

POLAR DECOMPOSITION OF THE ALUTHGE TRANSFORMATION IN HILBERT C^* -MODULES

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ABSTRACT. Let $T = U|T|$ and $S = V|S|$ be the polar decompositions of adjointable operators T and S , respectively on a Hilbert C^* -module. We determine these pairs of operators for which their products TS accepts the polar decomposition as $TS = UV|TS|$. Specially, we provide sufficient conditions for a certain operator T such that its Aluthge transform $\tilde{T} = |T|^{1/2}U|T|^{1/2}$ admits the polar decomposition.

1. Introduction and preliminaries

In 1990 A. Aluthge in [1] gave the definition of the Aluthge transformation $\tilde{T} = |T|^{1/2}U|T|^{1/2}$ of an operator T whose polar decomposition is $T = U|T|$ where $|T| = (T^*T)^{1/2}$. Also he discussed on the polar decomposition of Aluthge transformation, but the complete solution of this problem has not been obtained. Further in 2004 Ito, Yamazaki and Yanagida [7] obtained the polar decomposition of Aluthge transformation in the setting of Hilbert spaces. After that, many lectures began to discuss the properties of T and \tilde{T} such as p -hyponormal, log-hyponormal, spectrum, numerical range etc. Most results on \tilde{T} show that it generally has better properties than T and many authors have obtained results by using it. In this paper we make an investigation on the polar decomposition of \tilde{T} of a certain operator T acting on the Hilbert C^* -modules. In order to prove it, we need to investigate polar decomposition theory on the product of operators. In 1983 Furuta [4, 5] obtained the polar decomposition of the product of operators acting on Hilbert spaces. Precisely, he presented suitable conditions that the partial isometry in the polar decomposition of product of two operators is the product of partial isometries in each polar decomposition of them. We have to study Furuta's theorem in the setting of Hilbert C^* -modules.

We recall the definition of a Hilbert C^* -module and introduce our notations. A pre-Hilbert C^* -module \mathcal{X} over a C^* -algebra \mathcal{A} , is a right \mathcal{A} -module together with an \mathcal{A} -valued inner product $\langle \cdot, \cdot \rangle: \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{A}$ satisfying the conditions:

2010 *Mathematics Subject Classification*: Primary 46L08; Secondary 47A05.

Key words and phrases: polar decomposition, Aluthge transformation, Hilbert C^* -module.
Communicated by Stevan Pilipović.

- (i) $\langle x, x \rangle \geq 0$ for $x \in \mathcal{X}$,
- (ii) $\langle x, x \rangle = 0$ if and only if $x = 0$.
- (iii) $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$ for $x, y, z \in \mathcal{X}, \alpha, \beta \in \mathbb{C}$,
- (iv) $\langle x, ya \rangle = \langle x, y \rangle a$ for $x, y \in \mathcal{X}, a \in \mathcal{A}$,
- (v) $\langle x, y \rangle^* = \langle y, x \rangle$ for $x, y \in \mathcal{X}$,

We can define a norm on \mathcal{X} by $\|x\|_{\mathcal{X}} = \|\langle x, x \rangle\|_{\mathcal{A}}^{\frac{1}{2}}$. A pre-Hilbert \mathcal{A} -module \mathcal{X} is called a right Hilbert C^* -module over \mathcal{A} if it is complete with respect to its norm. Each Hilbert space is a Hilbert \mathbb{C} -module and each C^* -algebra \mathcal{A} can be regarded as a Hilbert \mathcal{A} -modules via $\langle a, b \rangle = a^*b$ ($a, b \in \mathcal{A}$). Note that, some properties of Hilbert spaces do not hold in Hilbert C^* -modules. For example, a bounded operator T on Hilbert C^* -module \mathcal{X} might not admit a bounded operator T^* as its adjoint operator, that satisfy in the condition $\langle T(x), y \rangle = \langle x, T^*(y) \rangle$ for any $x, y \in \mathcal{X}$. But, it is easy to see that every adjointable operator T is a bounded linear \mathcal{A} -module mapping. Also, in general, a closed submodule F of \mathcal{X} might not be orthogonal complemented and $F^{\perp\perp}$ is usually larger than F . Recall that a closed submodule F of \mathcal{X} is said to be orthogonally complemented if $\mathcal{X} = F \oplus F^{\perp}$, where $F^{\perp} = \{x \in \mathcal{X} : \langle x, x' \rangle = 0 \text{ for all } x' \in F\}$. Lance [8] proved that if an adjointable operator $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ has closed range, then

- (i) $N(T)$ is a complemented of \mathcal{X} , with complement $R(T^*)$,
- (ii) $R(T)$ is a complemented of \mathcal{Y} , with complement $N(T^*)$.

