THE COSINE SERIES AND REGULAR VARIATION IN THE KARAMATA AND ZYGMUND SENSES

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ABSTRACT. The asymptotic behaviour of the coefficients of cosine series is related to the behaviour at the origin of its sum function, in terms of slowly varying functions (SVF's), and regularly varying sequences. Our work is motivated by the study of the sine series with monotone coefficients of Aljančić, Bojanić and Tomić (1956), which is in terms of SVF's in the Karamata sense. The direction of our approach to the cosine series is motivated by the recent presentation (using SVF's in the Zygmund sense) of Samorodnitsky (2016). An obituary for the first author by the second author, with specific relevance to our subject matter, is attached as Section 7.

1. Introduction

Let $f(x)$ be defined by

$$(1.1) \qquad f(x) = \frac{1}{2}a(0) + \sum_{n=1}^{\infty} a(n)\cos(nx)$$

whenever the series converges. Convergence occurs if $a(n) \downarrow 0$, $n \to \infty$, except possibly at $x = 0$, in the neighbourhood of which the symmetric function about 0, $f(x)$, may be unbounded.

Our focus in the present paper is this Fourier cosine series, and the interaction of the regular variation of its coefficients with the regular variation of $f(x)$, $x \to 0+$.

Some recent work on the cosine series has been in the context where the sequence $\{a(n)\}$ is the autocorrelation function of a second-order stationary time-series, in which case $f(x)$ is called the spectral density function. In his very recent book Samorodnitsky [**14**] devotes his Section 6.2 ("Spectral Domain Approaches") to a self-contained development of the theory in this context. When, for such a time series, $a(n) \sim n^{-\alpha}L(n)$, $n \to \infty$, where $0 < \alpha < 1$, then $\sum a(n) = \infty$ which is an expression in statistical terms, of "long range dependence" in the time series.

A specific feature of the spectral theory is that $f(x) \geqslant 0$, $-\pi < x < \pi$.

But there are close links to the general classical theory, and part of our motivation for this paper is to place the spectral theory into the classical theory without probabilistic discussion.

As an example, if the sequence $\{a(n)\}$ is convex on $N_+ = \{0, 1, 2, \ldots\}$ and satisfies on $N_+ : a(0) = 1$, and $a(n) \downarrow 0$, $n \to \infty$, the sequence defines the autocorrelation function of a second-order stationary time series [**8**, Theorem 7]. Then a classic result for trigonometric series is:

THEOREM 1.1 (Young's Theorem). *If $a(n) \to 0$, $n \to \infty$ and the sequence $a(0), a(1), \ldots$ is convex, then the series (1.1) converges, save possibly at $x = 0$, to a non-negative, integrable and continuous function $f(x)$, and is the Fourier series of f.*

A sequence $a(0), a(1), \ldots$ is said to be *convex* if $a(n+1) - 2a(n) + a(n-1) \geqslant 0$, $n \geqslant 1$.

The theorem thus describes convergence to, and nature of, the spectral density function $f(x)$.

For this theorem see [**20**, Chapter V, pp. 183–184, Theorems 1.5 and 1.8]. In [**18**, Chapter V, p. 109 footnote], this theorem is ascribed to Young [**17**] and Kolmogoroff [**13**].

Although we shall focus on Fourier cosine series for which $a(n) \downarrow 0$, $n \to \infty$, there is a parallel theory for the sine series which interacts with the cosine theory.

Write $g(x) = \sum_{n=1}^{\infty} \lambda(n) \sin(nx)$. $g(x)$ is a well-defined continuous function in $(-\pi, \pi)$, except possibly at zero, providing $\lambda(n) \downarrow 0$.

The following theorem on sine series is due to Aljančić, Bojanić and Tomić [**3**]. This is a key paper for our sequel, and we have included its digital address in our References list. All past papers in *Publications* have similar digital addresses.

THEOREM 1.2 (ABT Theorem). *Suppose $0 < \beta < 2$ and $\lambda(n) \downarrow 0$. Then*

$$(1.2) \quad \lambda(n) \sim n^{-\beta} L(n), n \to \infty \iff g(x) \sim \frac{\pi}{2\Gamma(\beta)\sin(\beta\pi/2)} x^{\beta-1} L(1/x), x \to 0+.$$

The implication from left to right in (1.2) when $1 < \beta < 2$ does not require monotonicity of $\lambda(n) > 0$

In the fundamental case where $L(x) = A$, $A > 0$ and $0 < \beta < 1$ Theorem 1.2 was proved, over two papers, by Hardy [**9, 10**], and it was this result that Aljančić, Bojanić and Tomić [**3**] extended, as their Théorème 1, to slowly varying functions in the sense of Karamata (whose landmark paper, Karamata [**12**], also appeared in 1931). Some of their technology is based on Hardy's, but their proof of both directions of their theorem, in their Section 4, is long and intricate, and relies in part on certain monotonicity properties of regularly varying functions (in the Karamata sense), which are mentioned in our Section 3.

