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ESTIMATES FOR HOLOMORPHIC FUNCTIONS CONCERNED WITH JACK'S LEMMA

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Abstract. We investigate a boundary version of the Schwarz lemma for classes $\mathcal{H}(\alpha)$. Also, we estimate a modulus of the angular derivative of $f(z)$ function at the boundary point b with $f(b) = b/\sqrt[\beta]{2}$, $0 < \beta \leqslant 1$. The sharpness of these inequalities is also proved.

1. Introduction

The classical Schwarz lemma gives information about the behavior of a holomorphic function on the unit disc $D = \{z : |z| < 1\}$ at the origin, subject only to the relatively weak hypotheses that the function map the unit disc to the disc and the origin to the origin. In its most basic form, the familiar Schwarz lemma says this [**8**, p. 329]:

Lemma 1.1 (Schwarz lemma). *Let $f\colon D \to D$ be a holomorphic function that fixes the origin 0. Then $|f(z)| \leqslant |z|$ for all $z \in D$, and $|f'(0)| \leqslant 1$. If $|f(z)| = |z|$ for any $z \neq 0$ or if $|f'(0)| = 1$, then f is a rotation of the unit disc.*

For historical background about the Schwarz lemma and its applications on the boundary of the unit disc, we refer to [**2**, **7**].

The following lemma, known as Jack's Lemma, is needed in the sequel [**9**].

Lemma 1.2 (Jack's lemma). *Let $f(z)$ be a non-constant and analytic function in the unit disc D with $f(0) = 0$. If $|f(z)|$ attains its maximum value on the circle $|z| = r$ at the point z_0, then $\frac{z_0 f'(z_0)}{f(z_0)} = k$, where $k \geqslant 1$ is a real number.*

Let $\mathcal{A}$ denote the class of functions $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$ that are holomorphic in the unit disc D. Also, let $\mathcal{H}(\alpha)$ be the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ satisfying $\left|\frac{2\alpha f(z)}{z f'(z)} - 1\right| < 1$ $(z \in D)$ for some $\alpha \in (0, 1)$. The certain holomorphic functions which is in the class of $\mathcal{H}(\alpha)$ on the unit disc D are considered in this paper. The subject of the present paper is to discuss some

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properties of the function $f(z)$ which belongs to the class of $\mathcal{H}(\alpha)$ by applying Jack's Lemma.

Let $f(z) \in \mathcal{H}(D)$ be a function with $f(0) = 0$. Then f is starlike if and only if $f'(0) \neq 0$ and $\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0$, $z \in D$. By $S^*(\alpha)$, we denote the family of starlike univalent functions of order α. $f \in S^*(\alpha)$ if $\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha$. We say that $f \in \mathcal{H}(\alpha)$ $(0 < \alpha < 1)$ if $f \in \mathcal{A}$ and $\mathcal{A}_\alpha(z) = \frac{2\alpha f(z)}{zf'(z)}$ maps the unit disc into $B(1,1)$. Using that conformal mapping $B(w) = (1+w)^{-1}$ maps D onto $\operatorname{Re} w > \frac{1}{2}$, one can check the classes $S^*(\alpha)$ and $\mathcal{H}(\alpha)$ coincide.

Let $f(z) \in \mathcal{H}(\alpha)$ $\left(\frac{1}{2} \leqslant \alpha < 1\right)$ and consider the function $h(z) = \left(\frac{z}{f(z)}\right)^\beta - 1$, where $0 < \beta \leqslant 1$. It is a holomorphic function in D and $h(0) = 0$. Now, let us show that $|h(z)| < 1$ in D. From the definition for $h(z)$, we have

$$h(z) + 1 = \left(\frac{z}{f(z)}\right)^\beta \tag{1.1}$$

If we take the derivative on (1.1), we obtain

$$\beta\left(\frac{z}{f(z)}\right)^{\beta-1}\left(\frac{f(z) - zf'(z)}{(f(z))^2}\right) = h'(z),$$
$$\beta\left(1 - \frac{zf'(z)}{f(z)}\right) = zh'(z)\left(\frac{f(z)}{z}\right)^\beta = \frac{zh'(z)}{1+h(z)}.$$

