expressions such as $U(x+a,y+b) - U(x,y)$, $F_e(x+a,y+b) - F_e(x,y)$ and $T_1(x+a,y+b) - T_1(x,y)$. Results for both these cases are future work for forthcoming papers.

Acknowledgments. Part of the work by Rein Vesilo was done when on sabbatical at KU Leuven, Campus Brussels. He wishes to extend his gratitude to the host university and acknowledge the support of Macquarie University

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(Received 11 10 2016)

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