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HARMONIC BETA-CONVEX FUNCTIONS INVOLVING HYPERGEOMETRIC FUNCTIONS

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ABSTRACT. We introduce and study a new class harmonic convex functions, which is called harmonic beta-convex functions. This new class includes several new and previously known classes of harmonic convex functions as special cases. We obtain some new integral inequalities involving hypergeometric functions. Some special cases are also discussed. Results obtained in this paper continue to hold for these cases. Ideas and techniques of this paper may stimulate further research.

1. Introduction

Convexity theory provides us a sound basis for studying a wide class of unrelated problems in a unified and general framework. Integral inequalities present a very active and fascinating field of research. These integral inequalities are useful in physics, where upper and lower bounds for natural phenomena described by integrals such as mechanical work are required. It is known that integral inequalities are closely related with convex functions and their variant forms. In recent years, convex sets and convex functions have been generalized and extended in several directions using different techniques and ideas. In fact, it is known that a function f is a convex function, if and only if, it satisfies an integral inequality, which is known as Hermite–Hadamard inequality. A very significant and important generalization of the convex functions is called the harmonic convex functions. Some authors established new integral inequalities related to harmonically convex functions. For recent results, refinements, generalizations, and new integral inequalities, see [**1**–**9**, **12**–**19**, **22**].

Motivated and inspired by the on going research, we introduce and investigate a new class of harmonic convex functions, which is called the harmonic beta-convex functions. It is shown that harmonic beta-convex functions are quite general and

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unifying ones. We obtain some new estimates for the harmonic beta-convex functions involving the Euler beta function and the Hypergeometric function. Some special special cases are also discussed. Results obtained in this paper represent significant improvement and refinement of the known results. The ideas and technique of this paper may stimulate further research in this dynamic field.

2. Preliminaries

In this section, we discuss some new and known results.

DEFINITION 2.1. [**20**]. A set $I = [a, b] \subseteq \mathbb{R} \setminus \{0\}$ is said to be a harmonic convex set, if

$$\frac{xy}{tx + (1-t)y} \in I, \qquad \forall x, y \in I,\ t \in [0, 1].$$

DEFINITION 2.2. [**5**]. A function $f\colon I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be harmonic convex function, if

$$f\Big(\frac{xy}{tx + (1-t)y}\Big) \leqslant (1-t)f(x) + tf(y), \qquad \forall x, y \in I,\ t \in [0, 1].$$

It has been shown by Noor and Noor [**10**] that the minimum of a differentiable harmonic convex functions on the harmonic convex sets can be characterized by a class of variational inequalities, which is called the harmonic variational inequalities. For the properties and other aspects of the harmonic convex functions, see [**8**,**10**,**11**] and the references therein.

We now introduce a new class of the harmonic convex functions, which is called harmonic beta-convex functions.

DEFINITION 2.3. A function $f\colon I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be harmonic beta-convex function, where $p, q > -1$, if

$$f\Big(\frac{xy}{tx + (1-t)y}\Big) \leqslant (1-t)^p t^q f(x) + t^p (1-t)^q f(y), \qquad \forall x, y \in I,\ t \in (0, 1). \tag{2.1}$$

The function f is said to be harmonic beta-concave function, if $-f$ is harmonic beta-convex function. We say that f is harmonic beta-mid convex, if (2.2) is assumed only for $t = \frac{1}{2}$, that is

$$f\Big(\frac{2xy}{x+y}\Big) \leqslant \frac{f(x) + f(y)}{2^{p+q}}.$$

REMARK 2.1. In the above definition, if $(p, q) = (0, 0), (1, 0), (-1, 0), (-s, 0), (s, 0), (1, 1)$, we obtain harmonic P-function, ordinary harmonic convex function, Godunova–Levin harmonic convex function, s-Godunova–Levin harmonic convex function, harmonic s-convex function and harmonic tgs-convex function respectively.

We remark that, if $p = q$, then Definition 2.3 reduces to:

DEFINITION 2.4. A function $f\colon I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be generalized harmonic beta-convex function, where $p, q > -1$, if

$$f\Big(\frac{xy}{tx + (1-t)y}\Big) \leqslant (1-t)^p t^q [f(x) + f(y)], \qquad \forall x, y \in I,\ t \in (0, 1). \tag{2.2}$$

It is worth mentioning that these generalized harmonic beta-convex functions have been introduced and studied by Noor [**8**]. For recent developments, see [**8**, **10**, **12**–**14**, **17**] and the references therein.

