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EXPONENTIAL SPLINE APPROACH FOR THE SOLUTION OF NONLINEAR FOURTH-ORDER BOUNDARY VALUE PROBLEMS

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Abstract. Exponential sextic spline function is used for the numerical solution of nonlinear fourth-order two-point boundary value problems. Spline relations are derived and direct methods of order two, four and six are obtained. Convergence analysis of the methods is discussed. The proposed method is tested on linear and nonlinear problems. Comparisons are made to confirm the reliability and accuracy of the proposed technique.

1. Introduction

We consider the nonlinear fourth-order two-point boundary value problems of the form

$$\begin{gathered} u^{(4)} = g(x,u), \quad x \in [a,b], \\ u(a) = A_0, \quad u(b) = B_0, \quad u'(a) = A_1, \quad u'(b) = B_1, \end{gathered} \tag{1.1}$$

where $g(x,u)$ is continuous on the interval $x \in [a,b]$ and A_i, B_i $(i = 0,1)$ are finite real arbitrary constants.

Fourth-order nonlinear boundary value problem (1.1) arise in the mathematical modelling of viscoelastic and inelastic flows, deformation of beams and plate deflection theory. For a brief introduction on the subject by using spline functions for the treatment of ordinary differential equations, the reader is referred to Ahlberg et al. [**3**]. Theorems which list the conditions for the existence and uniqueness of solution of such problems are thoroughly discussed in a book by Agarwal [**1**], though no numerical methods are contained therein.

Agarwal and Akrivis [**2**] and Chawla and Katti [**5**] used a finite difference method for solving fourth-order nonlinear two-point boundary value problems prescribed with boundary conditions at the second derivative, such as

$$u(a) = A_0, \quad u(b) = B_0, \quad u''(a) = A_2, \quad u''(b) = B_2.$$

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Several authors [**7**, **14**–**19**] have discussed the linear fourth-order boundary value problem:

$$u^{(4)} + f(x)u = g(x),$$
$$u(a) = A_0, \quad u(b) = B_0, \quad u'(a) = A_1, \quad u'(b) = B_1,$$

where $f(x)$ and $g(x)$ are continuous functions on the interval $[a, b]$.

Khan et al. [**9**] used parametric quintic spline, Al-said and Noor [**4**] used uniform quartic polynomial splines and Noor and Tirmizi [**12**] developed some finite difference methods based on Pade's approximants for the solution of a system of fourth-order boundary value problems associated with obstacle, unilateral and contact problems. Recently, Khan and Khandelwal [**10**] derived a non-polynomial sextic spline function for the solution of linear fourth-order two-point boundary value problems with boundary conditions involving second derivatives and Zahra [**21**] solved linear and nonlinear fourth-order boundary value problems based on exponential quintic spline function.

The fourth-order nonlinear boundary value problems (1.1) were discussed by many authors using different techniques such as Sinc Galerkin method by Gamel et al. [**6**], B-spline method by Viswanadhan et al. [**8**], variational iteration technique by Noor and Mohyud-Din [**13**] and modified decomposition method by Wazwaz [**20**].

According to [**11**], the space $T_n = \text{span}\{1, x, x^2, x^3, x^4, e^{kx}, e^{-kx}\}$ generates an extended complete Chebyshev space on $[a, b]$. Thus it is possible to construct a spline approximation that has a polynomial and an exponential part. In this paper, we construct a new spline method that is based on an exponential spline function of the form $ae^{kx} + be^{-kx} + p_{n-2}(x)$, with $p_{n-2}(x) = \sum_{i=0}^{n-2} m_i x^i$ is an ordinary polynomial of degree $n-2$ and an exponential part to develop a family of numerical methods of order two, four and six for obtaining smooth approximations for the solution of problem (1.1). This approach has its advantage that it does not only provide continuous approximations to $u(x)$, but also for $u'(x), u''(x)$ and higher derivatives at every point of the range of integration. The exponential sextic spline function proposed in this paper has the form $T_6 = \text{span}\{1, x, x^2, x^3, x^4, e^{kx}, e^{-kx}\}$, where k is a free parameter which can be real or pure imaginary. It will be used to raise the accuracy of the method. Thus in each subinterval $x_i \leqslant x \leqslant x_{i+1}$, we have

$$\text{span}\{1, x, x^2, x^3, x^4, e^{kx}, e^{-kx}\} \quad \text{or} \quad \text{span}\{1, x, x^2, x^3, x^4, \cos kx, \sin kx\}$$
$$\text{or} \quad \text{span}\{1, x, x^2, x^3, x^4, \cosh kx, \sinh kx\}, \quad \text{or} \quad \text{span}\{1, x, x^2, x^3, x^4, x^5, x^6\},$$

when $k \to 0$.

The paper is organized into six sections. In Section 2, we present our method for the solution of nonlinear fourth-order two-point boundary value problems. In Section 3, methods of different orders are obtained. In Section 4, the exponential sextic spline solution approximating the analytic solution of boundary value problem is determined. Convergence analysis is briefly discussed in Section 5. In Section 6, several linear and nonlinear problems are considered to illustrate the accuracy and performance of the proposed method and the results are presented along with their comparison.

2. Exponential sextic spline

To develop the spline approximation for the fourth-order nonlinear boundary value problem (1.1), the interval $[a,b]$ is divided into n equal subintervals using the grid $x_i = a + ih$, $i = 0,1,\dots,n$, where $h = \frac{(b-a)}{n}$. The method is developed by using the exponential sextic spline of the form:

$$E_i(x) = a_i e^{k(x-x_i)} + b_i e^{-k(x-x_i)} + c_i(x-x_i)^4 + d_i(x-x_i)^3 + e_i(x-x_i)^2 + f_i(x-x_i) + g_i, \qquad i = 0,1,\dots,n, \tag{2.1}$$

where $a_i, b_i, c_i, d_i, e_i, f_i, g_i$ are real finite constants and k is a free parameter which will be used to raise the accuracy of the method. If $k \to 0$, then $E_i(x)$ reduces to sextic polynomial spline.

