PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 104(118) (2018), 43–51 DOI: https://doi.org/10.2298/PIM1818043K

ON THE STRUCTURE OF RANDOM HYPERGRAPHS

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Abstract. Let $\mathcal{H}_n$ be a countable random n-uniform hypergraph for $n > 2$, and $\mathbb{P}(\mathcal{H}_n) = \{f[\mathcal{H}_n] : f\colon \mathcal{H}_n \to \mathcal{H}_n$ is an embedding$\}$. We prove that a linear order L is isomorphic to the maximal chain in the partial order $\langle \mathbb{P}(\mathcal{H}_n) \cup \{\emptyset\}, \subset \rangle$ if and only if L is isomorphic to the order type of a compact set of reals whose minimal element is nonisolated.

1. Introduction

1.1. Background and the statement of the result. We completely characterize chains of isomorphic substructures of the Fraïssé limit of finite n-uniform hypergraphs for each $n > 1$, thus generalizing some results from [**8**] and [**7**] to higher dimensions. Fraïssé theory, the systematic study of ultrahomogeneous universal structures, was initiated in the mid 1950's by Roland Fraïssé [**2**]. Typical examples of Fraïssé limits are the rational line $\langle \mathbb{Q}, < \rangle$ and the countable random graph (i.e., Rado graph). A particularly active research area is the investigation of the automorphism groups of these structures (see [**4**] for the most notable example). Besides that, there has been interest in considering the embeddings of an ultrahomogeneous structure into itself (for a relational structure $\mathbb{X}$, denote $\mathrm{Emb}(\mathbb{X}) = \{f\colon \mathbb{X} \to \mathbb{X} : f$ is an embedding$\}$). See [**1**] for some results on the self-embeddings of ultrahomogeneous n-uniform hypergraphs or [**10**] for one of the most prominent results concerning self-embeddings of ultrahomogeneous structures. In this context, one usually investigates the set of isomorphic substructures of a structure $\mathbb{X}$, denoted $\mathbb{P}(\mathbb{X}) = \{f[X] : f \in \mathrm{Emb}(\mathbb{X})\} = \{A \subset \mathbb{X} : A \cong \mathbb{X}\}$.

The set $\mathbb{P}(\mathbb{X})$ is naturally ordered by inclusion, and we will be interested in order types of chains in these partial orders where $\mathbb{X}$ is the countable random n-uniform hypergraph (for all $n \geqslant 2$). Some recent results related to the ones in this paper can be found in [**6, 9, 8**]. The main result of this paper is the following.

2010 *Mathematics Subject Classification:* 03C15, 05C65, 03C50, 06A05.

Key words and phrases: Fraïssé theory, k-uniform hypergraph, maximal chain, isomorphic substructure, ultrahomogeneous structure, positive family.

Supported by the Ministry of Education and Science of Serbia, grant ON174006.

Communicated by Miloš Kurilić.

THEOREM 1.1. *Let $\mathcal{H}_n$, $n > 1$, be a countable random n-uniform hypergraph. Then a linear order L is isomorphic to a maximal chain in the partial order $\langle \mathbb{P}(\mathcal{H}_n) \cup \{\emptyset\}, \subset\rangle$ if and only if it is isomorphic to the order type of a compact set of reals whose minimum is nonisolated.*

1.2. Preliminaries. In this paper n will be reserved for natural numbers and $|X|$ denotes the cardinality of a set X, in particular, ω is the cardinality of a countably infinite set. For a set X and $n \geqslant 1$, by $[X]^n$ we denote the set of all n-element subsets of X, i.e., $[X]^n = \{y \subset X : |y| = n\}$. Also, $[X]^{<\omega}$ denotes the set of all finite subsets of X. If f maps A into B, then $f[A] = \{f(x) : x \in A\}$. The power set of X is denoted by $P(X)$. If J is a subset of the real line and $x \in \mathbb{R}$, then we denote $(-\infty, x)_J = (-\infty, x) \cap J$ and $(-\infty, x]_J = (-\infty, x] \cap J$.