THEOREM 2.1. [**8**]. *Let $f\colon I = [a,b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be harmonic beta-convex function with $a < b$. If $f \in L[a,b]$, then*

$$2^{p+q-1} f\Big(\frac{2ab}{a+b}\Big) \leqslant \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,dx \leqslant \frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(\alpha+\beta+2)}[f(a)+f(b)].$$

We recall the following special functions which are known as beta function and hypergeometric function respectively.

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \qquad x,y>0,$$

$${}_2F_1[a,b;c;z] = \frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt, \qquad c>b>0, |z|<1.$$

The generalized quadrature formula of Guass–Jacobi type has the form

$$\int_a^b (x-a)^\alpha (b-x)^\beta f(x)dx = \sum_{k=0}^m B_{m,k} f(\gamma_k) + R_m[f],$$

for some $B_{m,k}$, γ_k and rest term $R_m[f]$. For more information, see [**21**].

3. Integral Inequalities

We need the following result in order to obtain new integral inequalities related to harmonic beta-convex function involving hypergeometric functions.

LEMMA 3.1. [**6**]. *If $f\colon I = [a,b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is a function such that $f \in L[a,b]$, then the following equality holds for some fixed $\alpha, \beta > 0$.*

$$\int_a^b (x-a)^\alpha (b-x)^\beta f(x)dx = a^{\alpha+1}b^{\beta+1}(b-a)^{\alpha+\beta+1}\int_0^1 \frac{t^\alpha(1-t)^\beta}{A_t^{\alpha+\beta+2}} f\Big(\frac{ab}{A_t}\Big)dt,$$

where $A_t = ta + (1-t)b$.

We now derive the main results of this paper.

THEOREM 3.1. *Let $f\colon I = [a,b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a differentiable function on the interior I° of I. If $f \in L[a,b]$ and $|f|$ is harmonic beta-convex function on $[a,b]$ and $\alpha, \beta > 0$, then*

$$\begin{aligned}\int_a^b (x-a)^\alpha (b-x)^\beta f(x)dx \leqslant \Big(\frac{a}{b}\Big)^{\alpha+1}(b-a)^{\alpha+\beta+1}\Big(&|f(a)|\beta(\alpha+q+1,\beta+p+1)\\ &{}_2F_1[\alpha+\beta+2,\alpha+q+1;\alpha+\beta+p+q+2;1-\frac{a}{b}]\\ &+|f(b)|\beta(\alpha+p+1,\beta+q+1)\\ &{}_2F_1[\alpha+\beta+2,\alpha+p+1;\alpha+\beta+p+q+2;1-\frac{a}{b}]\Big).\end{aligned}$$

PROOF. Using Lemma 3.1 and harmonic beta-convexity of $|f|$, we have

$$\int_a^b (x-a)^\alpha (b-x)^\beta f(x)dx = a^{\alpha+1}b^{\beta+1}(b-a)^{\alpha+\beta+1}\int_0^1 \frac{t^\alpha(1-t)^\beta}{A_t^{\alpha+\beta+2}}\Big|f\Big(\frac{ab}{A_t}\Big)\Big|dt$$

$$\leqslant a^{\alpha+1}b^{\beta+1}(b-a)^{\alpha+\beta+1}\int_0^1 \frac{t^\alpha(1-t)^\beta}{A_t^{\alpha+\beta+2}}\big\{(1-t)^p t^q|f(a)| + t^p(1-t)^q|f(b)|\big\}dt$$

$$= a^{\alpha+1}b^{\beta+1}(b-a)^{\alpha+\beta+1}$$

$$\times\Bigg(|f(a)|\int_0^1 \frac{t^{\alpha+q}(1-t)^{\beta+p}}{A_t^{\alpha+\beta+2}}dt + |f(b)|\int_0^1 \frac{t^{\alpha+p}(1-t)^{\beta+q}}{A_t^{\alpha+\beta+2}}dt\Bigg)$$

$$= a^{\alpha+1}b^{\beta+1}(b-a)^{\alpha+\beta+1}\times\Big(|f(a)|\frac{\beta(\alpha+q+1,\beta+p+1)}{b^{\alpha+\beta+2}}$$

$${}_2F_1\Big[\alpha+\beta+2,\alpha+q+1;\alpha+\beta+p+q+2;1-\frac{a}{b}\Big]$$

$$+|f(b)|\frac{\beta(\alpha+p+1,\beta+q+1)}{b^{\alpha+\beta+2}}$$

$${}_2F_1\Big[\alpha+\beta+2,\alpha+p+1;\alpha+\beta+p+q+2;1-\frac{a}{b}\Big]\Big). \qquad \square$$