Let $u(x)$ be the exact solution of system (1.1) and u_i be an approximation to $u(x_i)$, obtained by the segment $E_i(x)$ of the mixed splines function passing through the points (x_i,u_i) and (x_{i+1},u_{i+1}). To obtain the necessary conditions for the coefficients introduced in (2.1), we do not only require that $E_i(x)$ satisfies (1.1) at x_i, x_{i+1} and that the boundary conditions are fulfilled, but also the continuity of the first, second, third, fourth and fifth derivatives at the common nodes (x_i,u_i). To determine the coefficients of equation (2.1) in terms of u_i, u_{i+1}, M_i, M_{i+1}, F_i and F_{i+1}, we first define

$$\begin{gathered} E_i(x_i) = u_i, \quad E_i(x_{i+1}) = u_{i+1}, \\ E_i'(x_i) = m_i, \quad E_i''(x_i) = M_i, \quad E_i''(x_{i+1}) = M_{i+1}, \\ E_i^{(4)}(x_i) = F_i, \quad E_i^{(4)}(x_{i+1}) = F_{i+1}. \end{gathered} \tag{2.2}$$

We obtain via a long but straightforward calculation

$$\begin{aligned}
a_i &= \frac{a\{-12(u_{i+1}-u_i)+h^2(4M_i+2M_{i+1})+12m_ih\}+h^4(b)F_{i+1}+h^4(-b-a/2)F_i}{\alpha},\\
b_i &= \frac{c\{12(u_{i+1}-u_i)-h^2(4M_i+2M_{i+1})-12m_ih\}+h^4(d)F_{i+1}+h^4(c/2-d)F_i}{\alpha},\\
c_i &= \frac{F_i - a_ik^4 - b_ik^4}{24},\\
d_i &= \frac{u(u_i-u_{i+1})+h(u)m_i+h^2(vM_i+wM_{i+1})+h^4(mF_i+nF_{i+1})}{h^3\alpha},\\
e_i &= \frac{M_i - a_ik^2 - b_ik^2}{2}, \quad f_i = m_i - a_ik + b_ik, \quad g_i = u_i - a_i - b_i, \quad \theta = kh
\end{aligned}$$

and $i = 0(1)n-1$ where

$$\begin{aligned}
&a = \theta^4 e^{-\theta} - \theta^4, \quad b = \frac{\theta^4}{2} - 4\theta^2 - 2\theta^2 e^{-\theta} + 12\theta - 12e^{-\theta} - 12,\\
&c = \theta^4 e^{\theta} - \theta^4, \quad d = \frac{-\theta^4}{2} + 4\theta^2 + 2\theta^2 e^{\theta} + 12\theta - 12e^{\theta} + 12,\\
&u = -\theta^8(e^{\theta} - e^{-\theta}),
\end{aligned}$$

$$
\begin{aligned}
v &= \frac{-5\theta^8}{12}(e^\theta - e^{-\theta}) + \theta^6(e^\theta - e^{-\theta}) - 2\theta^5(e^\theta + e^{-\theta} - 2),\\
w &= \frac{-\theta^8}{12}(e^\theta - e^{-\theta}) - \theta^6(e^\theta - e^{-\theta}) + 2\theta^5(e^\theta + e^{-\theta} - 2),\\
m &= \frac{5\theta^6}{12}(e^\theta - e^{-\theta}) - \theta^5(e^\theta + e^{-\theta}) + 2\theta^3(e^\theta + e^{-\theta} - 2),\\
n &= \frac{\theta^6}{12}(e^\theta - e^{-\theta}) + 2\theta^5 - 2\theta^3(e^\theta + e^{-\theta} - 2),\\
\alpha &= \frac{\theta^8}{2}(e^\theta - e^{-\theta}) - 6\theta^6(e^\theta - e^{-\theta}) + 12\theta^5(e^\theta + e^{-\theta} - 2).
\end{aligned}
$$

Using the continuity conditions of the first, third and fifth derivatives at the point (x_i, u_i), that is $E_{i-1}^{(m)}(x_i) = E_i^{(m)}(x_i)$, $m = 1, 3$ and 5 we derive the relations for $i = 1, 2, \ldots, n-1$

$$
m_i + m_{i+1} = \frac{2(u_i - u_{i+1})}{h} + \frac{h(M_i - M_{i-1})}{6} - \frac{h^3 p(F_i - F_{i-1})}{\alpha}, \tag{2.3}
$$

$$
\begin{aligned}
m_i + m_{i+1} = {} & \frac{(u_{i+1} - u_{i-1})}{h}\\
& - \frac{h[(\beta/6 + \alpha)M_{i+1} + (\beta/2 - 2\alpha)M_i + (\beta/3 + \alpha)M_{i-1}]}{\beta}\\
& + \frac{h^3(p_1F_{i+1} + q_1F_i + r_1F_{i-1})}{\beta},
\end{aligned} \tag{2.4}
$$

$$
\begin{aligned}
m_i + m_{i+1} = {} & \frac{(u_{i+1} - u_{i-1})}{h} - \frac{h(M_{i+1} + 3M_i + 2M_{i-1})}{6}\\
& + \frac{h^3(p_2F_{i+1} + q_2F_i + r_2F_{i-1})}{\gamma},
\end{aligned} \tag{2.5}
$$

where

$$
\begin{aligned}
p = {} & \frac{\theta^6}{12}(e^\theta - e^{-\theta}) - \frac{\theta^5}{2}(e^\theta + e^{-\theta} + 2) + 8\theta^3(e^\theta + e^{-\theta} + 1)\\
& - 24\theta^2(e^\theta - e^{-\theta}) + 24\theta(e^\theta + e^{-\theta} - 2),
\end{aligned}
$$