Hardy [**9, 10**] also proved the parallel result for cosine series: Suppose $0 < \alpha < 1$, $A > 0$ and $a(n) \downarrow 0$. Then

$$(1.3) \qquad a(n) \sim An^{-\alpha}, n \to \infty \iff f(x) \sim \frac{A\pi x^{\alpha-1}}{2\Gamma(\alpha)\cos(\alpha\pi/2)}, \quad x \to 0+.$$

The possibility of a similar result to Theorem 1.2 for the cosine series (1.1) was envisaged in Aljančić, Bojanić and Tomić [**2**], based on the Theorem 1.2, and the identity

$$(1.4) \qquad \sin(x)\left(\frac{1}{2}a(0) + \sum_{k=1}^{\infty} a(k)\cos(kx)\right) = \sum_{k=1}^{\infty} b(k)\sin(kx)$$

where $b(k) = \frac{1}{2}(a(k-1) - a(k+1))$, $1 \leqslant k$.

No proof was given for (1.4). And no generalization of Hardy's cosine series result (1.3) to regularly varying functions, analogous to Theorem 1.2, was given by those authors.

It is, however, in terms of *regularly varying functions in the sense of Zygmund* that cosine series analogues of Theorem 1.2 appear to have been first expressed.

A function $L(x)$, defined, positive and measurable on $x \geqslant B$, for some $B > 0$ is said to be slowly varying (SVF) (in the Karamata sense) if

$$(1.5) \qquad \lim_{x \to \infty} \frac{L(\lambda x)}{L(x)} = 1$$

for every $\lambda > 0$. A fundamental theorem of SVF theory is the Uniform Convergence Theorem (UCT), which asserts that the convergence in (1.5) is uniform in λ in any fixed interval (a, b), $0 < a < b < \infty$.

A function $L(x)$, defined and positive for $x \geqslant B$, for some $B > 0$ is said to be of the *Zygmund class of slowly varying functions* if $x^{\epsilon}L(x)$ eventually monotonically increases, and $x^{-\epsilon}L(x)$ eventually monotonically decreases, for every fixed $\epsilon > 0$. Since ([**20**, p. 186], [**15**, p. 49]) such a function satisfies, for any $\lambda > 0$,

$$\lim_{x \to \infty} \frac{L(\lambda x)}{L(x)} = 1,$$

slowly varying functions of the Zygmund class are slowly varying in the (more general) Karamata sense.

The term *slowly varying function* (SVF) and the notation $L(\cdot)$, denotes a slowly varying function at infinity in the sense of Karamata throughout this paper, unless otherwise stated.

A function $R(x)$, $x > 0$ is said to be regularly varying of index ρ if $R(x) = x^{\rho}L(x)$, where $-\infty < \rho < \infty$.

2. The Cosine Series

It is in terms of regularly varying functions in the sense of Zygmund that cosine series analogues of Theorem 1.2 appear to have been first expressed.

THEOREM 2.1. *Suppose $0 < \alpha < 1$. If $L(x), x \to \infty$, is SVF in the Zygmund sense then*

$$a(n) = n^{-\alpha}L(n), \ n \to \infty \ \Rightarrow \ f(x) \sim \frac{\pi x^{\alpha-1}L(1/x)}{2\Gamma(\alpha)\cos(\pi\alpha/2)}, \quad x \to 0+ \, .$$

Theorem 2.1 is contained in "(2.6) Theorem", of Zygmund [20, p. 187]. Under the conditions of Theorem 2.1 the function $f(x)$ as defined by (1.1) is a symmetric integrable function on $(-\pi, \pi)$, possibly infinite at $x = 0$ such that

$$(2.1) \qquad a(n) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx, \; n = 0, 1, 2, \dots.$$

The conditions $\lambda(n) \downarrow 0$, $\lambda(n) \sim n^{-\beta} L(n)$, $n \to \infty$, $0 < \beta < 2$, for the implication from left to right in (1.2) were quickly generalized (Bojanić and Karamata[5, Theorem 7]) by replacing them by the assumption that $\lambda(n) = n^{-\beta} L(n)$ where $L(x)$, $x > 0$, is a quasi-monotone slowly varying function. A slowly varying function $L(x)$ in the Zygmund sense is a quasi-monotone slowly varying function (Bojanić and Karamata [5, Section 4])

But we follow the consequences of Zygmund's [19, 20] approach inasmuch it is related to Samorodnitsky's one[14].

Samorodnitsky [14] gives a proof of Theorem 2.1 within his Section 6.2 ("Spectral Domain Approaches").

Samorodnitsky's [14] additional probabilistic condition, that $\{a(n)\}$ is a sequence of autocovariances of a stationary process, which renders $f(x) \geqslant 0$, $-\pi < x < \pi$, is not present in our non-probabilistic context here. To ensure this property for the limit function in Theorem 2.1 we use Theorem 1.1 (Young's Theorem):

COROLLARY 2.1. *If, in addition to the assumption in Theorem 2.1, we assume that the sequence $\{a(n)\}$ is convex on $n \geqslant 1$, then $f(x) \geqslant 0$, $-\pi < x < \pi$.*

THEOREM 2.2. *Define $a(n)$ by (2.1) where $f(x)$ is a non-negative symmetric integrable function on $(-\pi, \pi)$, possibly infinite at $x = 0$. Assume that*

$$(2.2) \qquad f(x) = \frac{\pi x^{\alpha-1} L(1/x)}{2\Gamma(\alpha) \cos(\pi\alpha/2)}, \quad 0 < x < \pi,$$

where $0 < \alpha < 1$, and the SVF $L(x)$, $x \to \infty$, belongs to the Zygmund class. Assume further that $f(x)$ has bounded variation on (ϵ, π) for any ϵ, $0 < \epsilon < \pi$. Then

$$(2.3) \qquad a(n) \sim n^{-\alpha} L(n), \quad n \to \infty.$$

Theorem 2.2 occurs as part of "(2.22) Theorem" in Zygmund [20, p. 190], where the proof is sketched. A detailed proof is given in Samorodnitsky [14].