From this we obtain

$$\frac{zf'(z)}{f(z)} = 1 - \frac{zh'(z)}{\beta(1+h(z))} = \frac{\beta + \beta h(z) - zh'(z)}{\beta + \beta h(z)}.$$

Thus, we take

$$\frac{2\alpha f(z)}{zf'(z)} - 1 = \frac{\beta(2\alpha-1) + (2\alpha-1)\beta h(z) + zh'(z)}{\beta + \beta h(z) - zh'(z)}.$$

Since $f(z) \in \mathcal{H}(\alpha)$, we have $\left|\frac{2\alpha f(z)}{zf'(z)} - 1\right| < 1$ $(z \in D)$. We suppose that there exists a $z_0 \in D$ such that $\max_{|z| \leqslant |z_0|} |h(z)| = |h(z_0)| = 1$. From Jack's lemma, we obtain

$$h(z_0) = e^{i\theta} \quad \text{and} \quad \frac{z_0 h'(z_0)}{h(z_0)} = k.$$

So, we have that

$$\left|\frac{2\alpha f(z_0)}{zf'(z_0)} - 1\right| = \left|\frac{\frac{\beta(2\alpha-1)}{h(z_0)} + \beta(2\alpha-1) + \frac{z_0 h'(z_0)}{h(z_0)}}{\frac{\beta}{h(z_0)} + \beta - \frac{z_0 h'(z_0)}{h(z_0)}}\right| = \left|\frac{\beta(2\alpha-1) + \beta(2\alpha-1)e^{-i\theta} + k}{\beta e^{-i\theta} + \beta - k}\right|.$$
$$\left|\frac{2\alpha f(z_0)}{zf'(z_0)} - 1\right| = \left|\frac{\beta(2\alpha-1) + k + \beta(2\alpha-1)(\cos\theta - i\sin\theta)}{\beta - k + \beta(\cos\theta - i\sin\theta)}\right|.$$

This indicates that

$$\left|\frac{2\alpha f(z_0)}{zf'(z_0)} - 1\right|^2 = \left|\frac{\beta(2\alpha-1) + k + \beta(2\alpha-1)(\cos\theta - i\sin\theta)}{\beta - k + \beta\cos\theta - i\beta\sin\theta}\right|^2$$

$$= \frac{(2\alpha\beta + k - \beta)^2 + \beta^2(2\alpha - 1)^2 + 2\beta(2\alpha - 1)(2\alpha\beta + k - \beta)\cos\theta}{(\beta - k)^2 + \beta^2 + 2\beta(\beta - k)\cos\theta}.$$

We can define the function $p(s)$ by

$$p(s) = \frac{(2\alpha\beta + k - \beta)^2 + \beta^2(2\alpha - 1)^2 + 2\beta(2\alpha - 1)(2\alpha\beta + k - \beta)s}{(\beta - k)^2 + \beta^2 + 2\beta(\beta - k)s}$$

with $s = \cos\theta$. Taking the differentiation of $p(s)$ for s, we take

$$p'(s) = \frac{2\beta(2\alpha - 1)(2\alpha\beta + k - \beta)[(\beta - k)^2 + \beta^2 + 2\beta(\beta - k)s]}{((\beta - k)^2 + \beta^2 + 2\beta(\beta - k)s)^2}$$
$$- \frac{[\beta^2 + 2\beta(\beta - k)][(2\alpha\beta + k - \beta)^2 + \beta^2(2\alpha - 1)^2 + 2\beta(2\alpha - 1)(2\alpha\beta + k - \beta)s]}{((\beta - k)^2 + \beta^2 + 2\beta(\beta - k)s)^2}.$$