$$
\begin{aligned}
p_1 &= -\theta^7 - \frac{\theta^6}{2}(e^\theta - e^{-\theta}) + 2\theta^5(e^\theta + e^{-\theta} - 2),\\
q_1 &= \theta^7(e^\theta + e^{-\theta}) - 4\theta^6(e^\theta - e^{-\theta}) - 24\theta^5 + 24\theta^4(e^\theta - e^{-\theta}) - 24\theta^3(e^\theta + e^{-\theta} - 2),\\
r_1 &= -\theta^7 - \frac{3\theta^6}{2}(e^\theta - e^{-\theta}) + 10\theta^5(e^\theta - e^{-\theta}) + 4\theta^5\\
&\quad - 24\theta^4(e^\theta - e^{-\theta}) + 24\theta^3(e^\theta + e^{-\theta} - 2),\\
p_2 &= -\theta^4 + 2\theta^2(e^\theta + e^{-\theta}) + 8\theta^2 - 12(e^\theta + e^{-\theta} - 2),\\
q_2 &= \theta^4(e^\theta + e^{-\theta}) - 6\theta^2(e^\theta + e^{-\theta} + 2) + 12\theta(e^\theta - e^{-\theta}),\\
r_2 &= -\theta^4 + 4\theta^2(e^\theta + e^{-\theta} + 1) - 12\theta(e^\theta - e^{-\theta}) + 12(e^\theta + e^{-\theta} - 2),\\
\beta &= -6\theta^8(e^\theta - e^{-\theta}) + 12\theta^7(e^\theta + e^{-\theta} - 2), \quad \gamma = 12\theta^4(e^\theta + e^{-\theta} - 2).
\end{aligned}
$$

From equations (2.3), (2.4) and (2.5) we obtain on equating the right-hand sides of the equality sign

$$(2.6)\quad u_{i+1}-2u_i+u_{i-1}=\frac{h^2[(\beta+6\alpha)M_{i+1}+(4\beta-12\alpha)M_i+(\beta+6\alpha)M_{i-1}]}{6\beta} - \frac{h^4[(p_1\alpha)F_{i+1}+(q_1\alpha+p\beta)F_i+(r_1\alpha-p\beta)F_{i-1}]}{\beta\alpha},$$

$$(2.7)\quad u_{i+1}-2u_i+u_{i-1}=\frac{h^2[M_{i+1}+4M_i+M_{i-1}]}{6} - \frac{h^4[(p_2\alpha)F_{i+1}+(q_2\alpha+p\gamma)F_i+(r_2\alpha-p\gamma)F_{i-1}]}{\alpha\gamma}.$$

From the above equations we deduce

$$(2.8)\quad h^2M_i=u_{i+1}-2u_i+u_{i-1} - \frac{h^4[(p_1\gamma-p_2(\beta+6\alpha))F_{i+1}+(r_1\gamma-r_2(\beta+6\alpha)+6p\gamma)F_{i-1}]}{6\alpha\gamma} - \frac{h^4[(q_1\gamma-q_2(\beta+6\alpha)-6p\gamma)]F_i}{6\alpha\gamma}.$$

Substituting for $M_j (j=i, i\pm 1)$ from equation (2.8) into (2.7), we arrive at the following useful relation:

$$(2.9)\quad u_{i+2}-4u_{i+1}+6u_i-4u_{i-1}+u_{i-2}=h^4(\lambda F_{i+2}+\mu F_{i+1}+\nu F_i+\mu F_{i-1}+\lambda F_{i-2});\quad i=2,3,\dots,n-2,$$

where $\lambda=\frac{A}{Z}$, $\mu=\frac{B}{Z}$, $\nu=\frac{C}{Z}$ and A, B, C and Z are given by

$$\begin{aligned}
A= & -3\theta^{12}(e^{\theta}-e^{-\theta})+144\theta^9-72\theta^9(e^{\theta}+e^{-\theta}-2)+360\theta^8(e^{\theta}-e^{-\theta})\\
& +36\theta^8(e^{\theta^2}-e^{-\theta^2})-864\theta^7(e^{\theta}+e^{-\theta})+864\theta^6(e^{\theta}-e^{-\theta})\\
& -432\theta^6(e^{\theta^2}-e^{-\theta^2})+5185\theta^5-3456\theta^5(e^{\theta}+e^{-\theta})+864\theta^5(e^{\theta^2}+e^{-\theta^2}),\\
B= & -30\theta^{12}(e^{\theta}-e^{-\theta})+3\theta^{12}(e^{\theta^2}+e^{-\theta^2})+432\theta^{10}(e^{\theta}-e^{-\theta})+1584\theta^9\\
& -864\theta^9(e^{\theta}+e^{-\theta})+72\theta^9(e^{\theta^2}+e^{-\theta^2})-576\theta^8(e^{\theta}-e^{-\theta})-576\theta^8(e^{\theta^2}-e^{-\theta^2})\\
& -1728\theta^7+864\theta^7(e^{\theta^2}+e^{-\theta^2})-3456\theta^6(e^{\theta}-e^{-\theta})+1728\theta^6(e^{\theta^2}-e^{-\theta^2})\\
& -20736\theta^5+13824\theta^5(e^{\theta}+e^{-\theta})-3456(e^{\theta^2}+e^{-\theta^2}),\\
C= & -6\theta^{12}(e^{\theta}-e^{-\theta})+30\theta^{12}(e^{\theta^2}-e^{-\theta^2})-432\theta^{10}(e^{\theta^2}-e^{-\theta^2})+1728\theta^9\\
& -1584\theta^9(e^{\theta}+e^{-\theta})+720\theta^9(e^{\theta^2}+e^{-\theta^2})+432\theta^8(e^{\theta}-e^{-\theta})\\
& +1080\theta^8(e^{\theta^2}-e^{-\theta^2})+1728\theta^7(e^{\theta}+e^{-\theta})-1728\theta^7(e^{\theta^2}+e^{-\theta^2})\\
& +5184\theta^6(e^{\theta}-e^{-\theta})-2592\theta^6(e^{\theta^2}-e^{-\theta^2})+31104\theta^5-20736\theta^5(e^{\theta}+e^{-\theta})\\
& +5184\theta^5(e^{\theta^2}+e^{-\theta^2}),\\
Z= & -72\theta^{12}(e^{\theta}-e^{-\theta})+36\theta^{12}(e^{\theta^2}-e^{-\theta^2})+864\theta^{10}(e^{\theta}-e^{-\theta})
\end{aligned}$$

$$- 432\theta^{10}(e^{\theta^2} - e^{-\theta^2}) + 5184\theta^9 - 3456\theta^9(e^{\theta} + e^{-\theta}) + 864\theta^9(e^{\theta^2} + e^{-\theta^2}).$$

If $\theta \to 0$, then $(\lambda, \mu, \nu) \to \frac{1}{360}(1, 56, 246)$, the spline defined by (2.9) reduces to a sextic spline and the above spline relations reduce to the corresponding ordinary sextic-spline relations [**3**].

