PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 104(118) (2018), 251–264 DOI: https://doi.org/10.2298/PIM1818251K

APPROXIMATION OF SIGNALS (FUNCTIONS) OF Lip(α, p), $(p \geqslant 1)$-CLASS BY TRIGONOMETRIC POLYNOMIALS

Kejal Khatri and Vishnu Narayan Mishra*

ABSTRACT. Given a function f in the class Lip(α, p) $(0 < \alpha \leqslant 1, p \geqslant 1)$, Mittal and Singh (2014) approximated such an f by using trigonometric polynomials, which are the n^{th} terms of either certain Riesz mean or Nörlund mean transforms of the Fourier series representation for f. They showed that the degree of approximation is $O((\lambda(n))^{-\alpha})$ and extended two theorems of Leindler (2005) where he had weakened the conditions on $\{p_n\}$ given by Chandra (2002) to more general classes of triangular matrix methods. We obtain the same degree of approximation for a more general class of lower triangular matrices.

1. Introduction

Let $\sigma_n(f)$ denote the nth term of the $(C, 1)$ transform of the partial sums of the Fourier series of a 2π-periodic function f. In 1937 Quade [**27**] proved that, if $f \in$ Lip(α, p) for $0 < \alpha \leqslant 1$, then $\|f - \sigma_n(f)\|_p = O(n^{-\alpha})$ for either $p > 1$ and $0 < \alpha \leqslant 1$ or $p = 1$ and $0 < \alpha < 1$. He also showed that, if $p = \alpha = 1$, then $\|f - \sigma_n(f)\|_1 = O(n^{-1} \log(n + 1))$. In 2002, Chandra [**2**] extended the work of Quade [**27**] and proved three interesting on trigonometric approximation theorems and gave numerous attractive corollaries. In 2005, Leindler [**9**] extended the work of Chandra [**2**] and investigated trigonometric polynomials associated with on $f \in$ Lip(α, p) $(0 < \alpha \leqslant 1, p \geqslant 1)$ to approximate f in L_p norm to the degree of $O(n^{-\alpha})$ $(0 < \alpha \leqslant 1)$. Mittal et al. [**18**] studied on using infinite matrices to approximate functions of class Lip(α, p) using trigonometric polynomials. In continuous, Mishra et al. [**15**] studied linear operators to approximate signals of Lip(α, p) $(p \geqslant 1)$-class respectively. The degree of approximation of functions belonging to various classes through trigonometric Fourier approximation using different summability matrices with monotone rows has been proved by various

2010 *Mathematics Subject Classification:* Primary 40C05, 40D25, 40G05; Secondary 41A25, 42A10.

Key words and phrases: Lip$(\alpha, p), (p \geqslant 1)$-class, means of Fourier series, almost monotone sequence, modulus of continuity, more general classes of triangular matrix methods, C_λ.

Communicated by Gradimir Milovanović.

*Corresponding author

investigators like Khan [**5, 6, 7**], Mohapatra and Chandra [**21, 22**], Mishra [**11**], Mishra et al. [**13, 14**], Mishra and Khatri [**16**], Mishra et al. [**17**]. In 2012, Mishra and Mishra [**12**] studied on trigonometric approximation of signals (functions) in L_p $(p \geqslant 1)$ norm. In 2014, Mursaleen and Mohiuddine [**23**], Deepmala [**4**] and Khatri [**8**] discussed convergence methods for double sequences, a Study on fixed point theorems for nonlinear contractions and its applications and a study on approximation of functions in Banach spaces using summability methods respectively. Recentely, Değer et al. [**3**] extended two theorems of Chanda [**2**] using C_λ obtained by deleting a set of rows from Cesàro matrix C_1. Very recently, Mittal and Singh [**20**] generalized two theorems on summability matrix N_p of Değer et al. [**3**]. In 2017, Mishra [**10**] studied on existence and behavior of solutions to some nonlinear integral equations with applications. In this paper, we extend the theorem on lower triangular matrices of Mittal and Singh [**20**] where they have extended two theorems of Leindler [**9**] using C_λ-method obtained by deleting a set of rows from Cesàro matrix C_1. Here, we have obtained error of approximation which depends on strictly increasing sequence of positive integers i.e., the error of approximation is of order $(\lambda)^{-\alpha}$.

Analysis of signals or time functions is of great importance, because it conveys information or attributes of some phenomenon. The engineers and scientists use properties of Fourier approximation for designing digital filters. Especially, Psarakis and Moustakides [**26**] presented a new L_2 based method for designing the Finite Impulse Response (FIR) digital filters and get corresponding optimum approximations having improved performance. Some interesting applications can be seen in [**25**].

Let f be a 2π-periodic signal and let $f \in L_p[0, 2\pi] = L_p$ for $p \geqslant 1$. Then the Fourier series of function (signal) f at any point x is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx) \equiv \sum_{k=0}^{\infty} A_k(f; x). \tag{1.1}$$

Denote by $s_n(f; x)$, $n = 0, 1, \ldots$ the nth partial sums of the series (1.1) at the point x, that is, $s_n(f; x) = \sum_{k=0}^{\infty} A_k(f; x)$, a trigonometric polynomial of degree (or order) n, where $A_0(f; x) = \frac{a_0}{2}$, $A_k(f; x) = (a_k \cos kx + b_k \sin kx)$, $k = 1, 2, \ldots$. We define

$$\tau_n^\lambda(f; x) = \sum_{k=0}^{\lambda(n)} a_{\lambda(n),k} s_k(f; x) \quad \forall n \geqslant 0, \tag{1.2}$$

where $T \equiv (a_{\lambda(n),k})$ is a linear operator represented by a lower triangular regular matrix with nonnegative entries and row sums 1. The forward difference operator Δ is defined by $\Delta_k a_{n,k} = a_{n,k} - a_{n,k+1}$. Such a matrix T is said to have monotone rows if, for each n, $\{a_{\lambda(n),k}\}$ is either nonincreasing or nondecreasing in k, $0 \leqslant k \leqslant n$. Series (1.2) is said to be T-summable to s, if $\tau_n^\lambda(f; x) \to s$ as $n \to \infty$. The $T-$

operator reduces to the Nörlund (N_p)-operator, if

$$a_{\lambda(n),k}=\begin{cases}\frac{p_{\lambda(n)-k}}{P_{\lambda(n)}}, & 0\leqslant k\leqslant n,\\ 0, & k>n,\end{cases}$$

where $P_{\lambda(n)}=\sum_{k=0}^{\lambda(n)}p_k\neq 0$ and $p_{-1}=0=P_{-1}$. In this case, the transform $\tau_n^\lambda(f;x)$ reduces to the Nörlund transform $N_n^\lambda(f;x)$.