Since $\beta - k < 0$, from $0 < \beta \leqslant 1$, $k \geqslant 1$, we obtain $p(s)$ is monotone increasing for s, where $\frac{1}{2} \leqslant \alpha < 1$. Therefore, we have $p(s) \geqslant p(-1) = 1$. This contradicts the condition $f(z) \in \mathcal{H}(\alpha)$. This means that there is no point $z_0 \in D$ such that

$$\max_{|z| \leqslant |z_0|} |h(z)| = |h(z_0)| = 1.$$

Hence,

$$|h(z)| = \left| \left(\frac{z}{f(z)} \right)^{\beta} - 1 \right| < 1.$$

Therefore, we conclude that $|h(z)| < 1$ in D. From the Schwarz lemma, we obtain $|h(z)| \leqslant |z|$ and $|h'(0)| \leqslant 1$. Therefore, we obtain

$$|f(z)| \leqslant \frac{|z|}{(1 - |z|)^{\frac{1}{\beta}}} \quad \text{and} \quad |f''(0)| \leqslant \frac{2}{\beta}.$$

Equality is achieved in $|f(z)| \leqslant \frac{|z|}{(1-|z|)^{\frac{1}{\beta}}}$ (for some nonzero $z \in D$) or in $|f''(0)| \leqslant \frac{2}{\beta}$ if and only if $h(z) = ze^{i\theta}$, that is

$$f(z) = \frac{z}{(1 + ze^{i\theta})^{\frac{1}{\beta}}},$$

where θ is a real number. That proves

LEMMA 1.3. *If $f(z) \in \mathcal{H}(\alpha)$ $\left(\frac{1}{2} \leqslant \alpha < 1\right)$, then we have*

$$|f(z)| \leqslant \frac{|z|}{(1 - |z|)^{\frac{1}{\beta}}},$$

$$|f''(0)| \leqslant \frac{2}{\beta}, \tag{1.2}$$

where $0 < \beta \leqslant 1$. Equality is achieved in $|f(z)| \leqslant \frac{|z|}{(1-|z|)^{\frac{1}{\beta}}}$ (for some nonzero $z \in D$) or in (1.2) if and only if f is of the form

$$f(z) = \frac{z}{(1 + ze^{i\theta})^{\frac{1}{\beta}}},$$

where $0 < \beta \leqslant 1$ and θ is a real number.

This lemma yields an "$\mathcal{H}(\alpha)$ version" of the classical Schwarz lemma for holomorphic function of one complex variable. It is an elementary consequence of the Schwarz lemma that if f extends continuously to some boundary point b with $|b| = 1$, and if $|f(b)| = 1$ and $f'(b)$ exists, then $|f'(b)| \geqslant 1$.

Osserman [**17**] has given inequalities which are called the boundary Schwarz lemma. He has first showed that

$$|f'(b)| \geqslant \frac{2}{1 + |f'(0)|} \tag{1.3}$$

and

$$|f'(b)| \geqslant 1 \tag{1.4}$$

under the assumption $f(0) = 0$ where f is a holomorphic function mapping the unit disc into itself and b is a boundary point to which f extends continuously and $|f(b)| = 1$. n addition, the equality in (1.4) holds if and only if $f(z) = ze^{i\theta}$, θ is real. Also, $b = 1$ in the inequality (1.3) equality occurs for the function $f(z) = z\frac{z+\epsilon}{1+\epsilon z}$, $0 < \epsilon < 1$.

Let f be a holomorphic function in D, $f(0) = 0$ and $f(D) \subset D$. If, in addition, the function f has an angular limit $f(b)$ at $b \in \partial D$, $|f(b)| = 1$, then by the Julia–Wolff lemma the angular derivative $f'(b)$ exists and $1 \leqslant |f'(b)| \leqslant \infty$ (see [**18**]).

Inequality (1.4) and its generalizations have important applications in geometric theory of functions (see, e.g., [**8**,**18**]). Therefore, the interest to such type results is not vanished recently (see, e.g., [**1**, **2**, **5**, **6**, **10**, **11**, **16**, **17**, **19**, **20**] and references therein).