Relation (2.9) gives $(n-3)$ linear algebraic equations in $(n-1)$ unknowns u_i, $i = 1, 2, \ldots, n-1$. We require two more equations, one at each end of the range of integration. These two equations are given by

$$\tag{2.10} \begin{gathered} \sum_{k=0}^{3} b_k u_k + chu_0' + h^4 \sum_{k=0}^{5} d_k u_k^{(4)} + t_1 = 0, \quad i = 1, \\ \sum_{k=n-3}^{n} b_k u_k - chu_n' + h^4 \sum_{k=n-5}^{n} d_k u_k^{(4)} + t_{n-1} = 0, \quad i = n-1, \end{gathered}$$

where b_k, c and d_k are arbitrary parameters to be determined.

To obtain the local truncation error t_i; $i = 2, 3, \ldots, n-2$, associated with the scheme (2.9), we first rewrite it in the form

$$\begin{aligned} & u_{i+2} - 4u_{i+1} + 6u_i - 4u_{i-1} + u_{i-2} \\ & \quad = h^4(\lambda u_{i+2}^{(4)} + \mu u_{i+1}^{(4)} + \nu u_i^{(4)} + \mu u_{i-1}^{(4)} + \lambda u_{i-2}^{(4)}) + t_i; \end{aligned}$$

$i = 2, 3, \ldots, n-2$.

Using Taylor's series expansion, the terms $u_{i+2}^{(4)}$, $u_{i+1}^{(4)}$, etc. are expanded around the point x_i and the expression for t_i, $i = 2, 3, \ldots, n-2$ is obtained:

$$\begin{aligned} t_i = (1 - 2\lambda - 2\mu - \nu)h^4 u_i^{(4)} &+ \Big(\frac{1}{6} - 4\lambda - \mu\Big)h^6 u_i^{(6)} \\ &+ \Big(\frac{1}{80} - \frac{16}{12}\lambda - \frac{1}{12}\mu\Big)h^8 u_i^{(8)} \\ &+ \Big(\frac{17}{30240} - \frac{64}{360}\lambda - \frac{1}{360}\mu\Big)h^{10} u_i^{(10)} + O(h^{12}). \end{aligned}$$

Thus for different choices of parameters λ, μ, ν in scheme (2.9), we can obtain the class of methods.

3. Class of methods

3.1. Second-order methods. In order to obtain the second-order methods we find that

$$(b_0, b_1, b_2, b_3, d_0, d_1, d_2, d_3, d_4, d_5) = \Big(-\frac{11}{2}, 9, -\frac{9}{2}, 1, \frac{3}{20}, -\frac{9}{10}, 0, 0, 0, 0\Big), \quad c = -3,$$

$$\begin{aligned} & (b_{n-3}, b_{n-2}, b_{n-1}, b_n, d_{n-5}, d_{n-4}, d_{n-3}, d_{n-2}, d_{n-1}, d_n) \\ & \qquad = \Big(1, -\frac{9}{2}, 9, -\frac{11}{2}, 0, 0, 0, 0, -\frac{9}{10}, \frac{3}{20}\Big), \end{aligned}$$

and the local truncation error is

$$\tag{3.1} t_i = \frac{7}{40} h^6 u_i^{(6)} + O(h^7), \qquad i = 1,\ n-1.$$

(i) If we choose $(\lambda,\mu,\nu)=\frac{1}{120}(1,26,66)$, we obtain the second order method with truncation error

$$t_i=-\frac{1}{12}h^6u_i^{(6)}+O(h^8),\qquad i=2,3,\ldots,n-2. \tag{3.2}$$

(ii) If we choose $(\lambda,\mu,\nu)=(0,0,1)$, we obtain the second-order method with truncation error

$$t_i=\frac{1}{6}h^6u_i^{(6)}+O(h^8),\qquad i=2,3,\ldots,n-2. \tag{3.3}$$

3.2. Fourth-order methods. In order to obtain the fourth-order methods we find that

$$(b_0,b_1,b_2,b_3,d_0,d_1,d_2,d_3,d_4,d_5)=\Big(-\frac{11}{2},9,-\frac{9}{2},1,-\frac{1}{35},-\frac{151}{280},-\frac{13}{70},\frac{1}{280},0,0\Big),$$

$c=-3,$

$$(b_{n-3},b_{n-2},b_{n-1},b_n,d_{n-5},d_{n-4},d_{n-3},d_{n-2},d_{n-1},d_n)$$
$$=\Big(1,-\frac{9}{2},9,-\frac{11}{2},0,0,\frac{1}{280},-\frac{13}{70},-\frac{151}{280},-\frac{1}{35}\Big),$$

and the local truncation error is

$$t_i=\frac{1}{6720}h^8u_i^{(8)}+O(h^9),\qquad i=1,\ n-1. \tag{3.4}$$

(i) If we choose $(\lambda,\mu,\nu)=\frac{1}{6}(0,1,4)$, we obtain the fourth-order method with truncation error

$$t_i=-\frac{1}{720}h^8u_i^{(8)}+O(h^{10}),\qquad i=2,3,\cdots,n-2. \tag{3.5}$$

(ii) If we choose $(\lambda,\mu,\nu)=\frac{1}{360}(1,56,246)$, we obtain the fourth-order method with truncation error