The basic theory of Hilbert C^* -modules can be found in [8]. Throughout this paper, \mathcal{A} denotes a C^* -algebra, \mathcal{X} and \mathcal{Y} are Hilbert \mathcal{A} -modules, $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ is the set of all adjointable operators between \mathcal{X} and \mathcal{Y} and write $\mathcal{B}(\mathcal{X})$ for $\mathcal{B}(\mathcal{X}, \mathcal{X})$. We used $D(\cdot)$, $N(\cdot)$ and $R(\cdot)$ for domain, kernel and range of operators, respectively. An operator $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ (the set of all linear operators) for which $D(T)$ is a dense submodule of \mathcal{X} is called a densely defined operator. First, for the beginning, we state a background for the theory of polar decomposition of operators. The polar decomposition of operators is an important theoretical and computational tool that shows an operator as a product of a partial isometry and a positive element. It is known that every bounded operator T on Hilbert spaces can be decomposed as $T = U|T|$ where U is a partial isometry with $N(U) = N(|T|)$. Also, Kato [6] proved that a densely defined closed operator between Hilbert spaces has polar decomposition. In general adjointable operators on Hilbert C^* -modules do not have polar decomposition. In this respect, Wegge-Olsen stated a necessary and sufficient condition for adjointable operator T to admit a polar decomposition as follows:

THEOREM 1.1. [10, Theorem 15.3.7] *Suppose $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$. Then the following conditions are equivalent:*

- (i) T has a unique polar decomposition $T = U|T|$, where $U \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ is a partial isometry for which $N(U) = N(T) = \overline{N(|T|)}$, $N(U^*) = N(T^*) = \overline{N(|T^*|)}$.
- (ii) $\mathcal{X} = N(T) \oplus \overline{R(T^*)}$ and $\mathcal{Y} = N(T^*) \oplus \overline{R(T)}$.

*In this situation, U^*U is the projection onto $\overline{R(|T|)} = \overline{R(T^*)}$ and UU^* is the projection onto $\overline{R(|T^*|)} = \overline{R(T)}$ and $U^*U|T| = |T|$, $U^*T = |T|$, $UU^*T = T$.*

2. Polar decomposition of the product of two operators

In this section we obtain necessary and sufficient conditions when the operator TS admits a polar decomposition in the setting of Hilbert C^* -modules.

THEOREM 2.1. *Let \mathcal{X} is a Hilbert \mathcal{A} -module and $T, S \in \mathcal{B}(\mathcal{X})$ with $N(T) = N(S^*)$. Then the operator TS admits a polar decomposition if and only if the operators T and S have the polar decompositions.*

PROOF. Suppose the operators T and S admit the polar decompositions, then by Theorem 2.1 we have

$$(2.1) \quad \begin{aligned} \mathcal{X} &= N(T) \oplus \overline{R(T^*)} = N(T^*) \oplus \overline{R(T)}, \\ \mathcal{X} &= N(S) \oplus \overline{R(S^*)} = N(S^*) \oplus \overline{R(S)}. \end{aligned}$$