Inequality (1.3) is a particular case of a result due to Dubinin in [**5**], who strengthened the inequality $|f'(b)| \geqslant 1$ by involving zeros of the function f. The uniqueness part of the boundary Schwarz lemma was established in 1994 by Burns and Krantz [**3**]. In 2001, Chelts [**4**] generalized a boundary version of Schwarz's lemma proven by Burns and Krantz and provide sufficient conditions on the local behavior of f near a nite set of boundary points that requires f to be a nite Blaschke product.

Tang, Liu and Lu [**20**] established a new type of the classical boundary Schwraz lemma for holomorphic self-mappings of the unit polydisk D^n in $\mathbb{C}^n$. They extended the classical Schwarz lemma at the boundary to high dimensions.

Also, [**10**] showed some inequalities at a boundary point for different forms of holomorphic functions and found the condition for equality and in [**11**] a holomorphic self map defined on the closed unit disc with fixed points only on the boundary of the unit disc.

Similar types of results which are related with the subject of the paper can be found in [**12**–**14**]. In addition, the concerning results in more general aspects are discussed by Mateljević in [**15**] where they were announced on ResearchGate.

In [**1**], we gave an estimate below $|f'(b)|$ according to the first nonzero Taylor coefficient of f about two zeros, namely $z = 0$ and $z_1 \neq 0$. In [**16**], we obtained such type of results for other than the above mentioned class.

In [**2**], Boas discussed the classical Schwarz lemma, the boundary versions discovered by Gaston Julia and JuliusWolff, and some applications.

2. Main Results

In this section, we discuss different versions of the boundary Schwarz lemma. Assuming the existence of angular limit on a boundary point, we obtain some estimations from below for the moduli of derivatives of holomorphic functions from a certain class. We also show that these estimations are sharp. Then, we obtain more general results by taking into account the coefficients a_2 and a_3.

THEOREM 2.1. *Let* $f(z) \in \mathcal{H}(\alpha)$ $\left(\frac{1}{2} \leqslant \alpha < 1\right)$. *Assume that, for some* $b \in \partial D$, f *has an angular limit* $f(b)$ *at* b, $f(b) = \frac{b}{\sqrt[\beta]{2}}$, $0 < \beta \leqslant 1$. *Then*

$$|f'(b)| \geqslant \Big(\frac{1-2\beta}{2\beta}\Big)\frac{1}{\sqrt[\beta]{2}}. \tag{2.1}$$

The equality in (2.1) *holds if and only if*

$$f(z) = \frac{z}{(1+ze^{i\theta})^{\frac{1}{\beta}}},$$

where $0 < \beta \leqslant 1$ *and* θ *is a real number.*

PROOF. Let

$$h(z) = \Big(\frac{z}{f(z)}\Big)^{\beta} - 1.$$

Then $h(z)$ is a holomorphic function in the unit disc D and $h(0) = 0$. By Jack's lemma and since $f(z) \in \mathcal{H}(\alpha)$ $\left(\frac{1}{2} \leqslant \alpha < 1\right)$, we take $|h(z)| < 1$ for $|z| < 1$. Also, we have $|h(b)| = 1$ for $b \in \partial D$.

From (1.4), we obtain

$$\begin{aligned} 1 \leqslant |h'(b)| &= \left|\beta\Big(\frac{b}{f(b)}\Big)^{\beta-1}\Big(\frac{f(b)-bf'(b)}{(f(b))^2}\Big)\right| = \beta\big(\sqrt[\beta]{2}\big)^{\beta-1}\left|\frac{\frac{b}{\sqrt[\beta]{2}} - bf'(b)}{\big(\frac{b}{\sqrt[\beta]{2}}\big)^2}\right| \\ &= 2\beta\sqrt[\beta]{2}\left|\frac{1}{\sqrt[\beta]{2}} - f'(b)\right| \leqslant 2\beta\sqrt[\beta]{2}\left(\frac{1}{\sqrt[\beta]{2}} + |f'(b)|\right) \end{aligned}$$