A *relational structure* $\mathbb{X} = \langle X, \{\rho_i : i \in I\}\rangle$ consists of a set X and relations ρ_i $(i \in I)$. Often, when there can be no confusion, we do not make distinction between denoting the structure $\mathbb{X}$ and the underlying set X. We say that a structure $\mathbb{Y} = \{Y, \{\sigma_i : i \in I\}\}$ is a *substructure* of $\mathbb{X}$ if and only if $Y \subset X$ and for each $i \in I$ we have $\sigma_i = Y^{\mathrm{ar}(\rho_i)} \cap \rho_i$. A mapping $f : \mathbb{X} \to \mathbb{Y}$ is an *embedding* of a relational structure $\mathbb{X}$ into a relational structure $\mathbb{Y}$ of the same signature (denoted $f : \mathbb{X} \hookrightarrow \mathbb{Y}$) if and only if f is 1-1 and it holds $(k_i = \mathrm{ar}(\rho_i))$

$$\forall i \in I \; \forall \langle a_1, \ldots, a_{k_i}\rangle \in X^{k_i} \; (\langle a_1, \ldots, a_{k_i}\rangle \in \rho_i \Leftrightarrow \langle f(a_1), \ldots, f(a_{k_i})\rangle \in \sigma_i) .$$

We say that a relational structure $\mathbb{X}$ is *ultrahomogeneous* if and only if any isomorphism ϕ between finite substructures of $\mathbb{X}$ can be extended to an automorphism of $\mathbb{X}$. Further, we say that a relational structure $\mathbb{X}$ is *universal* for a class of structures $\mathcal{K}$ if and only if for each $\mathbb{K} \in \mathcal{K}$ there is an embedding $f : \mathbb{K} \to \mathbb{X}$. We use the following characterization of ultrahomogeneity (see [**3**, Theorem 12.1.2.]).

LEMMA 1.1. *Let $\mathbb{X}$ be a countable relational structure. Then $\mathbb{X}$ is ultrahomogeneous if and only if for any finite substructure F of $\mathbb{X}$, any embedding $f : F \to \mathbb{X}$, and any element $y \in \mathbb{X} \setminus F$, there exists an embedding $g : F \cup \{y\} \to \mathbb{X}$ which is an extension of f.*

Now we mention a few notions related to order theory. We say that a linear order is *complete* if and only if it is Dedekind-complete and has minimum and maximum (the reader may find this definition of completeness nonstandard, but we use it in order to shorten some statements). We say that a linear order L is *boolean* if and only if it is complete and has dense jumps, i.e., complete and for any $x, y \in L$ if $x < y$, then there are $s, t \in L$ such that $x \leqslant s < t \leqslant y$ and $(s, t)_L = \emptyset$.

We will also need the notions of a filter and a set dense in a partial order. Let $\langle P, \leqslant\rangle$ be a partial order, a set $D \subset P$ is *dense* in P if for any $p \in P$ there is $q \in D$ such that $q \leqslant p$. A set $G \subset P$ is a *filter* in P if and only if for all $x, y \in G$ there is $q \in G$ such that $q \leqslant x, y$ (i.e., elements of G are pairwise compatible in G) and for any $x \in G$ if $y > x$, then also $y \in G$. The following is a well-known fact.

LEMMA 1.2 (Rasiowa–Sikorski). *Let $\langle P, \leqslant\rangle$ be a partially ordered set and $\mathcal{D} = \{D_n : n \in \mathbb{N}\}$ a countable family of sets dense in P. Then there is a filter G in P such that $G \cap D_n \neq \emptyset$, for all $n \in \mathbb{N}$.*

1.3. Maximal chains. First note that a linear order L is isomorphic to the order type of a compact (nowhere dense compact) set of reals whose minimum is nonisolated if and only if it is complete (boolean), $\mathbb{R}$ embeddable and has a nonisolated minimum. For a proof of this fact see [**5**].

Recall that a positive family on a countable set X is a family $\mathcal{P} \subset P(X)$ satisfying (see also [**5**]):

(P1) $\emptyset \notin \mathcal{P}$;
(P2) $A \in \mathcal{P} \wedge B \in [A]^{<\omega} \Rightarrow A \setminus B \in \mathcal{P}$;
(P3) $A \in \mathcal{P} \wedge A \subset B \subset X \Rightarrow B \in \mathcal{P}$;
(P4) $\exists A \in \mathcal{P}\ |X \setminus A| = \omega$.

For example, each nonprincipal ultrafilter on ω is a positive family on ω. Also, the family of all dense subsets of the rational line $\mathbb{Q}$ is a positive family on $\mathbb{Q}$. Positive families play an important role in investigation of maximal chains in the posets of the form $\langle \mathbb{P}(\mathbb{X}) \cup \{\emptyset\}, \subset\rangle$. Namely, Theorem 2.2. in [**9**] states that if there is a positive family $\mathcal{P}$ on X such that $\mathcal{P} \subset \mathbb{P}(\mathbb{X})$, then for each countable and complete linear order L whose minimum is nonisolated, there is a maximal chain in $\langle \mathbb{P}(\mathbb{X}) \cup \{\emptyset\}, \subset\rangle$ isomorphic to L. This allows us to reformulate Theorem 3.2. from [**8**] in the following slightly weaker manner.