We claim that $\overline{R(T)} \subseteq \overline{R(TS)}$. Let $y \in \overline{R(T)}$, then there exists a sequence (x_n) in \mathcal{X} such that $(T(x_n))$ converges to y . It is clear from the construction of $\mathcal{X} = N(S^*) \oplus \overline{R(S)}$ that for each element $x_n \in \mathcal{X}$, there are elements $x_{1n} \in N(S^*)$ and $x_{2n} \in \overline{R(S)}$ such that $x_n = x_{1n} + x_{2n}$. The closedness of $\overline{R(S)}$ together with $x_{2n} \in \overline{R(S)}$ implies that there exists a sequence $(y_{2n}^m)_m$ in \mathcal{X} such that $(S(y_{2n}^m))_m$ converges to x_{2n} , as $m \rightarrow \infty$. Moreover, $x_{1n} \in N(S^*) \subseteq N(T)$ yields that $T(x_n) = T(x_{1n} + x_{2n}) = T(x_{1n} + \lim_{m \rightarrow \infty} S(y_{2n}^m)) = 0 + TS(\lim_{m \rightarrow \infty} y_{2n}^m)$ converges to y , as m, n go to infinity. Hence $y \in \overline{R(TS)}$. Observe that equality (2.1) can be rewritten as $\mathcal{X} = N(T^*) \oplus \overline{R(T)} \subseteq N((TS)^*) \oplus \overline{R(TS)} \subseteq \mathcal{X}$. Next, by using $\mathcal{X} = N(T) \oplus \overline{R(T^*)}$, $N(T) \subseteq N(S^*)$ and the same reasoning, we can prove that $\overline{R(S^*)} \subseteq \overline{R((TS)^*)}$. Therefore $\mathcal{X} = N(S) \oplus \overline{R(S^*)} \subseteq N(TS) \oplus \overline{R((TS)^*)} \subseteq \mathcal{X}$. It follows from Theorem 1.1 that the operator TS has the polar decomposition.

Conversely, suppose the operator TS has the polar decomposition, then $\mathcal{X} = N(TS) \oplus \overline{R((TS)^*)} = N((TS)^*) \oplus \overline{R(TS)}$. Let $y \in \overline{R(TS)}$, then there exists a sequence (x_n) in \mathcal{X} such that $(TS(x_n))$ converges to y . Letting $(S(x_n)) = (y_n)$, so the sequence $(T(y_n))$ converges to y , this means that $y \in \overline{R(T)}$. Let $x \in N((TS)^*)$, then $T^*x \in N(S^*) = N(T)$ whence $TT^*x = 0$. The equality $0 = \|\langle TT^*x, x \rangle\|_{\mathcal{A}} = \|\langle T^*x, T^*x \rangle\|_{\mathcal{A}} = \|T^*x\|_{\mathcal{X}}^2$ implies that $x \in N(T^*)$. Finally $\mathcal{X} = N((TS)^*) \oplus \overline{R(TS)} \subseteq N(T^*) \oplus \overline{R(T)} \subseteq \mathcal{X}$. The proof of $\mathcal{X} = N(TS) \oplus \overline{R((TS)^*)} \subseteq N(S) \oplus \overline{R(S^*)} \subseteq \mathcal{X}$ is similar to the argument of above. \square

The following lemma gives a necessary and sufficient condition for operators that satisfy statement (i) of Lemma 2.1. The proof of this lemma is based on the polar decomposition property for operators acting between Hilbert C^* -modules.

LEMMA 2.1. *Let $T = U|T|$ and $S = V|S|$ be the polar decompositions of $T, S \in \mathcal{B}(\mathcal{X})$, respectively. Then the following assertions are equivalent:*

- (i) $[T, S] = 0$ and $[T, S^*] = 0$.
- (ii) The following equations are satisfied:

$$\begin{aligned} [S, U] = 0, \quad [S^*, U] = 0, \quad [U, |S|] = 0, \\ [T, V] = 0, \quad [T^*, V] = 0, \quad [V, |T|] = 0, \end{aligned}$$

$$[U, V] = 0, \quad [U^*, V] = 0, \quad [|T|, |S|] = 0.$$

PROOF. (i) \Rightarrow (ii) Assume that (i) holds. Hence $TS^*S = S^*TS = S^*ST$ and so $T|S| = |S|T$. Similarly to $|T|S = S|T|$.