Note the careful distinction in the placing of the signs $\sim$ and $=$ in Theorems 2.1 and 2.2.

Also note the the bounded variation condition on $f(x)$ in Theorem 2.2. We reproduce here the logic of Samorodnitsky [14, Example 6.2.9] to show that a constraint like this additional to (2.2) is required for the conclusion (2.3). Let $g(x)$ be a positive integrable function on $(0, \pi)$ satisfying (2.2). Define on $(0, \pi)$

$$f(x) = g(x)\mathbf{I}(0 < x \leqslant \pi/2) + g(\pi - x)\mathbf{I}(\pi/2 \leqslant x < \pi),$$

and extend the definition to $-\pi < x < \pi$ by putting $f(x) = f(-x)$, so $f(x)$ is a symmetric non-negative integrable function satisfying (2.2), infinite at $x = 0$. Now

from (2.1):

$$\pi a(n) = 2\int_0^\pi \cos(nx)f(x)\,dx = 2\int_0^{\pi/2}\cos(nx)g(x)\,dx + 2\int_{\pi/2}^\pi \cos(nx)g(\pi-x)\,dx$$

$$= 2\int_0^{\pi/2}\cos(nx)g(x)\,dx + 2\int_0^{\pi/2}\cos(n(\pi-y))g(y)\,dy$$

and since $\cos(n\pi - ny) = \cos(n\pi)\cos ny + \sin n\pi \sin(ny)$

$$\pi a(n) = 2(1+(-1)^n)\int_0^{\pi/2}\cos(nx)g(x)\,dx$$

so that $a(n) = 0$ when n is an odd number, and so (2.3) does not hold.

Samorodnitsky [**14**, Theorem 6.2.11(b)], assumes in his formulation of Theorem 2.2 above, that $f(x)$ is a spectral density from the outset. This would entail assuming from the outset that defining expression (1.1) for $f(x)$ converges for some sequence of autocorrelation coefficients. We can bring the more general Theorem 2.2 into this specific framework by use of the already mentioned [**8**, Theorem 7]:

COROLLARY 2.2. *If the sequence $\{a(n)\}$ defined by (2.1) is convex on $N_+ = \{0,1,2,\ldots\}$ and satisfies on $N_+ : a(0) = 1$, and $a(s) \downarrow 0$, $s \to \infty$, the sequence defines the autocorrelation function of a second-order stationary time series.*

Finally in regard to Theorems 2.1 and 2.2, Samorodnitsky [**14**] points out that in Theorem 2.1, one needs only to assume that $x^{-\alpha}L(x)$, $x > 0$, is eventually non-increasing, and in Theorem 2.2 that $x^{1-\alpha}L(x), x > 0$, is eventually non-decreasing.

Now, Yong [**16**] (in this journal, paper received August 10, 1967) proved, using an exactly parallel methodology to Aljančić, Bojanić and Tomić [**3**], the analogue of the ABT Theorem:

THEOREM 2.3 (Y Theorem). *Suppose $0 < \alpha < 1$ and $\alpha(n) \downarrow 0$, and $L(x)$, $x > 0$, is a slowly varying function. Then*

$$(2.4) \qquad \alpha(n) \sim n^{-\alpha}L(n), n \to \infty \Leftrightarrow f(x) \sim \frac{\pi x^{\alpha-1}L(1/x)}{2\Gamma(\alpha)\cos(\pi\alpha/2)}, \quad x \to 0+ \,.$$

Notice that there is no analogue of the statement for the sine series that the implication from left to right in (1.2) when $1 < \beta < 2$ does not require monotonicity of $\lambda(n) > 0$. We shall use this property of the sine series to give a simple proof of the implication from left to right in (2.4), when this implication is expressed in terms of the sequence $\{a(n)\}$ as a *regularly varying sequence*, in Theorem 3.1. This will be based on identity (1.4).

A note added at the end of Yong's paper acknowledges that the implication from left to right in (2.4) had already been proved by Adamović [**1**]. This occurs within Adamović's [**1**, Théorème 1], a comprehensive generalization in this direction, of Aljančić, Bojanić and Tomić's [**3**] theorem.

3. Discussion

Both Adamović [1] and Yong [16] contain Zygmund [19] in their reference lists. Zygmund [20] is essentially a reprinting of [19], so Zygmund cites neither paper. Eventually Yong's paper is cited alongside that of Aljančić, Bojanić and Tomić [3] in the context of the Fourier series with monotone coefficients, in the treatise of Bingham, Goldie, and Teugels [4], who approach this context as a special application of quasi-monotone slowly varying functions.