THEOREM 3.2. *Let $f\colon I=[a,b]\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be a differentiable function on the interior I° of I. If $f\in L[a,b]$ and $|f|^\lambda$ is harmonic beta-convex function on $[a,b]$ and $\alpha,\beta>0$, $\lambda\geqslant 1$, then*

$$\int_a^b (x-a)^\alpha (b-x)^\beta f(x)dx \leqslant \Big(\frac{a}{b}\Big)^{\alpha+1}(b-a)^{\alpha+\beta+1}$$

$$\Big(\beta(\alpha+1,\beta+1){}_2F_1\Big[\alpha+\beta+2,\alpha+1;\alpha+\beta+2;1-\frac{a}{b}\Big]\Big)^{1-\frac{1}{\lambda}}$$

$$\Big(|f(a)|^\lambda\beta(\alpha+q+1,\beta+p+1)$$

$${}_2F_1\Big[\alpha+\beta+2,\alpha+q+1;\alpha+\beta+p+q+2;1-\frac{a}{b}\Big]$$

$$+|f(b)|^\lambda\frac{\beta(\alpha+p+1,\beta+q+1)}{b^{\alpha+\beta+2}}$$

$${}_2F_1\Big[\alpha+\beta+2,\alpha+p+1;\alpha+\beta+p+q+2;1-\frac{a}{b}\Big]\Big)^{\frac{1}{\lambda}}$$

PROOF. Using Lemma 3.1, harmonic beta-convexity of $|f|^\lambda$ and power mean inequality, we have

$$\int_a^b (x-a)^\alpha (b-x)^\beta f(x)dx = a^{\alpha+1}b^{\beta+1}(b-a)^{\alpha+\beta+1}\int_0^1 \frac{t^\alpha(1-t)^\beta}{A_t^{\alpha+\beta+2}}\Big|f\Big(\frac{ab}{A_t}\Big)\Big|dt$$

$$\leqslant a^{\alpha+1}b^{\beta+1}(b-a)^{\alpha+\beta+1}\Big(\int_0^1 \frac{t^\alpha(1-t)^\beta}{A_t^{\alpha+\beta+2}}dt\Big)^{1-\frac{1}{\lambda}}\Big(\int_0^1 \frac{t^\alpha(1-t)^\beta}{A_t^{\alpha+\beta+2}}\Big|f\Big(\frac{ab}{A_t}\Big)\Big|^\lambda dt\Big)^{\frac{1}{\lambda}}$$

$$\leqslant a^{\alpha+1}b^{\beta+1}(b-a)^{\alpha+\beta+1}\Big(\int_0^1 \frac{t^\alpha(1-t)^\beta}{A_t^{\alpha+\beta+2}}dt\Big)^{1-\frac{1}{\lambda}}$$

$$\left(\int_0^1 \frac{t^\alpha(1-t)^\beta}{A_t^{\alpha+\beta+2}}\big\{(1-t)^p t^q|f(a)|^\lambda + t^p(1-t)^q|f(b)|^\lambda\big\}dt\right)^{\frac{1}{\lambda}}$$

$$= a^{\alpha+1}b^{\beta+1}(b-a)^{\alpha+\beta+1}\left(\int_0^1 \frac{t^\alpha(1-t)^\beta}{A_t^{\alpha+\beta+2}}dt\right)^{1-\frac{1}{\lambda}}$$

$$\left(|f(a)|^\lambda \int_0^1 \frac{t^{\alpha+q}(1-t)^{\beta+p}}{A_t^{\alpha+\beta+2}}dt + |f(b)|^\lambda \int_0^1 \frac{t^{\alpha+p}(1-t)^{\beta+q}}{A_t^{\alpha+\beta+2}}dt\right)^{\frac{1}{\lambda}}$$