For the proof of $[U, |S|] = 0$, we first prove that $[U, S] = 0$ and $[U, S^*] = 0$. Consider $x \in \mathcal{X} = N(T) \oplus \overline{R(|T|)}$, such that $x = x_1 + |T|x_2$, where $x_1 \in N(T)$ and $x_2 \in D(|T|)$. Since $x_1 \in N(T) = N(|T|)$, then $S|T|x_1 = 0$. The comutativity S and $|T|$ implies that $|T|Sx_1 = 0$, hence $Sx_1 \in N(|T|) = N(U)$, therefore $USx_1 = 0$.

$$\begin{aligned} (US - SU)(x_1 + |T|x_2) &= USx_1 - SUx_1 + US|T|x_2 - SU|T|x_2 \\ &= U|T|Sx_2 - STx_2 = TSx_2 - STx_2 = 0. \end{aligned}$$

Using the argeument above together with comutativity $|T|$ and S^* , deduce that $US^*x_1 = 0$. Then we have

$$\begin{aligned} (US^* - S^*U)(x_1 + |T|x_2) &= US^*x_1 - S^*Ux_1 + US^*|T|x_2 - S^*U|T|x_2 \\ &= U|T|S^*x_2 - S^*Tx_2 = TS^*x_2 - S^*Tx_2 = 0. \end{aligned}$$

To demonstrate $[V, |T|]$, it is sufficient to show that $[V, T] = 0$ and $[V, T^*] = 0$. We consider $z \in \mathcal{X} = N(S) \oplus \overline{R(|S|)}$, such that $z = z_1 + |S|z_2$, where $z_1 \in N(S)$ and $z_2 \in D(|S|)$, then $T|S|z_1 = |S|Tz_1 = 0$, whence $Tz_1 \in N(|S|) = N(V)$. This means that $VTz_1 = 0$, hence

$$\begin{aligned} (TV - VT)(z_1 + |S|z_2) &= TVz_1 - VTz_1 + TV|S|z_2 - VT|S|z_2 \\ &= TSz_2 - V|S|Tz_2 = TSz_2 - STz_2 = 0. \end{aligned}$$

Also, we have $(T^*V - VT^*)(z_1 + |S|z_2) = 0$, consequently $[V, |T|] = 0$.

We show that $UVz = VUz$ for any $z \in \mathcal{X} = N(S) \oplus \overline{R(|S|)}$. To see this, suppose $z = z_1 + |S|z_2$, where $z_1 \in N(S)$ and $z_2 \in D(|S|)$. The equality $N(S) = N(|S|)$ yields that $|S|z_1 = 0$, hence $U|S|z_1 = |S|Uz_1 = 0$. Therefore $Uz_1 \in N(|S|) = N(V)$, i.e., $VUz_1 = 0$. In addition $VU|S|z_2 = V|S|Uz_2 = SUz_2 = USz_2 = UV|S|z_2$. Consequently

$$\begin{aligned} (UV - VU)(z_1 + |S|z_2) &= UVz_1 - VUz_1 + UV|S|z_2 - VU|S|z_2 \\ &= UV|S|z_2 - UV|S|z_2 = 0. \end{aligned}$$

Finally, assume that $t \in \mathcal{X} = N(T^*) \oplus \overline{R(|T^*|)}$, such that $t = t_1 + |T^*|t_2$, where $t_1 \in N(T^*) = N(|T^*|)$ and $t_2 \in D(|T^*|)$. Hence $t_1 \in N(T^*) = N(U^*)$, whence $VU^*t_1 = 0$.

On the other hand $t_1 \in N(T^*) = N(|T^*|)$. Since V commutes with $|T^*|$, we get $V|T^*|t_1 = |T^*|Vt_1 = 0$, then $Vt_1 \in N(|T^*|) = N(U^*)$, that is $U^*Vt_1 = 0$. Therefore,

$$\begin{aligned} (U^*V - VU^*)(t_1 + |T^*|t_2) &= U^*Vt_1 - VU^*t_1 + U^*V|T^*|t_2 - VU^*|T^*|t_2 \\ &= U^*|T^*|Vt_2 - V|T^*|t_2 = T^*Vt_2 - VT^*t_2 = 0. \end{aligned}$$

Hence we obtain $[U^*, V] = 0$.

$$(ii) \Rightarrow (i) \quad TS = U|T|V|S| = UV|T||S| = VU|S||T| = V|S|U|T| = ST. \quad \square$$

The following theorem presents the polar decomposition of the product of operators.