Therefore, we take

$$1 \leqslant 2\beta\sqrt[\beta]{2}\left(\frac{1}{\sqrt[\beta]{2}} + |f'(b)|\right)$$

and

$$|f'(b)| \geqslant \Big(\frac{1-2\beta}{2\beta}\Big)\frac{1}{\sqrt[\beta]{2}}.$$

If $|f'(b)| = \left(\frac{1-2\beta}{2\beta}\right)\frac{1}{\sqrt[\beta]{2}}$, then $|h'(b)| = 1$ and so by Osserman [**17**], $h(z) = ze^{i\theta}$ for some real θ. It means that

$$f(z) = \frac{z}{(1+ze^{i\theta})^{\frac{1}{\beta}}}. \qquad \square$$

THEOREM 2.2. *Let* $f(z) \in \mathcal{H}(\alpha)$ $\left(\frac{1}{2} \leqslant \alpha < 1\right)$. *Assume that, for some* $b \in \partial D$, f *has an angular limit* $f(b)$ *at* b, $f(b) = \frac{b}{\sqrt[\beta]{2}}$, $0 < \beta \leqslant 1$. *Then*

$$|f'(b)| \geqslant \frac{1}{\sqrt[\beta]{2}}\Big(\frac{2-2\beta-\beta^2|f''(0)|}{2\beta+\beta^2|f''(0)|}\Big). \tag{2.2}$$

Inequality (2.2) *is sharp with equality for the function*

$$f(z) = z\frac{(1+az)^{\frac{1}{\beta}}}{(z^2+2az+1)^{\frac{1}{\beta}}},$$

where $a = \frac{\beta|f''(0)|}{2}$, $a \in [0,1]$ *and* $0 < \beta \leqslant 1$ (*see* (1.2)).

Proof. Let $h(z)$ be the same as in the proof of Theorem 2.1. From (1.3), we obtain

$$\frac{2}{1+|h'(0)|} \leqslant |h'(b)| \leqslant 2\beta\sqrt[\beta]{2}\Big(\frac{1}{\sqrt[\beta]{2}} + |f'(b)|\Big).$$

Since

$$h'(0) = -\beta\frac{f''(0)}{2} \quad \text{and} \quad |h'(0)| = \beta\frac{|f''(0)|}{2}$$

we take

$$\frac{2}{1+\beta\frac{|f''(0)|}{2}} \leqslant 2\beta\sqrt[\beta]{2}\Big(\frac{1}{\sqrt[\beta]{2}} + |f'(b)|\Big).$$

Therefore, we obtain inequality (2.2).

Now, we shall show that inequality (2.2) is sharp. Let

$$f(z) = z\frac{\sqrt[\beta]{1+az}}{\sqrt[\beta]{z^2+2az+1}}.$$

Then,

$$f'(1) = \frac{1}{\sqrt[\beta]{2}}\Big(\frac{\beta+\beta a-1}{\beta+\beta a}\Big).$$

Since $a = \frac{\beta|f''(0)|}{2}$, we take

$$|f'(1)| = \frac{1}{\sqrt[\beta]{2}}\Big(\frac{2-2\beta-\beta^2|f''(0)|}{2\beta+\beta^2|f''(0)|}\Big) \qquad \square$$

Inequality (2.2) can be strengthened as below by taking into account a_3 which is the third coefficient in the expansion of the function $f(z)$.

Theorem 2.3. *Let* $f(z) \in \mathcal{H}(\alpha)$ $\big(\frac{1}{2} \leqslant \alpha < 1\big)$. *Assume that, for some* $b \in \partial D$, f *has an angular limit* $f(b)$ *at* b, $f(b) = \frac{b}{\sqrt[\beta]{2}}$, $0 < \beta \leqslant 1$. *Then*

$$|f'(b)| \geqslant \frac{1}{\beta}\frac{1}{\sqrt[\beta]{2}}\Big(\frac{1}{2} - \beta + \frac{2(1-\beta|a_2|)^2}{2(1-\beta^2|a_2|^2)+\beta|(1+\beta)a_2^2-2a_3|}\Big). \tag{2.3}$$

The equality in (2.3) *occurs for the function* $f(z) = \frac{z}{(1+z)^{\frac{1}{\beta}}}$, *where* $0 < \beta \leqslant 1$.