The T-operator reduces to the weighted (Riesz) ($\bar{N}_p$)-operator, if

$$a_{\lambda(n),k}=\begin{cases}\frac{p_k}{P_{\lambda(n)}}, & 0\leqslant k\leqslant n,\\ 0, & k>n,\end{cases}$$

where $P_{\lambda(n)}=\sum_{k=0}^{\lambda(n)}p_k\neq 0$ and $p_{-1}=0=P_{-1}$. In this case, the transform $\tau_n(f;x)$ reduces to the Nörlund transform $\bar{N}_n^\lambda(f;x)$ (or $R_n^\lambda(f;x)$).

A signal (function) $f\in\mathrm{Lip}\,\alpha$, if $f(x+t)-f(x)=O(|t^\alpha|)$ for $0<\alpha\leqslant 1$, $t>0$ A signal (function) $f\in\mathrm{Lip}(\alpha,p)$ for $p\geqslant 1$, $0\leqslant\alpha\leqslant 1$, if

$$\left(\int_0^{2\pi}|f(x+t)-f(x)|^p dx\right)^{\frac{1}{p}}=O(t^\alpha).$$

The integral modulus of continuity of function $f\in L_p[0,2\pi]$ is defined by

$$\omega_p(\delta;f)=\sup_{0<|h|\leqslant\delta}\left(\frac{1}{2\pi}\int_0^{2\pi}|f(x+h)-f(x)|^p dx\right)^{\frac{1}{p}}.$$

If, for $\alpha>0$, one has $\omega_p(\delta;f)=O(\delta^\alpha)$, then $f\in\mathrm{Lip}(\alpha,p),(p\geqslant 1)$.

The L_p-norm of f is defined by

$$\|f\|_p=\left(\frac{1}{2\pi}\int_0^{2\pi}|f(x)|^p dx\right)^{\frac{1}{p}}\quad (f\in L_p(p\geqslant 1)).$$

Also

$$s_n(f)=\frac{1}{\pi}\int_0^{2\pi}f(x+t)D_n(t)dt,\qquad \sigma_n^\lambda(f;x)=\frac{1}{\lambda(n)+1}\sum_{k=0}^{\lambda(n)}s_k(f;x),$$

where $D_n(t)=\frac{\sin(n+1/2)t}{2\sin(t/2)}$, is the Dirichlet Kernel of degree (or order) n.

A positive sequence $c:=\{c_n\}$ is called almost monotone decreasing (or increasing) if there exists a constant $K:=K(c)$, depending on the sequence c only, such that for all $n\leqslant m$, $c_n\leqslant Kc_m\quad(Kc_n\geqslant c_m)$. Such sequences will be denoted by $c\in$ AMDS and $c\in$ AMIS, respectively. A sequence which is either AMDS or AMIS is called monotone and will be denoted by $c\in$ AMS. Let $\mathbb{E}$ be an infinite subset of $\mathbb{N}$ and $\mathbb{E}$ as the range of strictly increasing sequence of positive integers; say $\mathbb{E}=\{\lambda(n)\}_{n=1}^\infty$. The Cesàro submethod C_λ is defined as

$$(C_\lambda x)_n=\frac{1}{\lambda(n)}\sum_{k=1}^{\lambda(n)}x_k,\quad (n=1,2,3,\dots),$$

where x_k is a sequence of real or complex numbers. Therefore, the C_λ-method yields a subsequence of the Cesàro method C_1, and hence it is regular for any λ. C_λ is obtained by deleting a set of rows from the Cesàro matrix. The basic properties of C_λ-method can be found in [**1, 24**].

$$N_n^\lambda(f;x) = \frac{1}{P_{\lambda(n)}} \sum_{k=0}^{\lambda(n)} p_{\lambda(n)-k} s_k(f;x), \qquad P_{\lambda(n)} = \sum_{r=0}^{\lambda(n)} p_r \neq 0, \quad p_{-1} = 0 = P_{-1}.$$

$$R_n^\lambda(f;x) = \frac{1}{P_{\lambda(n)}} \sum_{k=0}^{\lambda(n)} p_k s_k(f;x), \qquad P_{\lambda(n)} = \sum_{r=0}^{\lambda(n)} p_r \neq 0, \quad p_{-1} = 0 = P_{-1}.$$

$$A_{\lambda(n),k} = \sum_{r=k}^{\lambda(n)} a_{n,r}, \;\; t_n^\lambda = \sum_{k=0}^{\lambda(n)} a_{n,k} = A_{\lambda(n),0}, \;\; b_{n,k} = \frac{A_{\lambda(n),k} - A_{\lambda(n),0}}{k} \; \forall 1 \leqslant k \leqslant n,$$

$$\Delta_k a_{n,k} = a_{n,k} - a_{n,k+1}, \quad \Delta f_k g_k = g_k \Delta f_k + f_{k+1} \Delta g_k,$$

$[x]$-denotes the greatest integer not exceeding x.

A signal (function) f is approximated by trigonometric polynomials τ of order (or degree) n and the degree of approximation $E_n^\lambda(f)$ is given by

$$E_n^\lambda(f) = \min_n \|f(x) - \tau_n^\lambda(f;x)\|_p.$$

This method of approximation is called trigonometric Fourier approximation (TFA).