Regularly varying functions in the sense of Zygmund are introduced in Section 2 ("The order of magnitude of functions represented by series with monotone coefficients") of Volume 1, Chapter V ("Special Trigonometric Series") of Zygmund's [19, 20] definitive treatment of trigonometric series. In his notes to Chapter V, Section 2, Zygmund says:

"In this section we give a few fundamental results, aiming at simplicity rather than generality. The definition of a slowly varying function as we introduce it here, occurs in Hardy and Rogosinski [11], though the authors do not use it systematically. It seems the most convenient for our purposes, though it differs from the generally adopted definition... of Karamata."

In his next paragraph Zygmund mentions Aljančić, Bojanić and Tomić [3] twice, but pursues his development in his Section 2 in the direction we have described above in Theorems 2.1 and 2.2, elegantly completed in the exposition of Samorodnitsky [14].

There are important common features in the two approaches, Theorems 1.2 and 2.3 on the one hand, and Theorems 2.1 and 2.2 on the other.

The proofs of the implication from left to right in Theorems 1.2 and 2.3, and the proof of Theorem 2.1, all depend heavily on the monotonically-decreasing-to-zero nature of the coefficients, well beyond this assumption ensuring convergence of the Fourier series.

The step which uses this monotonicity is sketched in just two lines in Aljančić, Bojanić and Tomić [3], over pp. 110–111, and even less briefly by Yong [16], and with much detail omitted by Samorodnitsky [14]. We feel that it is worthwhile to present it completely here for the cosine series, since one of our aims is a clarifying synthesis of previous writings.

We eventually need to obtain a suitable bound for $\sum_2 = \sum_{k=r+1}^{\infty} a(k)\cos kx$. We derive a slightly more general result. First note the trigonometric identity:

$$\sum_{k=0}^{n} \cos kx = \cos \frac{nx}{2} \sin \frac{(n+1)x}{2} \Big/ \sin \frac{x}{2}.$$

Hence for $j = r, r+1, \ldots, n$, for any $j \leqslant n$ and small $x > 0$

(3.1)
$$\left| \sum_{j+1}^{n} \cos kx \right| = \left| \cos \frac{nx}{2} \sin \frac{(n+1)x}{2} \Big/ \sin \frac{x}{2} - \cos \frac{jx}{2} \sin \frac{(j+1)x}{2} \Big/ \sin \frac{x}{2} \right| \leqslant \frac{2}{\sin \frac{x}{2}}.$$

Next note the Abelian identity

$$(3.2) \qquad \sum_{k=r+1}^{n} f_k g_k = f_r \sum_{k=r+1}^{n} g_k + \sum_{j=r}^{j=n-1} (f_{j+1} - f_j) \sum_{k=j+1}^{n} g_k.$$

This identity can be verified by interchange of order of summation on the right-hand side. It is the summation analogue of integration by parts.

Thus from (3.2)

$$(3.3) \qquad \left| \sum_{k=r+1}^{n} f_k g_k \right| \leqslant |f_r| \left| \sum_{k=r+1}^{n} g_k \right| + \sum_{j=r}^{j=n-1} |f_{j+1} - f_j| \left| \sum_{k=j+1}^{n} g_k \right|.$$

Now put $f_k = a(k)$, $g_k = \cos kx$, recalling that $a(k) \downarrow 0$.

So, from (3.3) and (3.1), for $0 < x < \pi$,

$$\left| \sum_{k=r+1}^{n} a(k) \cos kx \right| \leqslant |a(r)| \left| \sum_{k=r+1}^{n} \cos kx \right| + \sum_{j=r}^{n-1} |a(j) - a(j+1)| \left| \sum_{k=j+1}^{n-1} \cos kx \right|$$

$$\leqslant a(r) \frac{2}{\sin \frac{x}{2}} + \sum_{j=r}^{j=n-1} (a(j) - a(j+1)) \frac{2}{\sin \frac{x}{2}}$$

$$= \{a(r) + (a(r) - a(n))\} \frac{2}{\sin \frac{x}{2}} \leqslant \frac{4a(r)}{\sin \frac{x}{2}} \leqslant \frac{4\pi a(r)}{x}$$

since $y/ \sin y \leqslant \pi/2$, $0 < y < \pi/2$. $\qquad \square$

The difference in the approaches in Theorem 2.1 from that in Theorems 1.2 (ABT) and 2.3 (Y) is that the Zygmund assumption implies monotonic approach to zero of $x^{-\alpha} L(x)$ from the outset, and avoids the need in the Theorem 1.2 (ABT) approach of the assumption of of monotonic approach to zero of the $\{a(n)\}$ and the use of the properties that $L^l(x) \sim L(x) \sim L^u(x)$, $x \to \infty$, where for $\sigma > 0$:

$$L^l(x) = x^{-\sigma} \inf_{t \geqslant x} (t^\sigma L(t)), \;\; L^u(x) = x^\sigma \sup_{t \geqslant x} (t^{-\sigma} L(t)).$$

Even so, it seems to us that a *direct sequential condition* on the coefficients of the Fourier series $\{a(n)\}$ is preferable to one involving a function $L(x)$ defined for all x sufficiently large, and that is the purpose of our main result, Theorem 3.1.