THEOREM 1.2. *Let $\mathbb{X}$ be a countable relational structure and $\langle \mathbb{Q}, <\rangle$ the rational line. If there exist a partition $\{J_m : m \in \omega\}$ of $\mathbb{Q}$ and a structure with the domain $\mathbb{Q}$ of the same signature as $\mathbb{X}$ such that:*

(i) *J_0 is a dense subset of $\langle \mathbb{Q}, <\rangle$,*
(ii) *J_m $(m \in \omega)$ are coinitial subsets of $\langle \mathbb{Q}, <\rangle$,*
(iii) *$(-\infty, x)_{J_0} \subset A \subset (-\infty, x)_{\mathbb{Q}}$ implies $A \cong \mathbb{X}$ for $x \in \mathbb{R} \cup \{\infty\}$,*
(iv) *$(-\infty, q]_{J_0} \subset C \subset (-\infty, q]_{\mathbb{Q}}$ implies $C \ncong \mathbb{X}$ for $q \in J_0$,*
(v) *there is a positive family $\mathcal{P}$ on $\mathbb{X}$ such that $\mathcal{P} \subset \mathbb{P}(\mathbb{X})$,*

then for each $\mathbb{R}$-embeddable complete linear order L with $\min L$ nonisolated, there is a maximal chain in $\langle \mathbb{P}(\mathbb{X}) \cup \{\emptyset\}, \subset\rangle$ isomorphic to L.

The next result, proved in [**9**], shows that ultrahomogeneous structures provide a nice framework for investigating maximal chains of their isomorphic substructures.

THEOREM 1.3. *Let $\mathbb{X}$ be a countable ultrahomogeneous structure of an at most countable relational language which contains at least one nontrivial isomorphic substructure, i.e., $\mathbb{P}(\mathbb{X}) \neq \{X\}$. Then for each linear order L the implication* (1) $\Rightarrow$ (2) *is true, where*

(1) *L is isomorphic to a maximal chain in the poset $\langle \mathbb{P}(\mathbb{X}) \cup \{\emptyset\}, \subset\rangle$;*
(2) *L is a complete $\mathbb{R}$-embeddable linear order with $\min L$ nonisolated.*

2. Random hypergraphs

For $n \geqslant 2$, an n-uniform hypergraph is a relational structure $\langle X, \rho\rangle$, satisfying $\operatorname{ar}(\rho) = n$ and such that $\langle x_0, \dots, x_{n-1}\rangle \in \rho$ implies $x_i \neq x_j$ for all $i \neq j$ in n and $\langle x_{\pi(0)}, \dots, x_{\pi(n-1)}\rangle \in \rho$ for all permutations π of n (see [**3**]). Note that this is equivalent to saying that n-uniform hypergraph is a pair $\langle X, \rho\rangle$ where X is a set and $\rho \subset [X]^n$, so we will sometimes refer to the first formulation, and sometimes, when it is more convenient, to the second. For example, if, in addition, $\mathbb{Z} = \langle Z, \sigma \rangle$ is an n-uniform hypergraph and $g : Z \to X$ is 1-1, then $g \in \mathrm{Emb}(\mathbb{Z}, \mathbb{X})$ iff

$$\forall K \in [Z]^n \ (K \in \sigma \ \Leftrightarrow \ g[K] \in \rho). \tag{2.1}$$

Recall that the class of countably many (up to isomorphism) finite structures is a Fraïssé class (see [**3**]) if it is hereditary, satisfies joint embedding and amalgamation property and contains structures of arbitrary large finite cardinality. It is well known that the class $\mathcal{K}_n$ of finite n-uniform hypergraphs ($n \geqslant 2$) is a Fraïssé class, hence the famous Fraïssé's theorem states there there is a unique up to isomorphism countable ultrahomogeneous relational structure whose age is exactly $\mathcal{K}_n$ (the age of a relational structure is the class of all of its finitely generated substructures).

Definition 2.1. For $n \geqslant 2$, the countable ultrahomogeneous n-uniform hypergraph universal for all finite n-uniform hypergraphs is called the countable random n-uniform hypergraph.

The countable random n-uniform hypergraph will be denoted $\mathcal{H}_n$. The following lemma gives a useful reformulation of the definition of the countable random n-uniform hypergraph. Note also that Fraïssé's theorem states that the countable random n-uniform hypergraph is universal even for the class of all countable n-uniform hypergraphs.