2. Known Results

Mittal et al. [**19**] proved the following theorem:

THEOREM 2.1. [**19**]. *Let $f \in \mathrm{Lip}(\alpha, p)$, and let $T = (a_{n,k})$ be an infinite regular triangular matrix.*

(i) *$p > 1$, $0 < \alpha < 1$, $\{a_{n,k}\} \in AMS$ in k and satisfies*

$$(n+1)\max\{a_{n,0}, a_{n,r}\} = O(1),$$

where $r := [n/2]$, then

$$\|\tau_n(f;x) - f(x)\|_p = O(n^{-\alpha}). \tag{2.1}$$

(ii) $$p > 1, \;\; \alpha = 1 \;\textit{and}\; \sum_{k=0}^{n-1} (n-k)|\Delta_k a_{n,k}| = O(1), \;\; \textit{or}$$

(iii) $$p > 1, \;\; \alpha = 1 \;\textit{and}\; \sum_{k=0}^{n} |\Delta_k a_{n,k}| = O(a_{n,0}), \;\; \textit{or}$$

(iv) $$p = 1, \;\; 0 < \alpha < 1, \; \sum_{k=0}^{n} |\Delta_k a_{n,k}| = O(a_{n,0}),$$

and also $(n+1)a_{n,0} = O(1)$, hold then (2.1) is satisfied.

Değer et al. [**3**] proved:

THEOREM 2.2. [**3**]. *Let $f \in \mathrm{Lip}(\alpha,p)$ and let $\{p_n\}$ be positive such that*

$$(\lambda(n)+1)p_{\lambda(n)} = O(P_{\lambda(n)}), \tag{2.2}$$

If either

(i) $p>1$, $0<\alpha \leqslant 1$ *and* (ii) $\{p_n\}$ *is monotonic,*

or

(i) $p=1$, $0<\alpha<1$ *and* (ii) $\{p_n\}$ *is nondecreasing sequence,*

then

$$\|f(x)-N_n^{\lambda}(f;x)\|_p = O(\lambda(n)^{-\alpha}). \tag{2.3}$$

THEOREM 2.3. [**3**]. *Let $f \in \mathrm{Lip}(\alpha,1)$, $0<\alpha<1$. If the positive $\{p_n\}$ satisfies condition* (2.2) *and nondecreasing, then*

$$\|f(x)-R_n^{\lambda}(f;x)\|_1 = O(\lambda(n)^{-\alpha}). \tag{2.4}$$

Recently, Mittal and Singh [**20**] have generalized two theorems 2.2 and 2.3 of Değer et al. [**3**] by dropping monotonicity on the elements of the matrix rows which in turn generalize theorems 2.1 and 2.2 respectively of Leindler [**9**] to a more general C_λ-method. They proved:

THEOREM 2.4. [**20**]. *If $f \in \mathrm{Lip}(\alpha,p)$ and let $\{p_n\}$ be positive. If one of the following conditions*

(i) $p>1$, $0<\alpha<1$, $\{p_n\} \in AMDS$,
(ii) $p>1$, $0<\alpha<1$ *and* $\{p_n\} \in AMIS$ *and* (2.2) *holds,*
(iii) $p>1$, $\alpha=1$ *and* $\sum_{k=1}^{\lambda(n)-1} k|\Delta p_k| = O(P_{\lambda(n)})$,
(iv) $p>1$, $\alpha=1$, $\sum_{k=0}^{\lambda(n)-1} |\Delta p_k| = O(P_{\lambda(n)}/\lambda(n))$ *and* (2.2) *holds,*
(v) $p=1$, $0<\alpha<1$, $\sum_{k=-1}^{\lambda(n)-1} |\Delta p_k| = O(P_{\lambda(n)}/\lambda(n))$

maintains, then (2.3) *holds.*

THEOREM 2.5. [**20**]. *Let $f \in \mathrm{Lip}(\alpha,1)$, $0<\alpha<1$. If the positive sequence $\{p_n\}$ satisfies* (2.2) *and the condition $\sum_{k=0}^{\lambda(n)-1} |\Delta p_k| = O(P_{\lambda(n)}/\lambda(n))$ holds, then* (2.4) *holds.*

3. Main result

In this paper, we extend the theorem on lower triangular matrices of Mittal and Singh [**20**] in which they have extended two theorems of Leindler [**9**] using C_λ–method obtained by deleting a set of rows from Cesàro matrix C_1. Our theorem also generalize the theorem of Mittal et al. [**19**] to T-matrix which in turn generalizes the results of Quade [**27**]. Here, we determine the degree of approximation which depends on strictly increasing sequence of positive integers i.e., the error of approximation is of order $(\lambda)^{-\alpha}$.

More precisely, we prove:

THEOREM 3.1. *Let $f \in \mathrm{Lip}(\alpha,p)$, and let $T=(a_{n,k})$ be an infinite regular triangular matrix.*

(i) $p > 1$, $0 < \alpha < 1$, $\{a_{n,k}\} \in AMS$ *in* k *and satisfies*

$$(\lambda(n)+1)\max\{a_{\lambda(n),0}, a_{\lambda(n),r}\} = O(1), \tag{3.1}$$

where $r := [\lambda(n)/2]$, *then*

$$\|f(x) - \tau_n^{\lambda}(f;x)\|_p = O(\lambda(n)^{-\alpha}). \tag{3.2}$$

(ii) $$p > 1,\ \alpha = 1 \ \textit{and} \ \sum_{k=0}^{\lambda(n)-1} (\lambda(n)-k)|\Delta_k a_{\lambda(n),k}| = O(1),\ \textit{or} \tag{3.3}$$

(iii) $$p > 1,\ \alpha = 1 \ \textit{and} \ \sum_{k=0}^{\lambda(n)-1} |\Delta_k a_{\lambda(n),k}| = O(a_{\lambda(n),0}),\ \textit{or} \tag{3.4}$$

(iv) $$p = 1,\ 0 < \alpha < 1,\ \sum_{k=0}^{\lambda(n)} |\Delta_k a_{\lambda(n),k}| = O(a_{\lambda(n),0}), \tag{3.5}$$

$$\textit{and also } (\lambda(n)+1)a_{\lambda(n),0} = O(1),\ \textit{hold then } (3.2) \textit{ is satisfied.} \tag{3.6}$$

We note that:

(i) If $\lambda(n) = n$, then our Theorem 3.1 reduces to theorem 2.1.