THEOREM 3.1. *Suppose* $0 < \alpha < 1$. *If* $a(n) > 0$, *then*

$$(3.4) \quad n \left\{ \frac{a(n-1)}{a(n)} - 1 \right\} \to \alpha, \;\; n \to \infty$$

$$\Rightarrow f(x) \sim \frac{\pi}{2\Gamma(\alpha) \cos(\pi\alpha/2)} x^{\alpha-1} L(1/x), \;\; x \to 0 + .$$

where $f(x)$ *is defined by* (1.1), *and* $L(x)$ *is an SVF.*

COROLLARY 3.1. *If additionally we assume that the sequence* $\{a(n)\}$ *is convex on* $n \geqslant 1$, *then* $f(x) \geqslant 0$, $-\pi < x < \pi$.

Next, Theorem 2.2 is elegant in that it imposes conditions only on the function $f(x)$. That a continuously differentiable positive function $L(x)$ is slowly varying in the Zygmund sense is easily verifiable through the practical sufficient condition:

$xL^{'}(x)/L(x) \to 0$, $x \to \infty$. On the other hand, the implication from right to left in Theorems 1.2 and 2.3 involves a minimal prior assumption that $a(n) \downarrow 0$, which however avoids the assumption of bounded variation on finite intervals of Theorem 2.2. Yong [**16**] imitates the complex technical methodology of Aljančić, Bojanić and Tomić [**3**] to prove this part of Theorem 2.3, but a proof can be more simply achieved by actually using the results of that 1956 paper in the way indicated in Aljančić, Bojanić and Tomić [**2**] which we have described at (1.4), providing we also assume convexity of $\{a(n)\}$. It is gratifying that the same methodology can be used as for Theorem 3.1, and that methodology is the purpose of Lemma 4.1 of the next section.

4. Preliminary Lemmas

We pursue the lead given by Aljančić, Bojanić and Tomić [**2**] in (1.4).

LEMMA 4.1. *For $n \geqslant 2$,*

(4.1)
$$\sin(x)(\frac{1}{2}a(0) + \sum_{k=1}^{n} a(k)\cos(kx)) = \sum_{k=1}^{n+1} b(k)\sin(kx)$$

where

$$b(k) = \frac{1}{2}(a(k-1) - a(k+1)), \quad 1 \leqslant k \leqslant n-1,$$
$$b(n) = \frac{1}{2}a(n-1), \quad b(n+1) = \frac{1}{2}a(n).$$

PROOF. Let

$$f_n(x) = \frac{1}{2}a(0) + \sum_{k=1}^{n} a(k)\cos(kx), \quad n \geqslant 2.$$

Now using the identity for $k = 0, 1, 2, \dots$

$$2\sin(x)\cos(kx) = \sin((k+1)x) - \sin((k-1)x),$$

it follows that we may write

$$\sin(x)f_n(x) = \frac{1}{2}a(0)\sin(x) + \sum_{k=1}^{n} a(k)\cos(kx)\sin(x)$$

$$= \frac{1}{2}\Big\{a(0)\sin(x) + \sum_{k=1}^{n} a(k)\sin((k+1)x) - \sum_{k=1}^{n} a(k)\sin((k-1)x)\Big\}$$

$$= \frac{1}{2}\Big\{a(0)\sin(x) + \sum_{h=2}^{n+1} a(h-1)\sin(hx) - \sum_{h=1}^{n-1} a(h+1)\sin(hx)\Big\}$$

$$= \frac{1}{2}\Big\{\sum_{h=1}^{n+1} a(h-1)\sin(hx) - \sum_{h=1}^{n-1} a(h+1)\sin(hx)\Big\} = \sum_{k=1}^{n+1} b(k)\sin(kx)$$

so (4.1) is proved. □

LEMMA 4.2. *Assume that $a(n) \downarrow 0$, and that*

$$(4.2) \qquad b(n) = \frac{1}{2}(a(n-1) - a(n+1)), \quad n = 1, 2, 3, \dots$$

Then the $\sum b(n)$ is a convergent series of positive numbers, and

$$(4.3) \qquad \sum_{k=n+1}^{\infty} b(k) \leqslant a(n) \leqslant \sum_{k=n}^{\infty} b(k), \quad n = 1, 2, 3 \dots$$

PROOF. From (4.2) we have for

$$2 \sum_{k=n}^{m} b(k) = \sum_{k=n}^{m} (a(k-1) - a(k+1)) = \sum_{k=n-1}^{m-1} a(k) - \sum_{k=n+1}^{m+1} a(k)$$
$$= a(n-1) + a(n) - a(m) - a(m+1).$$

As $m \to \infty$, we obtain

$$(4.4) \qquad a(n-1) + a(n) = 2 \sum_{k=n}^{\infty} b(k),$$

and since $a(n-1) \geqslant a(n)$ it follows that

$$(4.5) \qquad a(n) \leqslant \sum_{k=n}^{\infty} b(k).$$

Replacing n by $n+1$ in (4.4), we obtain $a(n) + a(n+1) = 2 \sum_{k=n+1}^{\infty} b(k)$. Since $a(n+1) \leqslant a(n)$ we see that

$$(4.6) \qquad a(n) \geqslant \sum_{k=n+1}^{\infty} b(k).$$

Now (4.3) follows from inequalities (4.5) and (4.6). $\qquad \square$

LEMMA 4.3. *If $L(n)$ is a slowly varying sequence, $\alpha > 0$ and*

$$C(n) = \sum_{k=n}^{\infty} k^{-\alpha-1} L(k), \ n = 1, 2, 3, \dots$$

then

$$(4.7) \qquad C(n) \sim \frac{1}{\alpha} n^{-\alpha} L(n), \quad n \to \infty.$$