$$t_i=-\frac{1}{240}h^8u_i^{(8)}+O(h^{10}),\qquad i=2,3,\cdots,n-2. \tag{3.6}$$

3.3. Sixth-order method. In order to obtain the sixth-order method we find

$$(b_0,b_1,b_2,b_3,d_0,d_1,d_2,d_3,d_4,d_5)$$
$$=\Big(-\frac{11}{2},9,-\frac{9}{2},1,-\frac{71}{2546},-\frac{19}{35},-\frac{599}{3360},-\frac{1}{240},\frac{9}{2240},-\frac{1}{1200}\Big),$$
$$(b_{n-3},b_{n-2},b_{n-1},b_n,d_{n-5},d_{n-4},d_{n-3},d_{n-2},d_{n-1},d_n)$$
$$=\Big(1,-\frac{9}{2},9,-\frac{11}{2},-\frac{1}{1200},\frac{9}{2240},-\frac{1}{240},-\frac{599}{3360},-\frac{19}{35},-\frac{71}{2546}\Big),\quad c=-3$$

and the local truncation error is

$$t_i=-\frac{1}{960}h^{10}u_i^{(10)}+O(h^{11}),\qquad i=1,\ n-1. \tag{3.7}$$

If we choose $(\lambda,\mu,\nu)=\frac{1}{720}(-1,124,474)$, we obtain the sixth-order method with truncation error

$$t_i=\frac{1}{3024}h^{10}u_i^{(10)}+O(h^{12}),\qquad i=2,3,\ldots,n-2. \tag{3.8}$$

Remarks 3.1. (1) When $(\lambda,\mu,\nu)\to\frac{1}{120}(1,26,66)$, then our method (2.9) reduces to Usmani's second-order method [**19**], Siddiqi and Akram [**16**], Rashidinia and Golbabaee [**15**] based on quintic polynomial spline.

(2) When $(\lambda,\mu,\nu)\to\frac{1}{48}(1,12,22)$, then our method (2.9) reduces to Al-said and Noor second-order method [**4**] based on quartic polynomial spline.

(3) When $(\lambda,\mu,\nu)\to\frac{1}{81}(1,14,51)$, then our method (2.9) reduces to Noor and Tirmizi second-order method [**12**].

(4) When $(\lambda,\mu,\nu)\to\frac{1}{360}(1,56,246)$, then our method (2.9) reduces to Usmani's second order method [**19**] based on sextic polynomial spline.

(5) When $(\lambda,\mu,\nu)\to\frac{1}{6}(0,1,4)$, then our method (2.9) reduces to Usmani's fourth-order finite difference method [**18**].

(6) When $(\lambda,\mu,\nu)\to\frac{1}{720}(-1,124,474)$, then our method (2.9) reduces to Usmani's sixth order finite difference method [**18**].

4. Exponential spline solution

Let $U=u(x_i)$, $\bar{U}=(u_i)$, $V=(v_i)$, $W=(w_i)$, $T=(t_i)$, $E=(e_i)=U-\bar{U}$, $i=1,2,\dots,n-1$ be an exact column vectors, where U, $\bar{U}$, T, E are exact, approximate, local truncation error and discretization error respectively.

For solving nonlinear fourth-order two-point boundary value problem $u^{(4)}=g(x,u)$, the family of numerical methods is described by the set of equations given by (2.9) and (2.10), and the solution vector $\bar{U}=[u_1,u_2,\dots,u_{n-1}]^T$, T denoting transpose, is obtained by solving a nonlinear algebraic system of order $(n-1)$ which has the form $M_0\bar{U}^{(i+1)}-h^4Bg(x,\bar{U}^{(i)})=V$, $i=0,1,2,3\dots$

In case of linear fourth-order boundary value problem $u^{(4)}+f(x)u=g(x)$, we can write the standard matrix equation for the method developed in the form

$$MU=W+T,\quad M\bar{U}=W,\quad ME=T,\tag{4.1}$$

where M is a matrix of order $(n-1)$ with $M=(M_0+h^4BF)$ and M_0,B, $F=\operatorname{diag}(f_i)$, $i=1,2,\dots,n-1$ are $(n-1)\times(n-1)$ matrices.

The five band matrix M_0 has the form

$$M_0=J^2=\begin{bmatrix}9 & -\frac{9}{2} & 1 & & & & & & \\ -4 & 6 & -4 & 1 & & & & & \\ 1 & -4 & 6 & -4 & 1 & & & & \\ & & & \ddots & & & & & \\ & & & & \ddots & & & & \\ & & & & & \ddots & & & \\ & & & & 1 & -4 & 6 & -4 & 1 \\ & & & & & 1 & -4 & 6 & -4 \\ & & & & & & 1 & -\frac{9}{2} & 9\end{bmatrix}$$

where $J = (J_{ij})$ is a tridiagonal symmetric matrix given by

$$J = \begin{bmatrix} -2 & 1 & & & & & & & \\ 1 & -2 & 1 & & & & & & \\ & 1 & -2 & 1 & & & & & \\ & & & \ddots & & & & & \\ & & & & \ddots & & & & \\ & & & & & \ddots & & & \\ & & & & & 1 & -2 & 1 & \\ & & & & & & 1 & -2 & 1 \\ & & & & & & & 1 & -2 \end{bmatrix}$$

and the matrix B has the form

$$B = \begin{bmatrix} -d_1 & -d_2 & -d_3 & -d_4 & -d_5 & & & & \\ \mu & \nu & \mu & \lambda & & & & & \\ \lambda & \mu & \nu & \mu & \lambda & & & & \\ & & & \ddots & & & & & \\ & & & & \ddots & & & & \\ & & & & & \ddots & & & \\ & & & & \lambda & \mu & \nu & \mu & \lambda \\ & & & & & \lambda & \mu & \nu & \mu \\ & & & & -d_{n-5} & -d_{n-4} & -d_{n-3} & -d_{n-2} & -d_{n-1} \end{bmatrix}$$