(ii) If T is a Nörlund (N_p) or weighted (R_p)-matrix, condition (3.1) (or (3.6)) reduces to (2.2), while conditions (3.3)–(3.5) reduce to conditions (iii), (iv) and (v) of Theorem 2.4, respectively. Thus our Theorem 3.1 generalizes Theorems 2.4 and 2.5.

(iii) It is easy to examine that the conditions of Theorems 3.1 claim less than the requirements of Theorems 2.4 and 2.5 for $A_{\lambda(n),0} = t_n$.

For example, the condition on the sum in (3.3) is always satisfied if the sequence $\{a_{n,k}\}$ is nondecreasing in k, then using (3.6), we get

$$\begin{aligned}\sum_{k=0}^{\lambda(n)-1} (\lambda(n)-k)|\Delta_k a_{n\lambda(n),k}| &= \sum_{k=0}^{\lambda(n)-1} (\lambda(n)-k)|a_{\lambda(n),k} - a_{\lambda(n),k+1}| \\ &= \sum_{k=0}^{\lambda(n)-1} (\lambda(n)-k)(a_{\lambda(n),k+1} - a_{\lambda(n),k}) \\ &= A_{\lambda(n),0} - (\lambda(n)+1)a_{\lambda(n),o} = O(1).\end{aligned}$$

If $\{a_{n,k}\}$ is nonincreasing in k and (3.6) holds then

$$\begin{aligned}\sum_{k=0}^{\lambda(n)} |\Delta_k a_{\lambda(n),k}| &= \sum_{k=0}^{\lambda(n)} |a_{\lambda(n),k} - a_{\lambda(n),k+1}| \\ &= \sum_{k=0}^{\lambda(n)} (a_{\lambda(n),k} - a_{\lambda(n),k+1}) = a_{\lambda(n),0} - a_{\lambda(n),\lambda(n)+1} = O(a_{\lambda(n),0})\end{aligned}$$

is also true. Our Theorem 3.1 under assumptions implies the results of Theorems 2.4 and 2.5.

4. Lemmas

We use the following lemmas in the proof of theorem 3.1:

LEMMA 4.1. [**27**]. *If $f\in \mathrm{Lip}(1,p)$, $p>1$ then*

$$\|\sigma_n(f;x)-s_n(f;x)\|_p=O(n^{-1}). \tag{4.1}$$

LEMMA 4.2. [**27**]. *Let, for $0<\alpha\leqslant 1$ and $p>1$, $f\in\mathrm{Lip}(\alpha,p)$. Then*

$$\|f(x)-s_n(f;x)\|_p=O(n^{-\alpha})\quad \forall n>0.$$

LEMMA 4.3. *Let T have AMS rows and satisfy* (3.1)*. Then, for $0<\alpha<1$,*

$$\sum_{k=1}^{\lambda(n)} a_{\lambda(n),k}k^{-\alpha}=O(\lambda(n)^{-\alpha})$$

PROOF. Let $r:=[\lambda(n)/2]$. Then, we have

$$\sum_{k=0}^{\lambda(n)} a_{\lambda(n),k}(k+1)^{-\alpha}=\sum_{k=1}^{r} a_{\lambda(n),k}k^{-\alpha}+\sum_{k=r+1}^{\lambda(n)} a_{\lambda(n),k}k^{-\alpha}.$$

CASE I. If $\{a_{n,k}\}$ is nondecreasing in k. Then, using (3.1), we get

$$\begin{aligned}\sum_{k=1}^{\lambda(n)} a_{\lambda(n),k}k^{-\alpha}&\leqslant a_{\lambda(n),r}\sum_{k=1}^{r}k^{-\alpha}+(r+1)^{-\alpha}\sum_{k=r+1}^{\lambda(n)} a_{\lambda(n),k}\\&\leqslant a_{\lambda(n),r}\sum_{k=0}^{r}k^{-\alpha}+(r+1)^{-\alpha}A_{(\lambda(n),0)}\\&=O((\lambda(n)+1)^{-1})O(\lambda(n)^{1-\alpha})+O(\lambda(n)^{-\alpha})=O(\lambda(n)^{-\alpha}).\end{aligned}$$

CASE II. If $\{a_{n,k}\}$ is nonincreasing in k. Then, using (3.1), we get

$$\sum_{k=1}^{\lambda(n)} a_{\lambda(n),k}k^{-\alpha}\leqslant a_{\lambda(n),0}\sum_{k=1}^{r}k^{-\alpha}+O(\lambda(n)^{-\alpha})=O(\lambda(n)^{-\alpha}).\qquad\square$$

5. Proof of Theorem 3.1

CASE I. If $p>1$, $0<\alpha<1$. Let $\{a_{n,k}\}$ be AMS in k. Then

$$\tau_n^{\lambda}(f;x)-f(x)=\sum_{k=0}^{\lambda(n)} a_{\lambda(n),k}s_k(f;x)-f(x)=\sum_{k=0}^{\lambda(n)} a_{\lambda(n),k}(s_k(f;x)-f(x)).$$

Using (3.1) and Lemmas 4.2 and 4.3,

$$\begin{aligned}\|\tau_n^{\lambda}(f;x)-f(x)\|_p&\leqslant\sum_{k=0}^{\lambda(n)} a_{\lambda(n),k}\|s_k(f;x)-f(x)\|_p\\&=\sum_{k=1}^{\lambda(n)} a_{\lambda(n),k}O(k^{-\alpha})=O(\lambda(n)^{-\alpha}).\end{aligned}$$

CASE III. If $p > 1$, $\alpha = 1$, we have

$$\tau_n^\lambda(f;x) - f(x) = \tau_n^\lambda(f;x) - s_n^\lambda(f;x) + s_n^\lambda(f;x) - f(x).$$

Now, using Lemma 4.2, we get

$$\begin{aligned}\|\tau_n^\lambda(f;x) - f(x)\|_p &\leqslant \|\tau_n^\lambda(f;x) - s_n^\lambda(f;x)\|_p + \|s_n^\lambda(f;x) - f(x)\|_p \\ &= \|\tau_n^\lambda(f;x) - s_n^\lambda(f;x)\|_p + O(\lambda(n)^{-1}).\end{aligned} \tag{5.1}$$