THEOREM 2.2. *Let $T = U|T|$ and $S = V|S|$ be the polar decompositions of $T, S \in \mathcal{B}(\mathcal{X})$, respectively. If $[T, S] = 0, [T, S^*] = 0$ and $N(T) = N(S^*)$, then $TS = UV|TS|$ is the polar decomposition of TS that is UV is a partial isometry with $N(UV) = N(|T||S|)$ and $|T||S| = |TS|$.*

PROOF. According to Theorem 2.1 the operator TS has a polar decomposition, we are going to prove that $TS = UV|TS|$. First, we show that $|TS| = |T||S|$,

$$|TS|^2 = S^*T^*TS = T^*S^*TS = T^*TS^*S = |T|^2|S|^2 = (|T||S|)^2.$$

The commutativity V with $|T|$ implies that $UV|TS| = UV|T||S| = U|T|V|S| = TS$. Let $x \in N(UV)$, then $Vx \in N(U) = N(T)$, hence $TVx = VTx = 0$. It follows that $Tx \in N(V) = N(S)$, therefore $STx = TSx = 0$. That is $N(UV) \subseteq N(TS)$. Also, $N(TS) \subseteq N(UV)$ can be shown by the same way.

Finally, UV is partial isometry. Indeed $N(UV)^\perp = N(|TS|)^\perp = \overline{R(|TS|)}$, then for any $x \in N(UV)^\perp$ there exists a sequence (y_n) in \mathcal{X} such that $x = \lim_{n \rightarrow \infty} |TS|(y_n)$.

$$\begin{aligned} \|UVx\|_{\mathcal{X}} &= \|UV \lim_{n \rightarrow \infty} |TS|(y_n)\|_{\mathcal{X}} = \|\lim_{n \rightarrow \infty} UV|T||S|(y_n)\|_{\mathcal{X}} \\ &= \|\lim_{n \rightarrow \infty} U|T|V|S|(y_n)\|_{\mathcal{X}} = \lim_{n \rightarrow \infty} \|TS(y_n)\|_{\mathcal{X}} \\ &= \lim_{n \rightarrow \infty} \| |TS|(y_n) \|_{\mathcal{X}} = \|\lim_{n \rightarrow \infty} |TS|(y_n)\|_{\mathcal{X}} = \|x\|_{\mathcal{X}}. \quad \square \end{aligned}$$

LEMMA 2.2. *Let $T = U|T|$ be the polar decomposition of $T \in \mathcal{B}(\mathcal{X})$. Then for any $q \geq 0$,*

- (i) $|T|^q = U^*U|T|^q$ is the polar decomposition of $|T|^q$,
- (ii) $|T^*|^q = UU^*|T^*|^q$ is the polar decomposition of $|T^*|^q$.

PROOF. Since U^*U is the initial projection on $\overline{R(|T|)}$ and $N(|T|^q) = N(|T|) = N(T)$ for all $q \geq 0$, it follows that $\overline{R(|T|^q)} = N(|T|^q)^\perp = N(|T|)^\perp = \overline{R(|T|)}$, hence $U^*U|T|^q = |T|^q$. \square

3. Polar decomposition of the Aluthge transformation

In this section we present a relationship between the polar decomposition of a binormal operator and its Aluthge transform. An operator $T \in \mathcal{B}(\mathcal{X})$ is said to be binormal if $[|T|, |T^*|] = 0$, where $[A, B] = AB - BA$. Binormality of operators was defined by Campbell. He obtained some properties of these operators in [2]. As a consequence, if T is a binormal operator, then we have $|T|^{1/2}|T^*|^{1/2} = ||T|^{1/2}|T^*|^{1/2}|$.

LEMMA 3.1. *Let $T = U|T|$ be the polar decomposition of a binormal operator $T \in \mathcal{B}(\mathcal{X})$ with $N(T) = N(T^*)$. Then $|T|^{1/2}|T^*|^{1/2} = U^*UUU^*||T|^{1/2}|T^*|^{1/2}|$ is the polar decomposition of $|T|^{1/2}|T^*|^{1/2}$.*

PROOF. Since $N(|T|^{1/2}) = N(|T|) = N(T) = N(T^*) = N(|T^*|^{1/2})$, it follows from Theorem 2.2 and Lemma 2.2. \square

REMARK 3.1. Let T be an adjointable operator in $\mathcal{B}(\mathcal{X})$ with the polar decomposition $T = U|T|$, then [3, Lemma 6.1] shows that $|T^*|^s = U|T|^sU^*$ holds for every positive number s . By multiplying U^* of the left-hand side of the above equality and projectivity U^*U onto $\overline{R(|T|)}$, we obtain $U^*|T^*|^s = |T|^sU^*$. Moreover, by multiplying both sides of the same equality by U and U^* , we have $|T|^s = U^*|T^*|^sU$ and similarly $U|T|^s = |T^*|^sU$.