For the $(n-1)$ column vector V, we have

$$v_i = \begin{cases} -b_0A_0 - cA_1 - h^4(d_0g_0), & i = 1, \\ h^4(\lambda g_0) - A_0, & i = 2, \\ 0, & 3 \leqslant i \leqslant n-3, \\ h^4(\lambda g_n) - B_0, & i = n-2, \\ -b_nB_0 + chB_1 - h^4(d_ng_n), & i = n-1. \end{cases}$$

and for the $(n-1)$ column vector W, we have

$$w_i = \begin{cases} -b_0A_0 - cA_1 - h^4d_0(g_0 - f_0A_0) & \\ -h^4(d_1g_1 + d_2g_2 + d_3g_3 + d_4g_4 + d_5g_5), & i = 1, \\ h^4(\lambda(g_0 - f_0A_0) + \mu g_1 + \nu g_2 + \mu g_3 + \lambda g_4) - A_0, & i = 2, \\ h^4(\lambda g_{i-2} + \mu g_{i-1} + \nu g_i + \mu g_{i+1} + \lambda g_{i+2}), & 3 \leqslant i \leqslant n-3, \\ h^4(\lambda(g_n - f_nB_0) + \lambda g_{n-4} + \mu g_{n-3} + \nu g_{n-2} + \mu g_{n-1}) - B_0, & i = n-2, \\ -b_nB_0 + chB_1 - h^4d_n(g_n - f_nB_0) & \\ -h^4(d_{n-5}g_{n-5}+d_{n-4}g_{n-4}+d_{n-3}g_{n-3}+d_{n-2}g_{n-2}+d_{n-1}g_{n-1}), & i = n-1. \end{cases}$$

5. Convergence analysis

In this section, we discuss the convergence analysis of the method (2.9) along with (2.10) based on exponential sextic spline. Our main purpose is to derive bounds on $\|E\|_\infty$. For this, the following lemma is needed [**18**].

LEMMA 5.1. *If H is a matrix of order N and $\|H\| < 1$, then there exist $(I + H)^{-1}$ and*

$$\|(I+H)^{-1}\|_\infty < \frac{1}{(1-\|H\|)_\infty}.$$

We can rewrite error equation (4.1) in the form

$$E = M^{-1}T = (M_0 + h^4 BF)^{-1}T = (1 + h^4 M_0^{-1} BF)^{-1} M_0^{-1} T.$$

We get

$$\|E\|_\infty \leqslant \frac{\|M_0^{-1}\|_\infty \|T\|_\infty}{1 - h^4 \|M_0^{-1}\|_\infty \|B\|_\infty \|F\|_\infty}, \tag{5.1}$$

provided that $h^4\|M_0^{-1}\|_\infty \|B\|_\infty \|F\|_\infty < 1$. Now $\|B\|_\infty = 1$ and $\|F\|_\infty \leqslant \|f\| = \max_{a\leqslant x\leqslant b} |f(x)|$. It was shown by Usmani [**18**] that

$$\|M_0^{-1}\|_\infty = \frac{(b-a)^4 + 8h^3(b-a)}{384h^4} = O(h^{-4}). \tag{5.2}$$

Now, using equations (3.1)–(3.8), we investigate the convergence analysis of second-order, fourth-order and sixth-order methods.

CASE (i) *Second-order method.* For $(\lambda, \mu, \nu) = \frac{1}{120}(1, 26, 66)$, then from equation (3.1), we have

$$\|T\|_\infty = \frac{7}{40} h^6 M_6, \quad M_6 = \max_{a\leqslant x\leqslant b} |u^{(6)}(x)|, \tag{5.3}$$

then from (5.1)–(5.3), it follows that

$$\|E\|_\infty \leqslant \frac{7\xi h^2 M_6}{40(1-\xi|f(x)|)} = K_2 h^2 = O(h^2), \tag{5.4}$$

where

$$\xi = \frac{(b-a)^4 + 8h^3(b-a)}{384} \quad \text{and} \quad K_2 = \frac{7\xi M_6}{40(1-\xi|f(x)|)},$$

which shows that the method developed for the solution of fourth-order boundary value problem is second-order convergent.

CASE (ii) *Fourth-order method.* For $(\lambda, \mu, \nu) = \frac{1}{6}(0, 1, 4)$, then from equation (3.4), we have

$$\|T\|_\infty = \frac{1}{6720} h^8 M_8, \quad M_8 = \max_{a\leqslant x\leqslant b} |u^{(8)}(x)|, \tag{5.5}$$

then from (5.1)–(5.2) and (5.5), it follows that

$$\|E\|_\infty \leqslant \frac{\xi h^4 M_8}{6720(1-\xi|f(x)|)} = K_4 h^4 = O(h^4), \tag{5.6}$$

where

$$\xi = \frac{(b-a)^4 + 8h^3(b-a)}{384} \quad \text{and} \quad K_4 = \frac{\xi M_8}{6720(1-\xi|f(x)|)},$$

which shows that the method developed for the solution of fourth-order boundary value problem is fourth-order convergent.

CASE (iii) *Sixth-order method.* For $(\lambda, \mu, \nu) = \frac{1}{720}(-1, 124, 474)$, then from equation (3.7), we have

$$\|T\|_\infty = \frac{1}{960} h^{10} M_{10}, \quad M_{10} = \max_{a \leqslant x \leqslant b} |u^{(10)}(x)|, \tag{5.7}$$

then from (5.1)–(5.2) and (5.7), it follows that

$$\|E\|_\infty \leqslant \frac{\xi h^6 M_{10}}{960(1-\xi|f(x)|)} = K_6 h^6 = O(h^6), \tag{5.8}$$

where

$$\xi = \frac{(b-a)^4 + 8h^3(b-a)}{384} \quad \text{and} \quad K_6 = \frac{\xi M_{10}}{960(1-\xi|f(x)|)},$$

which shows that the method developed for the solution of fourth-order boundary value problem is sixth-order convergent.