Now to prove our theorem, it remains to show that

$$\|\tau_n^\lambda(f;x) - s_n^\lambda(f;x)\|_p = O(\lambda(n)^{-1}). \tag{5.2}$$

Now, we write

$$\tau_n^\lambda(f;x) = \sum_{k=0}^{\lambda(n)} a_{\lambda(n),k} s_k(f;x) = \sum_{k=0}^{\lambda(n)} a_{\lambda(n),k} \Big(\sum_{i=0}^{k} u_i(f;x)\Big) = \sum_{k=0}^{\lambda(n)} A_{\lambda(n),k} u_k(f;x),$$

and thus, as $A_{\lambda(n),0} = 1$, we have

$$\tau_n^\lambda(f;x) - s_n^\lambda(f;x) = \sum_{k=1}^{\lambda(n)} (A_{\lambda(n),k} - A_{\lambda(n),0}) u_k(f;x).$$

Hence by Abeł's transformation, we obtain

$$\begin{aligned}\sum_{k=1}^{\lambda(n)} \Big(\frac{A_{\lambda(n),k} - A_{\lambda(n),0}}{k}\Big) k u_k(f;x) &= \sum_{k=1}^{\lambda(n)} b_{\lambda(n),k} k u_k(f;x) \\ = \sum_{k=1}^{\lambda(n)-1} (\Delta_k b_{\lambda(n),k}) \Big(\sum_{j=1}^{k} j u_j(f;x)\Big) &+ b_{\lambda(n),\lambda(n)} \sum_{j=1}^{\lambda(n)} j u_j(f;x).\end{aligned}$$

Thus by triangle inequality, we find

$$\begin{aligned}\|\tau_n^\lambda(f;x) - f(x)\|_p &\leqslant \sum_{k=1}^{\lambda(n)-1} |\Delta_k b_{\lambda(n),k}| \Big\| \sum_{j=1}^{k} j u_j(f;x)\Big\|_p \\ &\quad + |b_{\lambda(n),\lambda(n)}| \Big\| \sum_{j=1}^{\lambda(n)} j u_j(f;x)\Big\|_p \\ = \sum_{k=1}^{\lambda(n)-1} |\Delta_k b_{\lambda(n),k}| \Big\| \sum_{j=1}^{k} j u_j(f;x)\Big\|_p &+ |b_{\lambda(n),\lambda(n)}| \Big\| \sum_{j=1}^{\lambda(n)} j u_j(f;x)\Big\|_p\end{aligned} \tag{5.3}$$

Now

$$\begin{aligned}\sigma_n^\lambda(f;x) - s_n^\lambda(f;x) &= \frac{1}{\lambda(n)+1} \sum_{m=0}^{\lambda(n)} s_m(f;x) - s_n^\lambda(f;x) \\ &= \frac{1}{\lambda(n)+1} \sum_{m=0}^{\lambda(n)} s_m(f;x) - \sum_{k=0}^{\lambda(n)} u_k(f;x)\end{aligned}$$

$$= -\frac{1}{\lambda(n)+1} \sum_{j=1}^{\lambda(n)} ju_j(f;x).$$

Therefore by Lemma 4.1, we have

$$\begin{aligned}
(5.4) \qquad \Big\| \sum_{j=1}^{\lambda(n)} ju_j(f;x) \Big\|_p &= (\lambda(n)+1) \|\sigma_n^\lambda(f;x) - s_n^\lambda(f;x)\|_p \\
&= (\lambda(n)+1) O(\lambda(n)^{-1}) = O(1).
\end{aligned}$$

We note that

$$\begin{aligned}
|b_{\lambda(n),\lambda(n)}| &= \Big| \frac{A_{\lambda(n),\lambda(n)} - A_{\lambda(n),0}}{\lambda(n)} \Big| \\
&= \frac{|A_{\lambda(n),\lambda(n)} - A_{\lambda(n),0}|}{\lambda(n)} = \frac{(A_{\lambda(n),\lambda(n)} - A_{\lambda(n),0})}{\lambda(n)} = O(\lambda(n)^{-1}).
\end{aligned}$$

Thus

$$(5.5) \qquad \Big\| b_{\lambda(n),\lambda(n)} \sum_{j=1}^{\lambda(n)} ju_j(f;x) \Big\|_p = |b_{\lambda(n),\lambda(n)}| \Big\| \sum_{j=1}^{\lambda(n)} ju_j(f;x) \Big\|_p = O(\lambda(n)^{-1}).$$

We may write

$$\begin{aligned}
(5.6) \qquad \Delta_k b_{\lambda(n),k} &= \frac{\Delta_k (A_{\lambda(n),k} - A_{\lambda(n),0})}{k} + \frac{A_{\lambda(n),k+1} - A_{\lambda(n),0}}{k(k+1)} \\
&= \frac{1}{k(k+1)} \Big[(k+1)\Delta_k A_{\lambda(n),k} + \sum_{r=k+1}^{\lambda(n)} a_{\lambda(n),r} - \sum_{r=0}^{\lambda(n)} a_{\lambda(n),r} \Big] \\
&= \frac{1}{k(k+1)} \Big[(k+1) a_{\lambda(n),k} - \sum_{r=0}^{k} a_{\lambda(n),r} \Big].
\end{aligned}$$

Next, we shall verify by mathematical induction that

$$(5.7) \qquad \Big| \sum_{r=0}^{k} a_{\lambda(n),r} - (k+1) a_{\lambda(n),k} \Big| \leqslant \sum_{r=0}^{k-1} (r+1) |a_{\lambda(n),r} - a_{\lambda(n),r+1}|.$$

If $k=1$, then $\big| \sum_{r=0}^{1} a_{\lambda(n),r} - 2a_{\lambda(n),1} \big| = |a_{\lambda(n),0} - a_{\lambda(n),1}|$. Thus (5.7) holds.