Lemma 2.1. *If $n \geqslant 2$, $|X| = \omega$, $\mathbb{X} = \langle X, \Gamma \rangle$ for $\Gamma \subset [X]^n$, then*

$$\forall A \in \bigcup_{k \geqslant n-1} [X]^k \ \forall \mathcal{B} \subset [A]^{n-1} \ \exists q \in X \smallsetminus A \ \forall C \in [A]^{n-1} \ \left(\{q\} \cup C \in \Gamma \Leftrightarrow C \in \mathcal{B} \right),$$

if and only if $\mathbb{X} \cong \mathcal{H}_n$.

Proof. First we prove that if $\mathbb{X} \cong \mathcal{H}_n$, then $\mathbb{X}$ satisfies the assumption in the lemma. Without any loss of generality, we can work with $\mathcal{H}_n$ itself. Suppose that a finite set $A \subset \mathcal{H}_n$ of size $k \geqslant n-1$, and $\mathcal{B} \subset [A]^{n-1}$ are given. Take any $x \in \mathcal{H}_n \smallsetminus A$, and consider the following set $\rho = \{C \cup \{x\} : C \in \mathcal{B}\} \cup \Gamma \restriction A$. Then it is clear that $\rho \subset [A \cup \{x\}]^n$ and $\Gamma \restriction A = \rho \restriction A$, so $\langle A \cup \{x\}, \rho \rangle$ is a finite n-uniform hypergraph. Since $\mathcal{H}_n$ is universal for all finite n-unifrom hypergraphs, there is $E \subset \mathcal{H}_n$ and an isomorphism $f : \langle A \cup \{x\}, \rho \rangle \to \langle E, \Gamma \restriction E \rangle$. Let y denote the single point in the set $E \smallsetminus f[A]$ and let $g = f \restriction A$. By ultrahomogeneity of $\mathcal{H}_n$, Lemma 1.1 applied to $f[A] = E \smallsetminus \{y\}$, g^{-1}, and y, gives us an embedding $h : E \to \mathcal{H}_n$ which is an extension of g^{-1}. Denote $q = h(y)$, and note that since h is an isomorphism and $h[f[A]] = A$, it must be the case that $q \notin A$. Hence, for $C \in [A]^{n-1}$

$$\{q\} \cup C \in \Gamma \Leftrightarrow h^{-1}[C \cup \{q\}] \in \Gamma \Leftrightarrow \{y\} \cup g[C] \in \Gamma \Leftrightarrow \{x\} \cup C \in \rho \Leftrightarrow C \in \mathcal{B},$$

as required.

Next, we have to prove that the n-uniform hypergraph $\mathbb{X}$ satisfying the assumption in the lemma is ultrahomogeneous and universal for all finite n-uniform hypergraphs. We will be using the following claim.

Claim 2.1. *If $\mathbb{Y} = \langle Y, \sigma \rangle$ is an n-uniform hypergraph, $F \in [Y]^{<\omega}$, $y \in Y \smallsetminus F$, and $f : F \hookrightarrow \mathbb{X}$, then there is $a \in X \smallsetminus f[F]$ such that $g := f \cup \{\langle y, a \rangle\} : F \cup \{y\} \hookrightarrow \mathbb{X}$.*

PROOF. Let $\mathcal{B} = \{B \in [f[F]]^{n-1} : \{y\} \cup f^{-1}[B] \in \sigma\}$. Then by the assumption of the lemma, there is $a \in X \setminus f[F]$ such that:

$$\forall B \in [f[F]]^{n-1} \ (\{a\} \cup B \in \rho \Leftrightarrow B \in \mathcal{B}).$$

Now, defining $Z = F \cup \{y\}$ we prove (2.1) for $g := f \cup \{\langle y, a\rangle\}$. Take any $K \in [Z]^n$. There are two possibilities: either $y \in K$ or $y \notin K$. If $y \notin K$, since f is an embedding and g is an extension of f, we clearly have that $K \in \sigma$ if and only if $g[K] = f[K] \in \rho$. If $y \in K$, then $K = C \cup \{y\}$ for some $C \in [F]^{n-1}$. Now using the fact that f is one-to-one we get

$$K \in \sigma \Leftrightarrow C \cup \{y\} \in \sigma \Leftrightarrow f[C] \in \mathcal{B} \Leftrightarrow \{a\} \cup f[C] \in \rho \Leftrightarrow f[K] \in \rho,$$

as required. So g is an embedding extending f. □

Now we prove the ultrahomogeneity of $\mathbb{X}$ using Lemma 1.1. Let F be any finite substructure of $\mathbb{X}$, $f : F \hookrightarrow \mathbb{X}$ any embedding, and $y \in X \setminus F$ arbitrary. Applying Claim 2.1 to $Y := F \cup \{y\}$, y, and f, we obtain $a \in X \setminus f[F]$ and embedding $g := f \cup \{\langle y, a\rangle\} : F \cup \{y\} \hookrightarrow \mathbb{X}$ exactly as required in Lemma 1.1. Thus $\mathbb{X}$ is ultrahomogeneous.