We summarize the above results in the following theorem:

THEOREM 5.1. *Let $u(x)$ be the exact solution of fourth-order boundary value problem and let u_i, $i = 1, 2, \ldots, n$ be the numerical solution obtained by the difference scheme* (4.1). *Further, if $e_i = u(x_i) - u_i$, then*

(i) $\|E\| = O(h^2)$, *is a second-order method which is given by* (5.4).
(ii) $\|E\| = O(h^4)$, *is a fourth-order method which is given by* (5.6).
(iii) $\|E\| = O(h^6)$, *is a sixth-order method which is given by* (5.8).

neglecting all errors due to rounding off.

6. Numerical results and discussion

The numerical methods outlined in the previous sections are tested on the following linear and nonlinear problems.

Nonlinear problems. EXAMPLE 6.1. Consider the boundary value problem, which is discussed in [**2**, **5**],

$$u^{(4)} - 6\exp(-4u(x)) = -12(1+x)^{-4}, \qquad x \in [0,1],$$
$$u(0) = 0, \quad u(1) = \log 2, \quad u'(0) = 1, \quad u'(1) = 0.5.$$

The analytical solution of the above problem is $u(x) = \log(1+x)$.

EXAMPLE 6.2. Consider the boundary value problem:

$$u^{(4)} = \sin x + \sin^2 x - (u'')^2, \qquad x \in [0,1],$$
$$u(0) = 0, \quad u(1) = \sin(1), \quad u'(0) = 1, \quad u'(1) = \cos(1).$$

The analytical solution of the above problem is $u(x) = \sin(x)$.

TABLE 1. Observed maximum absolute errors, Example 6.1

Methods ↓	$\mathbf{h = \frac{1}{8}}$	$\mathbf{h = \frac{1}{16}}$	$\mathbf{h = \frac{1}{32}}$	$\mathbf{h = \frac{1}{64}}$
Second order method				
$(\lambda, \mu, \nu) = \frac{1}{120}(1, 26, 66)$	3.09×10^{-5}	1.17×10^{-5}	3.10×10^{-6}	7.83×10^{-7}
$(\lambda, \mu, \nu) = (0, 0, 1)$	1.01×10^{-4}	2.51×10^{-5}	6.25×10^{-6}	1.57×10^{-6}
Chawla and Katti [**5**]	5.3×10^{-4}	1.3×10^{-4}	3.4×10^{-5}	–
Fourth order method				
$(\lambda, \mu, \nu) = \frac{1}{6}(0, 1, 4)$	2.48×10^{-7}	1.84×10^{-8}	1.20×10^{-9}	7.54×10^{-11}
$(\lambda, \mu, \nu) = \frac{1}{360}(1, 56, 246)$	7.79×10^{-7}	5.66×10^{-8}	3.64×10^{-9}	2.28×10^{-10}
Chawla and Katti [**5**]	2.8×10^{-6}	1.4×10^{-7}	7.6×10^{-9}	–
Sixth order method				
$(\lambda, \mu, \nu) = \frac{1}{720}(-1, 124, 474)$	5.10×10^{-8}	7.49×10^{-9}	1.00×10^{-9}	1.26×10^{-10}
Chawla and Katti [**5**]	1.7×10^{-7}	4.1×10^{-9}	6.3×10^{-11}	–
Agarwal and Akrivis [**2**]	1.4×10^{-5}	8.3×10^{-7}	5.4×10^{-8}	–

TABLE 2. Observed maximum absolute errors, Example 6.2

Methods ↓	$\mathbf{h = \frac{1}{8}}$	$\mathbf{h = \frac{1}{16}}$	$\mathbf{h = \frac{1}{32}}$	$\mathbf{h = \frac{1}{64}}$
Second order method				
$(\lambda, \mu, \nu) = \frac{1}{120}(1, 26, 66)$	1.32×10^{-6}	3.91×10^{-7}	9.98×10^{-8}	2.50×10^{-8}
$(\lambda, \mu, \nu) = (0, 0, 1)$	3.25×10^{-6}	8.00×10^{-7}	2.00×10^{-7}	5.01×10^{-8}
Fourth order method				
$(\lambda, \mu, \nu) = \frac{1}{6}(0, 1, 4)$	3.95×10^{-10}	2.58×10^{-11}	1.71×10^{-12}	1.20×10^{-12}
$(\lambda, \mu, \nu) = \frac{1}{360}(1, 56, 246)$	1.19×10^{-9}	7.75×10^{-11}	4.96×10^{-12}	1.39×10^{-12}
Sixth order method				
$(\lambda, \mu, \nu) = \frac{1}{720}(-1, 124, 474)$	1.20×10^{-12}	2.15×10^{-13}	8.13×10^{-14}	1.10×10^{-12}

TABLE 3. Observed maximum absolute errors, Example 6.3

Methods ↓	$\mathbf{h = \frac{1}{8}}$	$\mathbf{h = \frac{1}{16}}$	$\mathbf{h = \frac{1}{32}}$	$\mathbf{h = \frac{1}{64}}$
Second order method				
$(\lambda, \mu, \nu) = \frac{1}{120}(1, 26, 66)$	1.11×10^{-6}	9.99×10^{-8}	1.67×10^{-8}	4.44×10^{-9}
$(\lambda, \mu, \nu) = (0, 0, 1)$	2.28×10^{-6}	3.77×10^{-7}	7.09×10^{-8}	1.49×10^{-8}
Fourth order method				
$(\lambda, \mu, \nu) = \frac{1}{6}(0, 1, 4)$	3.08×10^{-7}	3.88×10^{-8}	4.86×10^{-9}	6.06×10^{-10}
$(\lambda, \mu, \nu) = \frac{1}{360}(1, 56, 246)$	4.21×10^{-7}	5.59×10^{-8}	7.24×10^{-9}	9.17×10^{-10}
Sixth order method				
$(\lambda, \mu, \nu) = \frac{1}{720}(-1, 124, 474))$	2.44×10^{-7}	2.93×10^{-8}	3.55×10^{-9}	4.36×10^{-10}

EXAMPLE 6.3. Consider the boundary value problem

$$u^{(4)} = -6\exp(-4u(x)), \qquad x \in [0, 1],$$
$$u(0) = 1, \quad u(1) = \log(e+1), \quad u'(0) = \frac{1}{e}, \quad u'(1) = \frac{1}{(e+1)}.$$

The analytical solution of the above problem is $u(x) = \log(e + x)$.