$$= pa^{\alpha+1}b^{\beta+1}(b-a)^{\alpha+\beta+1}$$

$$\left(\frac{\beta(\alpha+1,\beta+1)}{b^{\alpha+\beta+2}}{}_2F_1\Big[\alpha+\beta+2,\alpha+1;\alpha+\beta+2;1-\frac{a}{b}\Big]\right)^{1-\frac{1}{\lambda}}$$

$$\Big(|f(a)|^\lambda \frac{\beta(\alpha+q+1,\beta+p+1)}{b^{\alpha+\beta+2}}$$

$${}_2F_1\Big[\alpha+\beta+2,\alpha+q+1;\alpha+\beta+p+q+2;1-\frac{a}{b}\Big]$$

$$+ |f(b)|^\lambda \frac{\beta(\alpha+p+1,\beta+q+1)}{b^{\alpha+\beta+2}}$$

$${}_2F_1\Big[\alpha+\beta+2,\alpha+p+1;\alpha+\beta+p+q+2;1-\frac{a}{b}\Big]\Big)^{\frac{1}{\lambda}},$$

which is the required result. □

THEOREM 3.3. *Let $f\colon I=[a,b]\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be a differentiable function on the interior I° of I. If $f\in L[a,b]$ and $|f|^\lambda$ is harmonic beta-convex function on $[a,b]$ and $\alpha,\beta>0$, then*

$$\int_a^b (x-a)^\alpha(b-x)^\beta f(x)dx$$

$$\leqslant \Big(\frac{a}{b}\Big)^{\alpha+1}(b-a)^{\alpha+\beta+1}\Big(\beta(\alpha\mu+1,\beta\mu+1)$$

$${}_2F_1\Big[(\alpha+\beta+2)\mu,\alpha\mu+1;(\alpha+\beta)\mu+2;1-\frac{a}{b}\Big]\Big)^{\frac{1}{\mu}}$$

$$\times\big(\big[|f(a)|^\lambda+|f(b)|^\lambda\big]\beta(p+1,q+1)\big)^{\frac{1}{\lambda}},$$

where $\frac{1}{\lambda}+\frac{1}{\mu}=1$.

PROOF. Using Lemma 3.1, harmonic beta-convexity of $|f|^\lambda$ and Hölder's integral inequality, we have

$$\int_a^b (x-a)^\alpha(b-x)^\beta f(x)dx$$

$$= a^{\alpha+1}b^{\beta+1}(b-a)^{\alpha+\beta+1}\int_0^1 \frac{t^\alpha(1-t)^\beta}{A_t^{\alpha+\beta+2}}\Big|f\Big(\frac{ab}{A_t}\Big)\Big|dt$$

$$\leqslant a^{\alpha+1}b^{\beta+1}(b-a)^{\alpha+\beta+1}\left(\int_0^1 \frac{t^{\alpha\mu}(1-t)^{\beta\mu}}{A_t^{(\alpha+\beta+2)\mu}}dt\right)^{\frac{1}{\mu}}\left(\int_0^1 \Big|f\Big(\frac{ab}{A_t}\Big)\Big|^\lambda dt\right)^{\frac{1}{\lambda}}$$

$$\leqslant a^{\alpha+1}b^{\beta+1}(b-a)^{\alpha+\beta+1}\Big(\int_0^1 \frac{t^{\alpha\mu}(1-t)^{\beta\mu}}{A_t^{(\alpha+\beta+2)\mu}}dt\Big)^{\frac{1}{\mu}}$$
$$\Big(\int_0^1 \big[(1-t)^p t^q |f(a)|^\lambda + t^p(1-t)^q|f(b)|^\lambda\big]dt\Big)^{\frac{1}{\lambda}}$$
$$= a^{\alpha+1}b^{\beta+1}(b-a)^{\alpha+\beta+1}\Big(\frac{\beta(\alpha\mu+1,\beta\mu+1)}{b^{(\alpha+\beta+2)\mu}}$$
$${}_2F_1\Big[(\alpha+\beta+2)\mu, \alpha\mu+1; (\alpha+\beta)\mu+2; 1-\frac{a}{b}\Big]\Big)^{\frac{1}{\mu}}$$
$$\times\big(\big[|f(a)|^\lambda + |f(b)|^\lambda\big]\beta(p+1,q+1)\big)^{\frac{1}{\lambda}}, \qquad \square$$