In order to finish the proof, we also have to show that $\mathbb{X}$ is universal for all finite n-uniform hypergraphs. We show that it is in fact universal for all at most countable n-uniform hypergraphs. Let $\mathbb{Y} = \langle Y, \sigma\rangle$ be an arbitrary at most countable n-uniform hypergraph. Fix an enumeration $Y = \{y_1, y_2, \dots\}$. If $|Y| < n$ then any 1-1 mapping $h : Y \to X$ is an embedding because in that case $[Y]^n = \emptyset$, and that implies $\sigma \cap [Y]^n = \rho \cap [h[Y]]^n = \emptyset$. If $|Y| \geqslant n$, then we define the embedding f by induction on l. First, pick any elements $x_1, \dots, x_{n-1} \in X$ and define $f_{n-1}(y_i) = x_i$ for $1 \leqslant i \leqslant n-1$. Note that f_{n-1} is an embedding according to the previous considerations in this paragraph. Assume that an embedding $f_l : \{y_1 \dots, y_l\} \to \mathbb{X}$ is given for $n-1 \leqslant l$. Applying Claim 2.1 to Y, $F = \{y_1, \dots, y_l\}$, $y_{l+1} \in Y \setminus F$ and f_l, we obtain $a \in X \setminus f[F]$ and an embedding $f_{l+1} := f_l \cup \{\langle y_{l+1}, a\rangle\} = F \cup \{y_{l+1}\} \hookrightarrow \mathbb{X}$. In this way an increasing sequence of embeddings $f_l : \{y_1, \dots, y_l\} \hookrightarrow \mathbb{X}$ is obtained, and it is clear that $f = \bigcup_{n-1 \leqslant l < |Y|} f_{l+1}$ is an embedding of $\mathbb{Y}$ into $\mathbb{X}$. Thus $\mathbb{X}$ is universal and the lemma is proved. □

3. Main theorem

In this section we prove the central result of this note by constructing a specific representation of $\mathcal{H}_n$ in order to easily locate its isomorphic substructures. So $n > 1$ is fixed for the rest of the paper. We essentially plan to use Theorem 1.2, so pick any partition $[0, 1) \cap \mathbb{Q} = \bigcup_{m \in \omega} J'_m$ into countably many sets, all of them being dense in $[0, 1) \cap \mathbb{Q}$. Now define the sets $J_m = J'_m + \mathbb{Z}$ for every $m \in \omega$. It is clear that the family $\{J_m : m \in \omega\}$ is a partition of the rational line into dense sets such that if $x \in J_m$, then $x + k \in J_m$ for any $k \in \mathbb{Z}$ and $m \in \omega$. In order to simplify some further statements, for $a \in \mathbb{Q}$ denote $M(a) = \{a - i : i \in n\}$.

Let $\mathbb{P}$ be the set of pairs $p = \langle H_p, \Gamma_p\rangle$ such that

$$H_p \in [\mathbb{Q}]^{<\omega} \ \wedge \ \Gamma_p \subset [H_p]^n, \tag{3.1}$$

$$\forall a,b \in H_p \ \left[\left(M(a) \subset H_p \ \wedge \ \forall B \in [M(a)]^{n-1} \ B \cup \{b\} \in \Gamma_p\right) \Rightarrow b > a\right]. \tag{3.2}$$

For $p_1, p_2 \in \mathbb{P}$, let

$$p_1 \leqslant p_2 \Leftrightarrow H_{p_1} \supset H_{p_2} \ \wedge \ \Gamma_{p_1} \cap [H_{p_2}]^n = \Gamma_{p_2}. \tag{3.3}$$

Thus, each element of $\mathbb{P}$ is a finite n-uniform hypergraph, $p_1 \leqslant p_2$ if and only if p_2 is a substructure of p_1.

Lemma 3.1. *The set $\mathbb{P}$ with the relation $\leqslant$ on $\mathbb{P}$ is a partially ordered set.*

Proof. The reflexivity is clear. For the transitivity, notice that if $p_1 \leqslant p_2$ and $p_2 \leqslant p_3$, we have $H_{p_3} \subset H_{p_2} \subset H_{p_1}$ and $\Gamma_{p_1} \cap [H_{p_2}]^n = \Gamma_{p_2}$ and $\Gamma_{p_2} \cap [H_{p_3}]^n = \Gamma_{p_3}$, and it is easy to see that $\Gamma_{p_1} \cap [H_{p_3}]^n = \Gamma_{p_3}$. To see that $\leqslant$ is antisymmetric notice that if $p_1 \leqslant p_2$ and $p_2 \leqslant p_1$, then from $H_{p_1} \subset H_{p_2} \subset H_{p_1}$ follows $H_{p_1} = H_{p_2}$ and then $\Gamma_{p_1} = \Gamma_{p_1} \cap [H_{p_1}]^n = \Gamma_{p_1} \cap [H_{p_2}]^n = \Gamma_{p_2}$, or equivalently $p_1 = p_2$. □