Let $A > 0$, then for any $\alpha > 0$ and $\beta > 0$ obviously we have

$$(U|T|^\beta A|T|^\beta U^*)^2 = U(|T|^\beta A|T|^\beta)^2 U^*.$$

By induction, the equality $(U|T|^\beta A|T|^\beta U^*)^{\frac{n}{m}} = U(|T|^\beta A|T|^\beta)^{\frac{n}{m}} U^*$ holds for all natural numbers n, m .

The continuity of an operator yields that $(U|T|^\beta A|T|^\beta U^*)^\alpha = U(|T|^\beta A|T|^\beta)^\alpha U^*$ as $\frac{n}{m} \rightarrow \alpha$.

In the following theorem we present some conditions under which the Aluthge transformation possesses the polar decomposition.

THEOREM 3.1. *Let $T = U|T|$ be the polar decomposition of a binormal operator $T \in \mathcal{B}(\mathcal{X})$ with $N(T) = N(T^*)$. Then $\tilde{T} = U^*UU|\tilde{T}|$ is the polar decomposition of \tilde{T} .*

PROOF. For this purpose, observe that

$$\begin{aligned} \tilde{T} &= |T|^{1/2}U|T|^{1/2} = |T|^{1/2}|T^*|^{1/2}U = U^*UUU^*||T|^{1/2}|T^*|^{1/2}U \\ &= U^*UUU^*(|T^*|^{1/2}|T||T^*|^{1/2})^{1/2}U = U^*UU(U^*|T^*|^{1/2}|T||T^*|^{1/2}U)^{1/2} \\ &= U^*UU(|T|^{1/2}U^*|T|^{1/2}|T|^{1/2}U|T|^{1/2})^{1/2} = U^*UU|\tilde{T}|. \end{aligned}$$

$$\begin{aligned} x \in N(U^*UU) &\Leftrightarrow U^*UUx = 0 \Leftrightarrow U^*UUU^*Ux = 0 \\ &\Leftrightarrow Ux \in N(U^*UUU^*) \Leftrightarrow Ux \in N(|T|^{1/2}|T^*|^{1/2}) \\ &\Leftrightarrow |T|^{1/2}|T^*|^{1/2}Ux = 0 \Leftrightarrow |T|^{1/2}U|T|^{1/2}x = 0 \Leftrightarrow x \in N(\tilde{T}). \end{aligned}$$

Applying the same procedure of Theorem 2.2 we conclude that U^*UU is the partial isometry of \tilde{T} . □

As an extension of $\tilde{T} = |T|^{1/2}U|T|^{1/2}$, we consider $\tilde{T} = |T|^qU|T|^q$ for a positive number q which is not necessarily $\frac{1}{2}$ and call it the generalized Aluthge transformation. In the following theorem we obtain the polar decomposition of it.

THEOREM 3.2. *Let $T = U|T|$ be the polar decomposition of a binormal operator $T \in \mathcal{B}(\mathcal{X})$ with $N(T) = N(T^*)$. Then the generalized Aluthge transformation $\tilde{T} = |T|^qU|T|^q$ accepts the polar decomposition.*

PROOF. Note that $[[|T|, |T^*]] = 0$ implies that $[|T|^q, |T^*|^q] = 0$, for all $q > 0$. By induction, the equality $[[|T|^{\frac{n}{m}}, |T^*|^{\frac{n}{m}}]] = 0$ holds for all positive integers n, m . Hence $[|T|^q, |T^*|^q] = 0$, as $\frac{n}{m} \rightarrow q$. The proof follows from Theorem 3.1 and the fact that $N(|T|) = N(|T|^q)$, for all $q > 0$. □

The binormality of a bounded operator on Hilbert spaces does not imply the binormality of the its Aluthge transform. See a counter example in the paper of Ito, Yamazaki, and Yanagida [7]. As an application of the previous theorem we state an interesting result as follows:

COROLLARY 3.1. *Let $T = U|T|$ be the polar decomposition of an operator $T \in \mathcal{B}(\mathcal{X})$. If T and \tilde{T} are binormal operators with $N(T) = N(T^*)$, then \tilde{T} and $(\tilde{T}) = |\tilde{T}|^{1/2}\tilde{U}|\tilde{T}|^{1/2}$ accept the polar decompositions.*

PROOF. We first prove that $N(\tilde{T}) = N((\tilde{T})^*)$. Since $N(T) = N(|T|)$, the definition of \tilde{T} implies that $N(T) \subseteq N(\tilde{T})$. The results of Remark 3.1 yield that

$$\begin{aligned} \|\tilde{T}x\|_{\mathcal{X}}^2 &= \|\langle |T|^{1/2}U|T|^{1/2}x, |T|^{1/2}U|T|^{1/2}x \rangle\|_{\mathcal{A}} \\ &= \|\langle x, |T|^{1/2}U^*|T|U|T|^{1/2}x \rangle\|_{\mathcal{A}} = \|\langle x, U^*|T^*|^{1/2}|T||T^*|^{1/2}Ux \rangle\|_{\mathcal{A}} \\ &= \|\langle x, U^*||T|^{1/2}|T^*|^{1/2}Ux \rangle\|_{\mathcal{A}} = \|\langle x, U^*(|T|^{1/2}|T^*|^{1/2})^2Ux \rangle\|_{\mathcal{A}} \\ &= \|\langle x, U^*(|T||T^*|)Ux \rangle\|_{\mathcal{A}} = \|\langle x, U^*|T^*||T|Ux \rangle\|_{\mathcal{A}}. \end{aligned}$$

Now let $x \in N(\tilde{T})$. By the above equality $U^*|T^*||T|Ux = 0$, so $UU^*|T^*||T|Ux = 0$. The projectivity of UU^* on $\overline{R(|T^*|)}$ and binormality of T imply that $|T^*||T|Ux = |T||T^*|Ux = 0$, that is $|T^*|Ux \in N(|T|)$. Since $N(|T|) = N(T) \subseteq N(T^*) = N(|T^*|)$, hence $|T^*|^2Ux = 0$, so $Ux \in N(|T^*|^2) = N(|T^*|) = N(U^*)$, whence $U^*Ux = |U|^2x = 0$. Therefore $N(\tilde{T}) = N(T)$.

Obviously $N(T^*) \subseteq N((\tilde{T})^*)$, by $N(T) = N(T^*)$.

$$\begin{aligned} \|(\tilde{T})^*x\|_{\mathcal{X}}^2 &= \|\langle |T|^{1/2}U^*|T|^{1/2}x, |T|^{1/2}U^*|T|^{1/2}x \rangle\|_{\mathcal{A}} \\ &= \|\langle x, |T|^{1/2}U|T|U^*|T|^{1/2}x \rangle\|_{\mathcal{A}} = \|\langle x, |T|^{1/2}|T^*||T|^{1/2}x \rangle\|_{\mathcal{A}} \\ &= \|\langle x, ||T^*|^{1/2}|T|^{1/2}|^2x \rangle\|_{\mathcal{A}} = \|\langle x, (|T^*|^{1/2}|T|^{1/2})^2x \rangle\|_{\mathcal{A}} \\ &= \|\langle x, |T||T^*|x \rangle\|_{\mathcal{A}}. \end{aligned}$$

Suppose that $x \in N((\tilde{T})^*)$. By the assumption and above equality, we reach that $|T^*|x \in N(|T|) = N(T) \subseteq N(T^*) = N(|T^*|)$, hence $x \in N(|T^*|^2) = N(|T^*|) = N(T^*)$, therefore $N(T^*) = N((\tilde{T})^*)$. Consequently $N((\tilde{T})^*) = N(\tilde{T})$. This means that \tilde{T} satisfies all assumptions of Theorem 3.1, hence the second Aluthge transformation (\tilde{T}) possesses the polar decomposition. \square

Acknowledgment. The author is grateful to the referees for their comments and valuable suggestions for improvement of this work.

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(Received 24 05 2015)
(Revised 04 09 2017)