Proof. Let $h(z)$ be the same as in the proof of Theorem 2.1 and $B(z) = z$. By the maximum principle for each $z \in D$, we have $|h(z)| \leqslant |B(z)|$. Therefore, $\varphi(z) = \frac{h(z)}{B(z)}$ is a holomorphic function in D and $|\varphi(z)| < 1$ for $|z| < 1$. In particular, we have

$$|\varphi(0)| = \beta|a_2|, \tag{2.4}$$

$$|\varphi'(0)| = \frac{\beta}{2}\big|(1+\beta)a_2^2 - 2a_3\big|.$$

Furthermore, the geometric meaning of the derivative and the inequality $|h(z)| \leqslant |B(z)|$ imply the inequality

$$\frac{bh'(b)}{h(b)} \geqslant |h'(b)| \geqslant |B'(b)| = \frac{bB'(b)}{B(b)}.$$

The function $T(z) = \frac{\varphi(z)-\varphi(0)}{1-\varphi(z)\overline{\varphi(0)}}$ is holomorphic in D, $|T(z)| < 1$ for $|z| < 1$, $T(0) = 0$ and $|T(b)| = 1$ for $b \in \partial D$. From (1.3), we obtain

$$\begin{aligned}\frac{2}{1+|T'(0)|} \leqslant |T'(b)| &\leqslant \frac{1+|\varphi(0)|}{1-|\varphi(0)|}\left|\frac{h'(b)}{B(b)} - \frac{h(b)B'(b)}{B^2(b)}\right| \\ &= \frac{1+|\varphi(0)|}{1-|\varphi(0)|}\{|h'(b)| - |B'(b)|\}.\end{aligned}$$

Since

$$|T'(0)| = \frac{\beta}{2}\frac{|(1+\beta)a_2^2 - 2a_3|}{1-\beta^2|a_2|^2},$$

we take

$$\begin{aligned}\frac{2}{1+\frac{\beta}{2}\frac{|(1+\beta)a_2^2-2a_3|}{1-\beta^2|a_2|^2}} &\leqslant \frac{1+\beta|a_2|}{1-\beta|a_2|}\{2\beta(1+2^{\frac{1}{\beta}}|f'(b)|)-1\}, \\ \frac{4(1-\beta^2|a_2|^2)}{2(1-\beta^2|a_2|^2)+\beta|(1+\beta)a_2^2-2a_3|}\frac{1-\beta|a_2|}{1+\beta|a_2|} &\leqslant 2\beta(1+2^{\frac{1}{\beta}}|f'(b)|)-1, \\ \frac{4(1-\beta|a_2|)^2}{2(1-\beta^2|a_2|^2)+\beta|(1+\beta)a_2^2-2a_3|} &\leqslant 2\beta(1+2^{\frac{1}{\beta}}|f'(b)|)-1, \\ 1+\frac{4(1-\beta|a_2|)^2}{2(1-\beta^2|a_2|^2)+\beta|(1+\beta)a_2^2-2a_3|} &\leqslant 2\beta(1+2^{\frac{1}{\beta}}|f'(b)|)\end{aligned}$$

and

$$|f'(b)| \geqslant \frac{1}{\beta}\frac{1}{\sqrt[\beta]{2}}\Big(\frac{1}{2} - \beta + \frac{2(1-\beta|a_2|)^2}{2(1-\beta^2|a_2|^2)+\beta|(1+\beta)a_2^2-2a_3|}\Big).$$

Now, we shall show that the inequality (2.3) is sharp. Let $f(z) = \frac{z}{(1+z)^{\frac{1}{\beta}}}$. Then

$$f'(z) = \frac{(1+z)^{\frac{1}{\beta}} - \frac{1}{\beta}(1+z)^{\frac{1}{\beta}-1}z}{(1+z)^{\frac{2}{\beta}}} \quad \text{and} \quad f'(1) = \frac{2^{\frac{1}{\beta}} - \frac{1}{\beta}2^{\frac{1}{\beta}-1}}{2^{\frac{2}{\beta}}} = \frac{1}{2^{\frac{1}{\beta}}}\Big(1 - \frac{1}{2\beta}\Big).$$

Since $|a_2| = \frac{1}{\beta}$, (2.3) is satisfied with equality. □

If $f(z) - z$ has no zeros different from $z = 0$ in Theorem 2.3, inequality (2.3) can be further strengthened. This is given by the following theorem.