PROOF. We need the following property of regularly varying sequences (Bojanić and Seneta [**7**, Theorem 5, p. 96]. For any $0 < \sigma$ and $n = 1, 2, 3, \dots$ let

$$L^l(n) = n^{-\sigma} \inf_{k \geqslant n}(k^{\sigma} L(k)), \quad L^u(n) = n^{\sigma} \sup_{k \geqslant n}(k^{-\sigma} L(k)).$$

Then

$$(4.8) \qquad L^l(n) \sim L(n), \quad L^u(n) \sim L(n), \quad n \to \infty.$$

Note that the sequences $n^{\sigma} L^l(n)$, $n^{-\sigma} L^u(n)$ are, respectively, monotone increasing and decreasing, although we do not need to use this property.

We show first that

(4.9)
$$\liminf_{n\to\infty}\frac{C(n)}{n^{-\alpha}L(n)}\geqslant\frac{1}{\alpha}.$$

Now, for $0<\sigma$, we have

$$C(n)=\sum_{k=n}^{\infty}k^{-\alpha-\sigma-1}k^{\sigma}L(k)\geqslant\inf_{k\geqslant n}k^{\sigma}L(k)\sum_{k=n}^{\infty}k^{-\alpha-\sigma-1},$$

$$\sum_{k=n}^{\infty}k^{-\alpha-\sigma-1}\geqslant\int_{n}^{\infty}t^{-\alpha-\sigma-1}dt=\frac{1}{\alpha+\sigma}n^{-\alpha-\sigma}.$$

Thus

$$\frac{C(n)}{n^{-\alpha}L(n)}\geqslant\frac{(\inf_{k\geqslant n}k^{\sigma}L(k))n^{-\alpha-\sigma}}{n^{-\alpha}L(n)(\alpha+\sigma)}=\frac{(n^{\sigma}L^{l}(n))n^{-\alpha-\sigma}}{n^{-\alpha}L(n)(\alpha+\sigma)}=\frac{L^{l}(n)}{L(n)(\alpha+\sigma)},$$

and since, from (4.8), $L^{l}(n)\sim L(n)$, $n\to\infty$, and $\sigma>0$ can be made as small as we like, (4.9) follows. Similarly, for $0<\sigma<\alpha$:

$$C(n)=\sum_{k=n}^{\infty}k^{-\alpha+\sigma-1}k^{-\sigma}L(k)\leqslant\sup_{k\geqslant n}k^{-\sigma}L(k)\sum_{k=n}^{\infty}k^{-\alpha+\sigma-1}$$

$$\leqslant\sup_{k\geqslant n}k^{-\sigma}L(k)\int_{n-1}^{\infty}t^{-\alpha-\sigma-1}dt$$

$$=\frac{(n^{-\sigma}L^{u}(n))(n-1)^{-\alpha+\sigma}}{(\alpha-\sigma)}$$

so that

$$\frac{C(n)}{n^{-\alpha}L(n)}\leqslant\frac{(n^{-\sigma}L^{u}(n))(n-1)^{-\alpha+\sigma}}{n^{-\alpha}L(n)(\alpha-\sigma)}=\frac{L^{u}(n)(1-1/n)^{-\alpha+\sigma}}{L(n)(\alpha-\sigma)}.$$

Using (4.8)

(4.10)
$$\limsup_{n\to\infty}\frac{C(n)}{n^{-\alpha}L(n)}\leqslant\frac{1}{\alpha},$$

since $\sigma>0$ can be made arbitrarily small. Thus (4.7) follows from (4.9), (4.10). □

5. Proof of Theorem 3.1

We use the implication from left to right of Theorem 1.2 in the case $1<\beta<2$, noting that this case does not require the prior assumption of monotonicity of $\lambda(n)$.

PROOF. The left-hand side of (3.4) implies that $a(n)\downarrow 0$, $n\geqslant n_{0}$; and that $\{a(n)\}$ is a regularly varying *sequence* of index $-\alpha$, from Theorem 4 of Bojanić and Seneta [7].

Now by Lemma 4.1, letting $n\to\infty$ in (4.1) :

$$\frac{1}{2}a(0)+\sum_{n=1}^{\infty}a(n)\cos(nx)=\frac{\sum_{k=1}^{\infty}b(k)\sin(kx)}{\sin(x)}$$

where

$$kb(k) = \frac{k}{2}(a(k-1) - a(k) + a(k) - a(k+1))$$

$$= \frac{1}{2}\Big\{a(k)k\Big(\frac{a(k-1)}{a(k)} - 1\Big) + a(k+1)k\Big(\frac{a(k)}{a(k+1)} - 1\Big)\Big\}$$

so that

$$\frac{kb(k)}{a(k)} = \frac{1}{2}\Big\{k\Big(\frac{a(k-1)}{a(k)} - 1\Big) + \frac{ka(k+1)}{(k+1)a(k)}(k+1)\Big(\frac{a(k)}{a(k+1)} - 1\Big)\Big\}$$

$$(5.1) \qquad \rightarrow \frac{\alpha}{2} + \lim_{k\to\infty}\frac{1}{2}\frac{ka(k+1)}{(k+1)a(k)}(k+1)\Big(\frac{a(k)}{a(k+1)} - 1\Big)$$

$$(5.2) \qquad = \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha.$$

Here (5.1) and (5.2) follow from the left-hand side of (3.4), since in particular we note that $a(k+1)/a(k) \rightarrow 1$.