Now, let us suppose that (5.7) holds for $k = m$ i.e.,

$$(5.8) \qquad \Big| \sum_{r=0}^{m} a_{\lambda(n),r} - (m+1) a_{\lambda(n),k} \Big| \leqslant \sum_{r=0}^{m-1} (r+1) |a_{\lambda(n),r} - a_{\lambda(n),r+1}|,$$

and we have to show that (5.7) is true for $k = m+1 (\leqslant \lambda(n))$.

For $k = m+1 (\leqslant \lambda(n))$ and using (5.8), we get

$$\Big| \sum_{r=0}^{m+1} a_{\lambda(n),r} - (m+2) a_{\lambda(n),m+1} \Big| = \Big| \sum_{r=0}^{m} a_{\lambda(n),r} - (m+1) a_{\lambda(n),m+1} \Big|$$

$$
\begin{aligned}
&= \Big| \sum_{r=0}^{m} a_{\lambda(n),r} - (m+1)a_{\lambda(n),m} \\
&\qquad\qquad + (m+1)a_{\lambda(n),m} - (m+1)a_{\lambda(n),m+1} \Big| \\
&\leqslant \sum_{r=0}^{m-1} (r+1)|a_{\lambda(n),r} - a_{\lambda(n),r+1}| + (m+1)|a_{\lambda(n),m} - a_{\lambda(n),m+1}| \\
&= \sum_{r=0}^{(m+1)-1} (r+1)|a_{\lambda(n),r} - a_{\lambda(n),r+1}|,
\end{aligned}
$$

which shows that (5.7) is true for $k = m+1$. Thus (5.7) holds good for any $1 \leqslant k \leqslant \lambda(n)$.

Using (3.2), (3.4), (5.6) and (5.7), we find

$$
\begin{aligned}
\sum_{k=1}^{\lambda(n)} |\Delta_k b_{\lambda(n),k}| &= \sum_{k=1}^{\lambda(n)} |\Delta_k [k^{-1}(A_{\lambda(n),k} - A_{\lambda(n),0})]| \\
&= \sum_{k=1}^{\lambda(n)} k^{-1}(k+1)^{-1} \Big| (k+1)a_{\lambda(n),k} - \sum_{r=0}^{k} a_{\lambda(n),r} \Big| \\
&= \sum_{k=1}^{\lambda(n)} k^{-1}(k+1)^{-1} \Big| \sum_{r=0}^{k} a_{\lambda(n),r} - (k+1)a_{\lambda(n),k} \Big| \\
&\leqslant \sum_{k=1}^{\lambda(n)} k^{-1}(k+1)^{-1} \sum_{r=0}^{k-1} (r+1)|a_{\lambda(n),r} - a_{\lambda(n),r+1}| \\
&= \sum_{k=1}^{\lambda(n)} k^{-1}(k+1)^{-1} \sum_{m=1}^{k} m|a_{\lambda(n),m-1} - a_{\lambda(n),m}| \\
&\leqslant \sum_{m=1}^{\lambda(n)} m|\Delta_m a_{\lambda(n),m-1}| \sum_{k=m}^{\infty} \frac{1}{k(k+1)} \\
&= \sum_{k=0}^{\lambda(n)-1} |\Delta_k a_{\lambda(n),k}| = O(a_{\lambda(n),0}) = O((\lambda(n))^{-1}).
\end{aligned}
\tag{5.9}
$$

The combining (5.3)–(5.5) and (5.9) yields (5.2). From (5.2) and (5.1), we get

$$
\|\tau_n^{\lambda}(f;x) - f(x)\|_p = O((\lambda(n))^{-1}).
$$

Case II. $p > 1$, $\alpha = 1$. For this, we first prove that the condition

$$
\sum_{k=0}^{\lambda(n)-1} (\lambda(n)-k)|\Delta_k a_{\lambda(n),k}| = O(1) \Rightarrow
\tag{5.10}
$$

$$
B_{\lambda(n)} \equiv \sum_{k=1}^{\lambda(n)} |\Delta_k b_{\lambda(n),k}| = \sum_{k=1}^{\lambda(n)} |\Delta_k \{k^{-1}(A_{\lambda(n),k} - A_{\lambda(n),0})\}| = O((\lambda(n))^{-1})
$$

is satisfied. For this, using (5.7) as in case (iii), we have

$$\begin{aligned}
(5.11)\qquad B_{\lambda(n)} &= \sum_{k=1}^{\lambda(n)} k^{-1}(k+1)^{-1}\left|(k+1)a_{\lambda(n),k} - \sum_{r=0}^{k} a_{\lambda(n),r}\right| \\
&\leqslant \sum_{k=1}^{\lambda(n)} k^{-1}(k+1)^{-1}\sum_{r=0}^{k-1}(r+1)|a_{\lambda(n),r} - a_{\lambda(n),r+1}| \\
&\leqslant \left(\sum_{k=1}^{r} + \sum_{k=r}^{\lambda(n)}\right) k^{-1}(k+1)^{-1}\sum_{m=1}^{k} m|a_{\lambda(n),m-1}| = B_1 + B_2, \text{ say.}
\end{aligned}$$

Now, using (3.3) and interchanging the order of summation, we get

$$\begin{aligned}
(5.12)\qquad B_1 &\equiv \sum_{k=1}^{r} k^{-1}(k+1)^{-1}\sum_{m=1}^{k} m|\Delta_m a_{\lambda(n),m-1}| \\
&\leqslant \sum_{m=1}^{r} m|\Delta_m a_{\lambda(n),m-1}|\sum_{k=m}^{\infty}\frac{1}{k(k+1)} \\
&= \sum_{m=1}^{r}|\Delta_m a_{\lambda(n),m-1}| = \sum_{m=\lambda(n)-r+1}^{\lambda(n)}|\Delta_{\lambda(n)-m}a_{\lambda(n),\lambda(n)-m}| \\
&\leqslant \sum_{m=r-1}^{\lambda(n)}|\Delta_{\lambda(n)-m}a_{\lambda(n),\lambda(n)-m}|\left(\frac{m}{r-1}\right) \\
&\leqslant \frac{1}{r-1}\sum_{m=1}^{\lambda(n)} m|\Delta_{\lambda(n)-m}a_{\lambda(n),\lambda(n)-m}| \\
&= \frac{1}{r-1}\sum_{k=0}^{\lambda(n)-1}(\lambda(n)-k)|\Delta_k a_{\lambda(n),k}| = \frac{1}{r-1}O(1) = O((\lambda(n))^{-1}).
\end{aligned}$$