THEOREM 2.4. *Let $f(z) \in \mathcal{H}(\alpha)$ $\big(\frac{1}{2} \leqslant \alpha < 1\big)$, $f(z) - z$ has no zeros in D except $z = 0$ and $a_2 < 0$. Assume that, for some $b \in \partial D$, f has an angular limit $f(b)$ at b, $f(b) = \frac{b}{\sqrt[\beta]{2}}$, $0 < \beta \leqslant 1$. Then*

$$|f'(b)| \geqslant \frac{1}{\beta}\frac{1}{\sqrt[\beta]{2}}\Big(\frac{1}{2} - \beta - \frac{2\beta|a_2|(\ln|\beta a_2|)^2}{4\beta|a_2|\ln|\beta a_2| - \beta|(1+\beta)a_2^2 - 2a_3|}\Big). \tag{2.5}$$

Equality in (2.5) *occurs for the function* $f(z) = z/(1+z)^{\frac{1}{\beta}}$, *where* $0 < \beta \leqslant 1$.

Proof. Let $a_2 < 0$ be in the expression of the function $f(z)$. Besides, let $h(z)$, $B(z)$ and $\varphi(z)$ be as in the proof of Theorem 2.3 and the function $f(z) - z$ has no zero points in D except $D - \{0\}$. Bearing in mind inequality (2.4), we denote by $\ln \varphi(z)$ the holomorphic branch of the logarithm normed by the condition

$$\ln \varphi(0) = \ln(-\beta a_2) = \ln |\beta a_2| + i \arg(-\beta a_2) = \ln |\beta a_2| < 0.$$

The composite function $\Phi(z) = \frac{\ln \varphi(z) - \ln \varphi(0)}{\ln \varphi(z) + \ln \varphi(0)}$ is holomorphic in the unit disc D, $|\Phi(z)| < 1$ for $|z| < 1$, $\Phi(0) = 0$ and $|\Phi(b)| = 1$ for $b \in \partial D$.

From (1.3), we obtain

$$\begin{aligned} \frac{2}{1+|\Phi'(0)|} \leqslant |\Phi'(b)| &= \frac{|2\ln\varphi(0)|}{|\ln\varphi(b)+\ln\varphi(0)|^2}\left|\frac{\varphi'(b)}{\varphi(b)}\right| = \frac{|2\ln\varphi(0)|}{|\ln\varphi(b)+\ln\varphi(0)|^2}|\varphi'(b)| \\ &= \frac{|2\ln\varphi(0)|}{|\ln\varphi(b)+\ln\varphi(0)|^2}\left|\frac{h'(b)}{B(b)} - \frac{h(b)B'(b)}{B^2(b)}\right| \\ &= \frac{-2\ln\varphi(0)}{\ln^2\varphi(0)+\arg^2\varphi(b)}\{|h'(b)| - |B'(b)|\}. \end{aligned}$$

In addition, it can be seen that

$$\begin{aligned} \Phi'(z) &= \frac{2\ln\varphi(0)}{(\ln\varphi(z)+\ln\varphi(0))^2}\frac{\varphi'(z)}{\varphi(z)}, \quad \Phi'(0) = \frac{1}{2\ln\varphi(0)}\frac{\varphi'(0)}{\varphi(0)}, \\ |\Phi'(0)| &= \frac{1}{|2\ln\varphi(0)|}\left|\frac{\varphi'(0)}{\varphi(0)}\right| = \frac{1}{-2\ln|\beta a_2|}\frac{\frac{\beta}{2}|(1+\beta)a_2^2 - 2a_3|}{|\beta a_2|}. \end{aligned}$$