Write $a(n) = n^{-\alpha}l(n)$, where $l(n)$ is a slowly varying sequence. Now put $L(x) = \alpha l([x])$, $x \in [1, \infty)$ where $[x]$ denotes, as usual, the integer part of x. Then for any $\lambda > 0$,

$$\frac{L(\lambda x)}{L(x)} = \frac{l([\lambda x])}{l([x])} \rightarrow 1, \ x \rightarrow \infty$$

from the definition of a regularly varying sequence (Bojanić and Seneta [**7**]). Thus $L(\cdot)$ is a slowly varying function in the Karamata sense.

From (5.2)

$$(5.3) \qquad b(k) \sim \alpha a(k)/k = \alpha k^{-\alpha-1}l(k) = k^{-\alpha-1}L(k).$$

Now

$$\sum_{k=1}^{\infty} b(k)\sin(kx)$$

has coefficients $b(k) = \frac{1}{2}(a(k-1) - a(k+1)) > 0$, and $b(k) \rightarrow 0$ since by assumption $a(k) \downarrow 0$, $k \rightarrow \infty$. Further, from (5.3) $b(k) \sim k^{-\beta}L(k)$, where $1 < \beta < 2$, so by Theorem 1.2 (3.4) holds. $\qquad\square$

6. Proof of Theorem 2.3, from right to left, under convexity

We are assuming $a(n) \downarrow 0$, which implies the convergence of (1.1) to $f(x)$, assuming that the sequence $\{a(n)\}$ is convex, and assuming further that

$$f(x) = \frac{1}{2}a(0) + \sum_{n=1}^{\infty} a(n)\cos(nx) \sim \frac{\pi x^{\alpha-1}L(1/x)}{2\Gamma(\alpha)\cos(\pi\alpha/2)}, \quad x \rightarrow 0+.$$

PROOF. Letting $n \rightarrow \infty$ in (4.1) of Lemma 4.1:

$$(6.1) \qquad \frac{1}{2}a(0) + \sum_{n=1}^{\infty} a(n)\cos(nx) = \frac{\sum_{k=1}^{\infty} b(k)\sin(kx)}{\sin(x)}$$

$$(6.2) \qquad \sim \frac{\pi x^{\alpha-1}L(1/x)}{2\Gamma(\alpha)\cos(\pi\alpha/2)}, \quad x \rightarrow 0+$$

where $b(k) = \frac{1}{2}(a(k-1) - a(k+1))$, $1 \leqslant k < \infty$. From (6.1) and (6.2), as $x \to 0+$

$$\sum_{k=1}^{\infty} b(k)\sin(kx) \sim \frac{\sin x}{x} \frac{\pi x^{\alpha} L(1/x)}{2\Gamma(\alpha)\cos(\pi\alpha/2)} \sim \frac{\pi x^{\alpha} L(1/x)}{2\Gamma(\alpha)\cos(\pi\alpha/2)}.$$

Now put $\gamma = \alpha + 1$, so $1 < \gamma < 2$, whence

$$\sum_{k=1}^{\infty} b(k)\sin(kx) \sim \frac{\pi x^{\gamma-1} L(1/x)(\gamma-1)}{2\Gamma(\gamma)\sin(\pi\gamma/2)}.$$

From the additionally assumed convexity of the sequence $\{a(n)\}$, we find that

$$a(k+2) - a(k+1) \geqslant a(k+1) - a(k) \geqslant a(k) - a(k-1)$$

so that

$$a(k+2) - a(k) \geqslant a(k+1) - a(k-1),$$

so that

$$b(k+1) \leqslant b(k).$$

We may now apply the implication from right to left of Theorem 1.2 (ABT) since we have $b(k) > 0, b(k) \downarrow 0$, $k \to \infty$ and $1 < \gamma < 2$. So we have

$$b(n) \sim (\gamma-1)n^{-\gamma}L(n) = \alpha n^{-\alpha-1}L(n), \quad n \to \infty.$$

Next, from Lemma 4.2, since $a(n) \downarrow 0$,

$$(6.3) \qquad \sum_{k=n+1}^{\infty} b(k) \leqslant a(n) \leqslant \sum_{k=n}^{\infty} b(k);$$

and from Lemma 4.3, since $\alpha > 0$:

$$(6.4) \qquad \sum_{k=n}^{\infty} k^{-\alpha-1}L(k) \sim \frac{1}{\alpha}n^{-\alpha}L(n), \quad n \to \infty.$$