Theorem 3.4. *Let $f\colon I=[a,b]\subseteq \mathbb{R}\setminus\{0\}\to\mathbb{R}$ be a differentiable function on the interior I° of I. If $f\in L[a,b]$ and $|f|^\lambda$ is harmonic beta-convex function on $[a,b]$ and $\alpha,\beta>0$, then*

$$\int_a^b (x-a)^\alpha(b-x)^\beta f(x)dx$$
$$\leqslant\Big(\frac{a}{b}\Big)^{\alpha+1}(b-a)^{\alpha+\beta+1}\beta^{\frac{1}{\mu}}(\alpha\mu+1,\beta\mu+1)\Big(|f(a)|^\lambda\beta(q+1,p+1)$$
$${}_2F_1[(\alpha+\beta+2)\lambda, q+1; p+q+2; 1-\frac{a}{b}] + |f(a)|^\lambda\beta(p+1,q+1)$$
$${}_2F_1[(\alpha+\beta+2)\lambda, p+1; p+q+2; 1-\frac{a}{b}]\Big)^{\frac{1}{\lambda}},$$

where $\frac{1}{\lambda}+\frac{1}{\mu}=1$.

Proof. Using Lemma 3.1, harmonic beta-convexity of $|f|^\lambda$ and the Holder's integral inequality, we have

$$\int_a^b (x-a)^\alpha(b-x)^\beta f(x)dx$$
$$= a^{\alpha+1}b^{\beta+1}(b-a)^{\alpha+\beta+1}\int_0^1 \frac{t^\alpha(1-t)^\beta}{A_t^{\alpha+\beta+2}}\Big|f\Big(\frac{ab}{A_t}\Big)\Big|dt$$
$$\leqslant a^{\alpha+1}b^{\beta+1}(b-a)^{\alpha+\beta+1}\Big(\int_0^1 t^{\alpha\mu}(1-t)^{\beta\mu}dt\Big)^{\frac{1}{\mu}}\Big(\int_0^1 \frac{1}{A_t^{(\alpha+\beta+2)\lambda}}\Big|f\Big(\frac{ab}{A_t}\Big)\Big|^\lambda dt\Big)^{\frac{1}{\lambda}}$$
$$\leqslant a^{\alpha+1}b^{\beta+1}(b-a)^{\alpha+\beta+1}\Big(\int_0^1 t^{\alpha\mu}(1-t)^{\beta\mu}dt\Big)^{\frac{1}{\mu}}$$
$$\Big(\int_0^1 \frac{1}{A_t^{(\alpha+\beta+2)\lambda}}\big\{(1-t)^p t^q|f(a)|^\lambda + t^p(1-t)^q|f(b)|^\lambda\big\}dt\Big)^{\frac{1}{\lambda}}$$
$$= a^{\alpha+1}b^{\beta+1}(b-a)^{\alpha+\beta+1}\Big(\int_0^1 t^{\alpha\mu}(1-t)^{\beta\mu}dt\Big)^{\frac{1}{\mu}}$$

$$\Big(|f(a)|^\lambda \int_0^1 \frac{(1-t)^p t^q}{A_t^{(\alpha+\beta+2)\lambda}} dt + |f(b)|^\lambda \int_0^1 \frac{t^p(1-t)^q}{A_t^{(\alpha+\beta+2)\lambda}} dt\Big)^{\frac{1}{\lambda}}$$

$$= a^{\alpha+1} b^{\beta+1} (b-a)^{\alpha+\beta+1} \beta^{\frac{1}{\mu}}(\alpha\mu+1, \beta\mu+1)\Big(|f(a)|^\lambda \frac{\beta(q+1,p+1)}{b^{(\alpha+\beta+2)\lambda}}$$

$${}_2F_1\Big[(\alpha+\beta+2)\lambda, q+1; p+q+2; 1-\frac{a}{b}\Big] + |f(a)|^\lambda \frac{\beta(p+1,q+1)}{b^{(\alpha+\beta+2)\lambda}}$$

$${}_2F_1\Big[(\alpha+\beta+2)\lambda, p+1; p+q+2; 1-\frac{a}{b}\Big]\Big)^{\frac{1}{\lambda}}, \qquad \square$$