Lemma 3.2. *If $A \in \bigcup_{k \geqslant n-1} [\mathbb{Q}]^k$, $\mathcal{B} \subset [A]^{n-1}$, and $m \in \mathbb{N}$, then the set $D_{\mathcal{B}}^{A,m}$ of all $p \in \mathbb{P}$ satisfying $A \subset H_p$ and*

$$\exists q \in (\max A, \max A + 1/m) \cap J_0 \cap H_p \ \forall C \in [A]^{n-1} \ \ (\{q\} \cup C \in \Gamma_p \Leftrightarrow C \in \mathcal{B}) \tag{3.4}$$

is dense in $\mathbb{P}$.

Proof. Take any $p \in \mathbb{P}$ and assume that $A \subset H_p$ (if not, define $H_{p_2} = H_p \cup A$ and $\Gamma_{p_2} = \Gamma_p$ and continue with p_2 instead p). Because J_0 is dense in $\mathbb{Q}$, there is

$$q \in \left((\max A, \max A + 1/m) \cap J_0\right) \smallsetminus \textstyle\bigcup_{a \in H_p} \bigcup_{k \in (-n,n) \cap \mathbb{Z}} \{a + k\}. \tag{3.5}$$

Define p_1 in the following way: $H_{p_1} = H_p \cup \{q\}$, while $\Gamma_{p_1} = \Gamma_p \cup \{\{q\} \cup C : C \in \mathcal{B}\}$. It is clear that if $p_1 \in \mathbb{P}$, then $p_1 \in D_{\mathcal{B}}^{A,m}$ and $p_1 \leqslant p$. Now we prove that $p_1 \in \mathbb{P}$. Assume the contrary, i.e., that for some $a, b \in H_{p_1}$:

$$b \leqslant a \ \wedge \ M(a) \subset H_{p_1} \ \wedge \ \forall C \in [M(a)]^{n-1} \ \{b\} \cup C \in \Gamma_{p_1}. \tag{3.6}$$

Since p satisfies (3.2), q must appear in (3.6), so there are three possibilities:

- $q = a$ which is not possible because in that case $q = (a-1)+1$ with $a - 1 \in H_p$. Contradiction with the choice of q.
- $q \in M(a) \smallsetminus \{a\}$. Then $q = (a-k)-1$ for some $k < n-1$ which is impossible because $a - k \in M(a) \smallsetminus \{q\} \subset H_p$, and again we have a contradiction with the choice of q.
- $q = b$. In this case, the definition of Γ_{p_1} implies $[M(a)]^{n-1} \subset \mathcal{B} \subset [A]^{n-1}$. This in turn implies $a \in M(a) \subset A$, but this implies $q = b > \max A \geqslant a$, which contradicts the first part of assumption (3.6).

Hence, $p_1 \in \mathbb{P}$ and the lemma is proved. □

Since there are only countably many positive integers and only countably many finite subsets of the rational line, there are countably many sets $D_{\mathcal{B}}^{A,m}$, and, according to Lemma 1.2, there is a filter G in $\mathbb{P}$ such that $G \cap D_{\mathcal{B}}^{A,m} \neq \emptyset$ for each $A \in \bigcup_{k \geqslant n-1} [\mathbb{Q}]^k$, $\mathcal{B} \subset [A]^{n-1}$, $m \in \mathbb{N}$. Define $\Gamma = \bigcup_{p \in G} \Gamma_p$. Because $\Gamma_p \subset [\mathbb{Q}]^n$ for all $p \in G$, we have that $\Gamma \subset [\mathbb{Q}]^n$ so $\langle \mathbb{Q}, \Gamma \rangle$ is a countable n-uniform hypergraph. Notice also that for each $p \in G$ we have that:

$$\Gamma \cap [H_p]^n = \Gamma_p. \tag{3.7}$$

It is clear that $\Gamma_p \subset [H_p]^n \cap \Gamma$ (from the definition of Γ), so assume that for some $p \in G$ there is some $B \in (\Gamma \cap [H_p]^n) \setminus \Gamma_p$. Because $B \in \Gamma$ there is some $p_1 \in G$ such that $B \in \Gamma_{p_1}$. Since G is a filter, there is some $p_2 \in G$ such that $p_2 \leqslant p, p_1$, i.e., p_2 is an extension of both p and p_1. Because $B \notin \Gamma_p$, from (3.3) we conclude that $B \notin \Gamma_{p_2}$. However, because $B \in \Gamma_{p_1}$, again from (3.3), we conclude that $B \in \Gamma_{p_2}$ which is a contradiction so (3.7) holds.