On the other hand

$$\begin{aligned}
(5.13)\qquad B_2 &\equiv \sum_{k=r}^{\lambda(n)} k^{-1}(k+1)^{-1}\sum_{m=1}^{k} m|\Delta_m a_{\lambda(n),m-1}| \\
&\leqslant \sum_{k=r}^{\lambda(n)} k^{-1}(k+1)^{-1}\left[\left(\sum_{m=1}^{r} + \sum_{m=r}^{k}\right) m|\Delta_m a_{\lambda(n),m-1}|\right] \\
&= B_{\lambda(n),1} + B_{\lambda(n),2}, \text{ say.}
\end{aligned}$$

Using arguments as B_1 and (3.3), we obtain

$$(5.14)\qquad B_{\lambda(n),1} \equiv \sum_{k=r}^{\lambda(n)} k^{-1}(k+1)^{-1}\sum_{m=1}^{r} m|\Delta_m a_{\lambda(n),m-1}|$$

$$
\begin{aligned}
&\leqslant \sum_{k=r}^{\lambda(n)} (k+1)^{-1} \sum_{m=1}^{r} |\Delta_m a_{\lambda(n),m-1}| \\
&= \sum_{k=r}^{\lambda(n)} (k+1)^{-1} \sum_{m=\lambda(n)-r+1}^{\lambda(n)} |\Delta_{\lambda(n)-m} a_{\lambda(n),\lambda(n)-m}| \\
&\leqslant \sum_{k=r}^{\lambda(n)} (k+1)^{-1} \sum_{m=r-2}^{\lambda(n)} |\Delta_{\lambda(n)-m} a_{\lambda(n),\lambda(n)-m}| \frac{m}{r-2} \\
&\leqslant \frac{1}{r-2} \sum_{k=r}^{\lambda(n)} (k+1)^{-1} \sum_{m=1}^{\lambda(n)} m|\Delta_{\lambda(n)-m} a_{\lambda(n),\lambda(n)-m}| \\
&= \frac{1}{r-2} \sum_{k=r}^{\lambda(n)} (k+1)^{-1} \sum_{k=0}^{\lambda(n)-1} (\lambda(n)-k)|\Delta_k a_{\lambda(n),k}| \\
&= \frac{1}{r-2} \sum_{k=r}^{\lambda(n)} (k+1)^{-1} O(1) = O(1/\lambda(n)),
\end{aligned}
$$

again using (3.3) and interchanging the order of summation, we have

$$
\begin{aligned}
(5.15) \quad B_{\lambda(n),2} &\equiv \sum_{k=r}^{\lambda(n)} k^{-1}(k+1)^{-1} \sum_{m=r}^{k} m|\Delta_m a_{\lambda(n),m-1}| \\
&\leqslant \sum_{k=r}^{\lambda(n)} (k+1)^{-1} \sum_{m=r}^{k} |\Delta_m a_{\lambda(n),m-1}| \\
&\leqslant \frac{1}{r+1} \sum_{m=r}^{\lambda(n)} |\Delta_m a_{\lambda(n),m-1}| \sum_{k=m}^{\lambda(n)} 1 \\
&= \frac{1}{r+1} \sum_{m=r}^{\lambda(n)} (\lambda(n)-m+1)|\Delta_m a_{\lambda(n),m-1}| \\
&= \frac{1}{r+1} \sum_{k=r-1}^{\lambda(n)-1} (\lambda(n)-k)|\Delta_k a_{\lambda(n),k}| = \frac{1}{r+1} O(1) = O((\lambda(n))^{-1}).
\end{aligned}
$$

From (5.11), (5.12), (5.13), (5.14) and (5.15), we get (5.10). Thus (5.1), (5.3), (5.4), the estimate of $b_{\lambda(n),\lambda(n)}$ and Lemma 4.2 again yield (3.2).

Case IV. If $p = 1$, $0 < \alpha < 1$, using (4.1), $a_{\lambda(n),\lambda(n)+1} = 0$ and Abel's transformation, we obtain

$$
\tau_n^{\lambda}(f;x) - f(x) = \sum_{k=0}^{\lambda(n)-1} (\Delta_k a_{\lambda(n),k}) \left\{ \sum_{r=0}^{k} (s_r(f;x) - f(x)) \right\}
$$

$$
\begin{aligned}
&+ (a_{\lambda(n),\lambda(n)} - a_{\lambda(n),\lambda(n)+1}) \sum_{r=0}^{\lambda(n)} (s_r(f;x) - f(x)) \\
&= \sum_{k=0}^{\lambda(n)} (\Delta_k a_{\lambda(n),k}) \left\{ \sum_{r=0}^{k} (s_r(f;x) - f(x)) \right\} \\
&= \sum_{k=0}^{\lambda(n)} (\Delta_k a_{\lambda(n),k})(k+1)(\sigma_k(f;x) - f(x)).
\end{aligned}
$$

Hence, by conditions (3.5), (3.6) and Lemma 4.1, we find

$$
\begin{aligned}
\|\tau_n^{\lambda}(f;x) - f(x)\|_1 &\leqslant \sum_{k=0}^{\lambda(n)} k|\Delta_k a_{\lambda(n),k}| \|(\sigma_k(f;x) - f(x))\|_1 \\
&= O\left\{ \sum_{k=0}^{\lambda(n)} k^{1-\alpha} |\Delta_k a_{\lambda(n),k}| \right\} = O((\lambda(n))^{1-\alpha}) \sum_{k=0}^{\lambda(n)} |\Delta_k a_{\lambda(n),k}| \\
&= O((\lambda(n))^{1-\alpha}) O(a_{\lambda(n),0}) \\
&= O((\lambda(n))^{1-\alpha}) O((\lambda(n))^{-1}) = O((\lambda(n))^{-\alpha}).
\end{aligned}
$$

This completes the proof of case (IV) and consequently the proof of Theorem 3.1 is complete.