Now, for $\epsilon > 0$ and arbitrarily small, and $k, n \geqslant k_0(\epsilon)$

$$(1-\epsilon)\alpha k^{-\alpha-1}L(k) \leqslant b(k) \leqslant (1+\epsilon)\alpha k^{-\alpha-1}L(k),$$

$$(1-\epsilon)\alpha \sum_{k=n}^{\infty} k^{-\alpha-1}L(k) \leqslant \sum_{k=n}^{\infty} b(k) \leqslant (1+\epsilon)\alpha \sum_{k=n}^{\infty} k^{-\alpha-1}L(k).$$

Thus from (6.4), for $n \geqslant n_0(\epsilon)$:

$$(1-\epsilon)^2 n^{-\alpha}L(n) \leqslant \sum_{k=n}^{\infty} b(k) \leqslant (1+\epsilon)^2 n^{-\alpha}L(n).$$

Since $\epsilon > 0$ is arbitrarily small, using $\limsup$ and $\liminf$ we have:

$$\lim \frac{\sum_{k=n}^{\infty} b(k)}{n^{-\alpha}L(n)} = 1.$$

Now, from (6.3), since $L(n+1)/L(n) \to 1$, $n \to \infty$, it follows that

$$a(n) \sim n^{-\alpha}L(n),$$

as required. $\qquad\qquad\square$

7. Obituary. Ranko Bojanić (1924–2017)

Ranko Bojanić, the last of the three authors of the fundamental paper Aljančić, Bojanić and Tomić [**3**], died on February 21, 2017, in Columbus, Ohio, USA, while this sequel to it was in preparation. This current work was motivated by the importance of its non-probabilistic subject matter in the theory of second order stationary stochastic processes. This was exemplified in the paper of Finlay, Fung and Seneta [**8**] and by the imminent appearance during our investigations, of the treatise of Samorodnitsky [**14**].

Ranko asked me (ES) to append an obituary to this paper in case its preparation was not complete at the time of his death, and to submit the paper to the *Publications de l'Institut Mathématique (Beograd)* where the paper [**3**] and a number of others of his early career, had appeared.

Ranko Bojanić was born in Yugoslavia on November 12, 1924. He was raised in its capital, Beograd (Belgrade), the son of the late Gojko and Jelena Bojanić. He was awarded the equivalent of a B.S. (B.Sc.) degree by the University of Beograd in September 1950, and a PhD by the Mathematics Institute, Serbian Academy of Sciences, in January 1953. His first paper was published in 1949, followed by the first in *Publications de l'Institut Mathématique (Beograd)* in 1950. His doctoral thesis: Asimptotika resenja jedne klase implicitnih diferencijalnih jednačina prvog reda, was published in 1952 in the *Zbornik radova Matematičkog Instituta* **2**, 37–142.

One of the strong group of disciples of Jovan Karamata, in the last years of his life in email correspondence with me, Ranko often spoke of how he had loved "talking problems through" in the company of his peers, especially Aljančić, Tomić, and Bajšanski, in those days of youth, and when possible later, after he had settled in the US.

He was at the the University of Skoplje from 6/54 to 6/56, and at University of Beograd from 5/56 to 6/58. His address in the three-author 1956 paper [**3**], the basis of our joint preceding study, was Skoplje.

The following passage, from an email to me of July 13, 2011, is particularly relevant not only to his subsequent career, but also to our joint study.

> I think that at that time we [three] were tired of working with trigonometric series. Just look at the proofs of theorems in our joint [1956] French paper. We also thought that working on cosine series will bring no essentially new and different results....
>
> I remember that in 1957 Aljančić and I were invited to a conference in Varenna, Italy, where Zygmund was the principal speaker. He gave a series of talks on kernel functions. He was very much interested in the results of our French paper. At that time the second edition of his Trigonometric Series was being prepared for printing and he asked me to read printed sheets of Chapter V during the conference. In addition to corrected misprints and errors, I did provide a simpler proof of Theorem 1.3 of that chapter and made several suggestions in Section 2 which Zygmund adopted. Zygmund appreciated very much my help and wrote very good letters

of recommendation on my behalf to Chandrasekharan in Bombay and Szegö in Palo Alto. (Karamata did also write such letters to Chandrasekharan and Szegö.) This is how I came to the School of Mathematics of the Tata Institute of Fundamental Research in June of 1958 and to Stanford University in the Fall of 1959.

Ranko was at the Tata Institute as a Visitor in the School of Mathematics from 1958 to 1960, at Stanford University 10/59 to 10/60, then at the University of Notre Dame 10/60 to 10/63. He came to Ohio State University (OSU) in 10/63 as Associate Professor, and was made full Professor in 10/66. He formally retired in 1995, and was made Professor Emeritus. By the time of his death, he had thus spent more than 50 years at OSU, for which he had deep affection.

He became renowned in the field of Approximation Theory, and was an Associate Editor of the *Journal of Approximation Theory*, which will shortly carry an obituary, including extensive recollections by his former students, colleagues and mathematical collaborators, including this author.

An almost complete list of his publications and collaborators is available on MathSciNet. He and I collaborated on two papers hitherto: [**6, 7**]. Both have found a number of descendants.

He will be greatly missed by his loving wife of 58 years, Olga Bojanić, by their children Mira and Ivan Bojanić, granddaughters, nephews, extended family members, colleagues, and many friends.

He was a warm, very social human being, emotional and sentimental in the best Slavic tradition. The world is, indeed, the poorer for his passing.

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