Using Lemma 2.1, we prove that $\langle \mathbb{Q}, \Gamma \rangle$ is isomorphic to the countable random n-uniform hypergraph $\mathcal{H}_n$. Take any finite $A \subset \mathbb{Q}$ such that $|A| \geqslant n-1$ and $\mathcal{B} \subset [A]^{n-1}$. The set $D^{A,1}_{\mathcal{B}}$ is dense in $\mathbb{P}$ so there is some $p \in G \cap D^{A,1}_{\mathcal{B}}$. This implies that $A \subset H_p$ and that there is some $q > \max A$ (which implies $q \notin A$) such that for all $C \in [A]^{n-1}$, we have $\{q\} \cup C \in \Gamma_p \Leftrightarrow C \in \mathcal{B}$. Finally (3.7) gives us

$$\forall C \in [A]^{n-1}\ \{q\} \cup C \in \Gamma \Leftrightarrow C \in \mathcal{B},$$

as required by Lemma 2.1. Hence $\mathbb{X} = \langle \mathbb{Q}, \Gamma \rangle \cong \mathcal{H}_n$.

Lemma 3.3. *There is a positive family $\mathcal{P}$ on $\mathbb{Q}$ such that $\mathcal{P} \subset \mathbb{P}(\mathbb{X})$.*

Proof. We will prove that

$$\mathcal{P} = \left\{ \mathbb{Q} \setminus \bigcup_{m \in \mathbb{Z}} F_m : \langle F_m : m \in \mathbb{Z} \rangle \in \prod_{m \in \mathbb{Z}} [[m, m+1)_{\mathbb{Q}}]^{<\omega} \right\}$$

is a positive family in $\mathbb{P}(\mathbb{Q}, \Gamma)$. Take any $Y \in \mathcal{P}$. We will show that $\langle Y, \Gamma \rangle$ satisfies the conditions of Lemma 2.1. Take any finite $A \subset Y$ such that $|A| \geqslant n-1$ and any $\mathcal{B} \subset [A]^{n-1}$. First we find $m_0 \in \mathbb{Z}$ such that $\max A \in [m_0, m_0+1)_{\mathbb{Q}}$. This m_0 clearly exists because A is a finite set. Also, because F_{m_0} is a finite set and $A \cap F_{m_0} = \emptyset$, there is an $m \in \mathbb{N}$ such that $(\max A, \max A + \frac{1}{m}) \cap F_{m_0} = \emptyset$, i.e., $(\max A, \max A + \frac{1}{m}) \cap \mathbb{Q} \subset Y$. Now, since the set $D^{A,m}_{\mathcal{B}}$ is dense in $\mathbb{P}$, there is some $p \in G \cap D^{A,m}_{\mathcal{B}}$, i.e., there is some $q \in Y$ such that

$$\forall C \in [A]^{n-1}\ (\{q\} \cup C \in \Gamma_p \Leftrightarrow C \in \mathcal{B}).$$

Now, in the same way as before (3.7) and Lemma 2.1 prove that $Y \in \mathbb{P}(\mathbb{X})$.

To conclude the proof, we also have to show that $\mathcal{P}$ is a positive family on $\mathbb{Q}$. The condition (P1) is clearly satisfied because only finitely many points are removed from each bounded interval in $\mathbb{Q}$ to obtain the elements of $\mathcal{P}$. For the same reason (P2) and (P3) are also satisfied. The set $\mathbb{Q} \setminus \mathbb{Z}$ is in $\mathcal{P}$ and witnesses that the condition (P4) is true. □

The following lemma shows that we can apply Theorem 1.2 in order to prove the main theorem.

Lemma 3.4. *It holds:*

(1) $(-\infty, x)_{J_0} \subset Y \subset (-\infty, x)_{\mathbb{Q}}$ *implies* $\langle Y, \Gamma \rangle \cong \langle \mathbb{Q}, \Gamma \rangle$ *for* $x \in \mathbb{R} \cup \{\infty\}$;
(2) $(-\infty, q]_{J_0} \subset Y_1 \subset (-\infty, q]_{\mathbb{Q}}$ *implies* $\langle Y_1, \Gamma \rangle \ncong \langle \mathbb{Q}, \Gamma \rangle$ *for* $q \in J_0$.