Acknowledgement. The authors are thankful to the anonymous learned referees for making valuable comments leading to the better presentation of the paper.

References

1. H. Armitage, I. J. Maddox, *A new type of Cesàro mean*, Analysis **9**(1–2) (1989), 195–204.
2. P. Chandra, *Trigonometric approximation of functions in L_p-norm*, J. Math. Anal. Appl. **275** (2002), 13–26.
3. U. Değer, İ. Dagadur, M. Küçükaslan, *Approximation by trigonometric polynomials to functions in L_p-norm*, Proc. Jangjeon Math. Soc. **15**(2) (2012), 203–213.
4. Deepmala, *A Study on Fixed Point Theorems for Nonlinear Contractions and its Applications*, Ph.D. Thesis, Pt. Ravishankar Shukla University, Chhatisgarh, India, 2014.
5. H. H. Khan, *On the degree of approximation of a functions belonging to the class* Lip(α, p), Indian J. Pure Appl. Math. **5** (1974), 132–136.
6. ———, *On the degree of approximation to a function belonging to weighted $W(L_p, \xi(t))$ class*, Aligarh Bull. Math. **3–4** (1973–1974), 83–88.
7. ———, *A note on a theorem of Izumi*, Commun. Fac. Sci. Univ. Ankara, Sér. A1, Math. Stat. **31** (1982), 123–127.
8. K. Khatri, *A study on approximation of functions in Banach spaces using summability methods*, Ph.D. Thesis, Sardar Vallabhbhai National Institute of Technology, India, 2014.
9. L. Leindler, *Trigonometric approximation in L_p-norm*, J. Math. Anal. Appl. **302** (2005), 129–136.
10. L. N. Mishra, *On existence and behavior of solutions to some nonlinear integral equations with Applications*, Ph.D. Thesis, National Institute of Technology, Assam, India, 2017.
11. V. N. Mishra, *Some problems on Approximations of functions in Banach spaces*, Ph.D. Thesis, Indian Institute of Technology Roorkee, Roorkee, India, 2007.

12. V. N. Mishra, L. N. Mishra, *Trigonometric approximation of signals (functions) in L_p $(p \geqslant 1)$ norm*, Int. J. Contemp. Math. Sci. **7**(19) (2012), 909–918.
13. V. N. Mishra, K. Khatri, L. N. Mishra, *Product $(N, p_n)(C, 1)$ summability of a sequence of Fourier coefficients*, Math. Sci., Springer **6**:38 (2012), https://doi.org/10.1186/2251-7456-6-38.
14. ———, *Approximation of functions belonging to* $\mathrm{Lip}(\xi(t), r)$ *class by* $(N, p_n)(E, q)$ *summability of conjugate series of Fourier series*, J. Inequal. Appl. **2012**:296 (2012), https://doi.org/10.1186/1029-242X-2012-296.
15. ———, *Using linear operators to approximate signals of* $\mathrm{Lip}(\alpha, p, p \geqslant 1)$*-class*, Filomat **27**(2) (2013), 355–365.
16. V. N. Mishra, K. Khatri, *Degree of approximation of functions $f \in H_\omega$ class by the $(N_p.E^1)$ means in the Hölder metric*, Int. J. Math. Math. Sci. **2014** (2014), Article ID 837408, 9 pages.
17. L. N. Mishra, V. N. Mishra, K. Khatri, Deepmala, *On the trigonometric approximation of signals belonging to generalized weighted Lipschitz $W(L_r, \xi(t)), (r \geqslant 1)-$ class by matrix $(C^1.N_p)$ operator of conjugate series of its Fourier series*, Appl. Math. Comput. **237** (2014), 252–263.
18. M. L. Mittal, B. E. Rhoades, V. N. Mishra, U. Singh, *Using infinite matrices to approximate functions of class* $\mathrm{Lip}(\alpha, p)$ *using trigonometric polynomials*, J. Math. Anal. Appl. **326** (2007), 667–676.
19. M. L. Mittal, B. E. Rhoades, S. Sonker, U. Singh, *Approximation of signals of class* $\mathrm{Lip}(\alpha, p)$ *by linear operators*, Appl. Math. Comput. **217**(9) (2011), 4483–4489.
20. M. L. Mittal, M. V. Singh, *Approximation of signals (functions) by trigonometric polynomials in L_p-norm*, Int. J. Math. Math. Sci. **2014** (2014), Article ID 267383, 6 pages.
21. R. N. Mohapatra, P. Chandra, *Hölder continuous functions and their Euler, Borel and Taylor means*, Math. Chron. **11** (1982), 81–96.
22. ———, *Degree of approximation of functions in the Hölder metric*, Acta. Math. Hung. **41** (1–2) (1983), 67–76.
23. M. Mursaleen, S. A. Mohiuddine, *Convergence Methods for Double Sequences and Applications*, Springer, New York, 2014.
24. J. A. Osikiewicz, *Equivalence results for Cesàro submethods*, Analysis, München **20**(1) (2000), 35–43.
25. J. G. Proakis, *Digital Communications*, McGraw-Hill, New York, 1995.
26. E. Z. Psarakis, G. V. Moustakides, *An L_2-based method for the design of 1-D zero phase FIR digital filters*, IEEE Trans. Circuits Syst., I, Fundam. Theory Appl. **44** (1997), 591–601.
27. E. S. Quade, *Trigonometric approximation in the mean*, Duke Math. J. **3** (1937), 529–542.

Applied Mathematics and Humanities Department
Sardar Vallabhbhai National Institute of Technology
Surat
India
kejal0909@gmail.com

(Received 07 01 2016)

Department of Mathematics
Indira Gandhi National Tribal University
Lalpur, Amarkantak, Anuppur, 484 887, Madhya Pradesh,
India
L. 1627 Awadh Puri Colony Beniganj
Phase-III
Opposite-Industrial Training Institute (I.T.I.)
Ayodhya Main Road Faizabad-224 001
(Uttar Pradesh) India
vishnunarayanmishra@gmail.com,
vishnu_narayanmishra@yahoo.co.in