TABLE 4. Observed maximum absolute errors, Example 6.4

Methods ↓	$h = \frac{1}{4}$	$h = \frac{1}{8}$	$h = \frac{1}{16}$	$h = \frac{1}{32}$	$h = \frac{1}{64}$
Second order method					
$(\lambda, \mu, \nu) = \frac{1}{120}(1, 26, 66)$	1.81×10^{-4}	4.97×10^{-5}	1.25×10^{-5}	3.13×10^{-6}	7.82×10^{-7}
$(\lambda, \mu, \nu) = (0, 0, 1)$	4.06×10^{-4}	1.00×10^{-4}	2.50×10^{-5}	6.26×10^{-6}	1.56×10^{-6}
Fourth order method					
$(\lambda, \mu, \nu) = \frac{1}{6}(0, 1, 4)$	2.82×10^{-8}	1.79×10^{-9}	1.12×10^{-10}	5.24×10^{-12}	–
$(\lambda, \mu, \nu) = \frac{1}{360}(1, 56, 246)$	1.28×10^{-6}	8.50×10^{-8}	5.37×10^{-9}	3.38×10^{-10}	2.53×10^{-11}
Sixth order method					
$(\lambda, \mu, \nu) = \frac{1}{720}(-1, 124, 474)$	1.01×10^{-8}	1.85×10^{-10}	2.10×10^{-12}	9.52×10^{-13}	5.63×10^{-11}
Gupta and Kumar [**7**]	–	1.29×10^{-7}	3.08×10^{-8}	7.54×10^{-9}	–
Rashidinia and Aziz [**14**]	–	1.43×10^{-8}	1.87×10^{-10}	1.68×10^{-10}	–
Rashidinia and Golbabaee [**15**]	1.77×10^{-3}	4.06×10^{-4}	1.00×10^{-4}	2.50×10^{-5}	6.22×10^{-6}
Siddiqi and Akram [**16**]	–	6.35×10^{-5}	1.33×10^{-5}	3.17×10^{-6}	7.84×10^{-7}
Siraj-ul-Islam et al. [**17**]	–	9.90×10^{-9}	1.89×10^{-10}	2.82×10^{-12}	–

TABLE 5. Observed maximum absolute errors, Example 6.5

Methods ↓	$h = \frac{1}{4}$	$h = \frac{1}{8}$	$h = \frac{1}{16}$	$h = \frac{1}{32}$	$h = \frac{1}{64}$
Second order method					
$(\lambda, \mu, \nu) = \frac{1}{120}(1, 26, 66)$	5.39×10^{-3}	1.72×10^{-3}	4.48×10^{-4}	1.12×10^{-4}	2.81×10^{-5}
$(\lambda, \mu, \nu) = (0, 0, 1)$	1.45×10^{-2}	3.58×10^{-3}	9.00×10^{-4}	2.25×10^{-4}	5.62×10^{-5}
Fourth order method					
$(\lambda, \mu, \nu) = \frac{1}{6}(0, 1, 4)$	1.27×10^{-5}	8.52×10^{-7}	5.43×10^{-8}	3.40×10^{-9}	1.94×10^{-10}
$(\lambda, \mu, \nu) = \frac{1}{360}(1, 56, 246)$	3.87×10^{-5}	2.57×10^{-6}	1.63×10^{-7}	1.02×10^{-8}	6.37×10^{-10}
Sixth order method					
$(\lambda, \mu, \nu) = \frac{1}{720}(-1, 124, 474)$	2.38×10^{-7}	5.15×10^{-9}	1.09×10^{-10}	7.84×10^{-12}	3.18×10^{-11}
Rashidinia and Aziz [14]	–	3.31×10^{-7}	5.15×10^{-9}	8.10×10^{-11}	1.12×10^{-12}

Linear problems. EXAMPLE 6.4. Consider the boundary value problem, which is discussed in [**7**, **14**–**17**],

$$u^{(4)} + 4u = 1, \qquad x \in [-1, 1],$$

$$u(-1) = u(1) = 0, \quad u'(-1) = -u'(1) = \frac{\sinh(2) - \sin(2)}{4(\cosh(2) + \cos(2))}.$$

The analytical solution of the above problem is

$$u(x) = 0.25\Big\{1 - 2\Big(\frac{\sin(1)\sinh(1)\sin(x)\sinh(x) + \cos(1)\cosh(1)\cos(x)\cosh(x)}{\cos(2) + \cosh(2)}\Big)\Big\}.$$

EXAMPLE 6.5. Consider the boundary value problem, which is discussed in [**14**],

$$u^{(4)} - xu = -(11 + 9x + x^2 - x^3)e^x, \qquad x \in [-1, 1],$$

$$u(-1) = 0, \quad u(1) = 0, \quad u'(-1) = \frac{2}{e}, \quad u'(1) = -2e.$$

The analytical solution of the above problem is $u(x) = (1-x^2)e^x$. The observed maximum absolute errors corresponding to the Examples 6.1–6.5 for our second, fourth and sixth-order methods are briefly summarized in Tables 1–5. Comparisons with other existing methods are also listed in Tables 1–5. These tables show that our methods are more accurate than the existing spline methods.

It is verified from Tables 1–5 that on reducing the step-size from h to $h/2$, the maximum observed error $\|E\|$ is approximately reduces by a factor $1/2^P$, where p

is the theoretical order of numerical method, except possibly when the round off errors are significant.

Conclusion

Exponential sextic spline functions are used to develop a class of numerical methods for finding the numerical solution of fourth-order linear and nonlinear boundary value problems, with two-point boundary conditions. The computations associated with the examples discussed above were performed by using MATLAB 7. The methods are computationally efficient and can be easily implemented on a computer. The present method enables us to approximate the solution at every point of the range of integration. Comparison of the method is also depicted through Tables 1–5 which show that our methods perform better than the existing finite difference and spline methods.

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