Proof. To prove (1) take any finite $A \subset Y$ such that $|A| \geqslant n-1$ and take $\mathcal{B} \subset [A]^{n-1}$. There is some $m \in \mathbb{N}$ such that $\max A + \frac{1}{m} < \sup Y = x$ (this can be done by the choice of Y and J_0). Now, because the set $D^{A,m}_{\mathcal{B}}$ is dense in $\mathbb{P}$, there is some $p \in G \cap D^{A,m}_{\mathcal{B}}$. So in the same way as before (using (3.7) and properties of $D^{A,m}_{\mathcal{B}}$), there is $q \in Y$ such that $\forall C \in [A]^{n-1}$ $(\{q\} \cup C \in \Gamma \Leftrightarrow C \in \mathcal{B})$. So by Lemma 2.1, $Y \cong \mathcal{H}_n \cong \mathbb{X}$.

To prove (2) consider the set $M(q) \subset Y_1$ (we know that $M(q) \subset Y_1$ by the choice of the partition $\{J_m : m \in \omega\}$). Suppose that $\langle Y_1, \Gamma\rangle$ is isomorphic to $\mathcal{H}_n$. This means that there is an element $b \in Y_1$ (in particular $b \leqslant q = \max Y_1$) such that $\forall C \in [M(q)]^{n-1}$ $(\{b\} \cup C \in \Gamma)$. According to the definition of Γ, for each $C \in [M(q)]^{n-1}$ there is some $p_C \in G$ such that $\{b\} \cup C \subset H_{p_C}$. Because G is a filter there is some $p \leqslant p_C$ for all $C \in [M(q)]^{n-1}$. Then $M(q) \subset H_p$ and $\forall C \in [M(q)]^{n-1}$ $(\{b\} \cup C \in \Gamma_p)$ but $b \leqslant \max Y_1 = q$. This contradicts the definition of $\mathbb{P}$ (condition (3.2) for p). $\square$

Now we can prove the main result of this note.

Theorem 3.1. *For a linear order L, the following conditions are equivalent.*

(1) *L is a complete, $\mathbb{R}$-embeddable linear order with $\min L$ nonisolated;*
(2) *L is isomorphic to a maximal chain in the poset $\langle \mathbb{P}(\mathcal{H}_n) \cup \{\emptyset\}, \subset\rangle$;*
(3) *L is isomorphic to a compact set K of reals such that $\min K \in K'$.*

Proof. The equivalence of (1) and (3) was shown in [**6**], while the implication (2) $\Rightarrow$ (1) follows from Theorem 1.3.

To prove (1) $\Rightarrow$ (2), note that from the choice of the partition $\{J_m : m \in \omega\}$ and according to Lemma 3.4, conditions (i)–(iv) of Theorem 1.2 are satisfied. Also, Lemma 3.3 proves that the condition (v) of Theorem 1.2 is satisfied. Hence, Theorem 1.2 implies that (1) $\Rightarrow$ (2) is proved. $\square$

3.1. Acknowledgement. The author would like to thank the referee for many valuable comments which improved the presentation of the results significantly.

References

1. M. El-Zahar and N. W. Sauer, *The divisibility properties of homogeneous hypergraphs*, Combinatorica **14** (1994), 159–165.
2. R. Fraissé, *Sur l'extension aux relations de quelques proprietés des ordres*, Ann. Sci. École Norm. Sup. **71** (1954), 363–388.
3. ______, *Theory of Relations*, Stud. Logic Found. Math. 145, North-Holland, 2000.
4. A. S. Kechris, V. Pestov, S. Todorčević, *Fraissé limits, Ramsey theory, and topological dynamics of automorphism groups*, Geom. Funct. Anal. **15** (2005), 106–189.
5. M. S. Kurilić, *Maximal chains in positive subfamilies of $P(\omega)$*, Order **29** (2012), 119–129.
6. ______, *Maximal chains of copies of the rational line*, Order **30** (2013), 737–748.
7. M. S. Kurilić, B. Kuzeljević, *Maximal chains of isomorphic subgraphs of the rado graph*, Acta Math. Hungar. **141** (2013), 1–10.
8. ______, *Maximal chains of isomorphic subgraphs of the countable ultrahomogenous graphs*, Adv. Math. **264** (2014), 762–775.
9. ______, *Maximal chains of isomorphic suborders of the countable ultrahomogeneous partial orders*, Order **32** (2015), 83–99.

10. L. Nguyen Van Thé, N. W. Sauer, *The Urysohn sphere is oscillation stable*, Geom. Funct. Anal. **19** (2009), 536–557.

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(Received 10 07 2015)
(Revised 13 03 2018)