Therefore, replacing $\arg^2 \varphi(b)$ by zero, we take

$$\begin{aligned} \frac{2}{1 - \frac{1}{2\ln|\beta a_2|}\frac{\frac{\beta}{2}|(1+\beta)a_2^2-2a_3|}{|\beta a_2|}} &\leqslant \frac{-2}{\ln|\beta a_2|}\{2\beta(1+2^{\frac{1}{\beta}}|f'(b)|) - 1\}, \\ \frac{-2|\beta a_2|(\ln|\beta a_2|)^2}{2\ln|\beta a_2|\beta a_2 - \frac{\beta}{2}|(1+\beta)a_2^2 - 2a_3|} &\leqslant \{2\beta(1+2^{\frac{1}{\beta}}|f'(b)|) - 1\}, \\ 1 - \frac{2|\beta a_2|(\ln|\beta a_2|)^2}{2\ln|\beta a_2|\beta a_2 - \frac{\beta}{2}|(1+\beta)a_2^2 - 2a_3|} &\leqslant 2\beta(1+2^{\frac{1}{\beta}}|f'(b)|) \end{aligned}$$

and we obtain (2.5) with an obvious equality case. □

We note that inequality (1.3) has been used in the proofs of Theorem 2.3 and Theorem 2.4. So, there are both a_2 and a_3 on the right-hand side of the inequalities. But, if we use (1.4) instead of (1.3), we obtain weaker, but more simple, inequality (not including a_3). It is formulated in the following theorem.

Theorem 2.5. *Under the hypotheses of Theorem* 2.4, *we have*

$$|f'(b)| \geqslant \frac{1}{\beta}\frac{1}{\sqrt[\beta]{2}}\Big(\frac{1}{2} - \beta - \frac{1}{4}\ln|\beta a_2|\Big) \tag{2.6}$$

The equality in (2.6) *holds if and only if*

$$f(z)=\frac{z}{\left(1+ze^{\frac{1+ze^{i\theta}}{1-ze^{i\theta}}\ln(-\beta a_2)}\right)^{\frac{1}{\beta}}},$$

where $a_2<0$ and θ is a real number.

Proof. From Theorem 2.4, using inequality (1.4) for the function $\Phi(z)$, we obtain

$$\begin{aligned}1\leqslant|\Phi'(b)|&=\frac{|2\ln\varphi(0)|}{|\ln\varphi(b)+\ln\varphi(0)|^2}\left|\frac{\varphi'(b)}{\varphi(b)}\right|=\frac{|2\ln\varphi(0)|}{|\ln\varphi(b)+\ln\varphi(0)|^2}|\varphi'(b)|\\&=\frac{|2\ln\varphi(0)|}{|\ln\varphi(b)+\ln\varphi(0)|^2}\left|\frac{h'(b)}{B(b)}-\frac{h(b)B'(b)}{B^2(b)}\right|\\&=\frac{-2\ln\varphi(0)}{\ln^2\varphi(0)+\arg^2\varphi(b)}\{|h'(b)|-|B'(b)|\}.\end{aligned}$$

Replacing $\arg^2\varphi(b)$ by zero, we take

$$1\leqslant\frac{-2}{\ln|\beta a_2|}\{2\beta(1+2^{\frac{1}{\beta}}|f'(b)|)-1\}\tag{2.7}$$

Therefore, we have inequality (2.6).

If $|f'(b)|=\frac{1}{\beta}\frac{1}{\sqrt[\beta]{2}}\left(\frac{1}{2}-\beta-\frac{1}{4}\ln|\beta a_2|\right)$ from (2.7) and $|\Phi'(b)|=1$, we obtain

$$f(z)=\frac{z}{\left(1+ze^{\frac{1+ze^{i\theta}}{1-ze^{i\theta}}\ln(-\beta a_2}\right)^{\frac{1}{\beta}}}.\qquad\square$$

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