THEOREM 3.5. *Let $f\colon I=[a,b]\subseteq \mathbb{R}\setminus\{0\}\to\mathbb{R}$ be a differentiable function on the interior I° of I. If $f\in L[a,b]$ and $|f|^\lambda$ is harmonic beta-convex function on $[a,b]$ and $\alpha,\beta>0$, then*

$$\int_a^b (x-a)^\alpha (b-x)^\beta f(x)dx \leqslant \Big(\frac{a}{b}\Big)^{\alpha+1} (b-a)^{\alpha+\beta+1} {}_2F_1^{\frac{1}{\mu}}\Big[(\alpha+\beta+2)\mu, 1; 2; 1-\frac{a}{b}\Big]$$
$$\big(|f(a)|^\lambda \beta(\alpha\lambda+q+1, \beta\lambda+p+1)$$
$$+ |f(b)|^\lambda \beta(\alpha\lambda+p+1, \beta\lambda+q+1)\big)^{\frac{1}{\lambda}},$$

where $\frac{1}{\lambda}+\frac{1}{\mu}=1$.

PROOF. Using Lemma 3.1, harmonic beta-convexity of $|f|^\lambda$ and the Holder's integral inequality, we have

$$\int_a^b (x-a)^\alpha (b-x)^\beta f(x)dx$$

$$= a^{\alpha+1} b^{\beta+1} (b-a)^{\alpha+\beta+1} \int_0^1 \frac{t^\alpha (1-t)^\beta}{A_t^{\alpha+\beta+2}} \Big| f\Big(\frac{ab}{A_t}\Big)\Big| dt$$

$$\leqslant a^{\alpha+1} b^{\beta+1} (b-a)^{\alpha+\beta+1} \Big(\int_0^1 \frac{1}{A_t^{(\alpha+\beta+2)\mu}} dt\Big)^{\frac{1}{\mu}} \Big(\int_0^1 t^{\alpha\lambda}(1-t)^{\beta\lambda} \Big| f\Big(\frac{ab}{A_t}\Big)\Big|^\lambda dt\Big)^{\frac{1}{\lambda}}$$

$$\leqslant a^{\alpha+1} b^{\beta+1} (b-a)^{\alpha+\beta+1} \Big(\int_0^1 \frac{1}{A_t^{(\alpha+\beta+2)\mu}} dt\Big)^{\frac{1}{\mu}}$$

$$\Big(\int_0^1 t^{\alpha\lambda}(1-t)^{\beta\lambda}[(1-t)^p t^q |f(a)|^\lambda + t^p(1-t)^q |f(b)|^\lambda] dt\Big)^{\frac{1}{\lambda}}$$

$$= a^{\alpha+1} b^{\beta+1} (b-a)^{\alpha+\beta+1} \Big(\int_0^1 \frac{1}{A_t^{(\alpha+\beta+2)\mu}} dt\Big)^{\frac{1}{\mu}}$$

$$\Big(|f(a)|^\lambda \int_0^1 t^{\alpha\lambda+q}(1-t)^{\beta\lambda+p} dt + |f(b)|^\lambda \int_0^1 t^{\alpha\lambda+p}(1-t)^{\beta\lambda+q} dt\Big)^{\frac{1}{\lambda}}$$

$$= a^{\alpha+1} b^{\beta+1} (b-a)^{\alpha+\beta+1} \Big(\frac{{}_2F_1[(\alpha+\beta+2)\mu, 1; 2; 1-\frac{a}{b}]}{b^{(\alpha+\beta+2)\mu}}\Big)^{\frac{1}{\mu}}$$

$$\big(|f(a)|^\lambda \beta(\alpha\lambda+q+1, \beta\lambda+p+1) + |f(b)|^\lambda \beta(\alpha\lambda+p+1, \beta\lambda+q+1)\big)^{\frac{1}{\lambda}}, \qquad \square$$

4. Conclusion

We have introduced and investigated the class of harmonic beta-convex function. We have derived some new integral inequalities via harmonic beta-convex functions. This new class is quite general and unifying one. Consequently, the results obtained in this paper continue to hold for all new and known classes of harmonic convex functions.

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