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FUNCTIONS OF GENERALIZED BOUNDED VARIATION AND ITS MULTIPLE FOURIER COEFFICIENTS

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Abstract. Here, generalizing the class $(\Lambda^1,\Lambda^2)^*BV^{(p)}([0,2\pi]^2)$ to the class $(\Lambda^1,\Lambda^2)^*BV^{(p,q)}([0,2\pi]^2)$ of functions of p,q-$(\Lambda^1,\Lambda^2)^*$-bounded variation, it is observed that the class is a Banach space with respect to the pointwise operations and the generalized variation norm. Moreover, we estimate the order of magnitude of multiple Fourier coefficients of a function of this class.

1. Introduction

Fülöp and Móricz [**3**] estimated the order of magnitude of multiple Fourier coefficients of functions of $BV(\bar{\mathbb{T}}^N)$ in the sense of Vitali and Hardy, where $\mathbb{T}=[0,2\pi)$, which is generalized [**6**] for the functions of the class $(\Lambda^1,\dots,\Lambda^N)BV^{(p)}(\bar{\mathbb{T}}^N)$. Here, generalizing the class $(\Lambda^1,\Lambda^2)^*BV^{(p)}(\bar{\mathbb{T}}^2)$ to the class $(\Lambda^1,\Lambda^2)^*BV^{(p,q)}(\bar{\mathbb{T}}^2)$ of functions of p,q-$(\Lambda^1,\Lambda^2)^*$-bounded variation, we prove that it is a Banach space with respect to the pointwise operations and the generalized variation norm. Moreover, we estimate the order of magnitude of multiple Fourier coefficients of a function of this class.

2. Notations and definitions

Consider function f on $\mathbb{R}^k$. For $k=1$ and $I=[a,b]$, define $\Delta f_a^b = f(I) = f(b)-f(a)$. For $k=2$, $I=[a,b]$ and $J=[c,d]$, define

$$\Delta f_{(a,c)}^{(b,d)} = f(I\times J) = f(I,d)-f(I,c) = f(b,d)-f(a,d)-f(b,c)+f(a,c).$$

Definition 2.1. Let $\mathbb{L}$ be the class of nondecreasing sequences $\Lambda=\{\lambda_n\}_{n=1}^{\infty}$ of positive numbers such that $\sum_n \frac{1}{\lambda_n}$ diverges. Given $\Lambda=(\Lambda^1,\Lambda^2)$, where $\Lambda^k =$

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$\{\lambda_n^k\}_{n=1}^{\infty} \in \mathbb{L}$, for $k = 1, 2$, and $p, q \geqslant 1$, a complex valued measurable function f defined on $\bar{\mathbb{T}}^2$ is said to be of p, q-Λ-bounded variation (that is, $f \in \Lambda BV^{(p,q)}(\bar{\mathbb{T}}^2)$) if

$$V_{\Lambda_{p,q}}(f, \bar{\mathbb{T}}^2) = \sup_{P=P_1 \times P_2} \left\{V_{\Lambda_{p,q}}(f, \bar{\mathbb{T}}^2, P)\right\} < \infty,$$

where $P_1 : 0 = x_0 < x_1 < \cdots < x_m = 2\pi$, $P_2 : 0 = y_0 < y_1 < \cdots < y_n = 2\pi$ and

$$V_{\Lambda_{p,q}}(f, \bar{\mathbb{T}}^2, P) = \left(\sum_j \frac{\left(\sum_i \frac{|\Delta f(x_i, y_j)|^p}{\lambda_i^1} \right)^{q/p}}{\lambda_j^2} \right)^{1/q}, \quad \text{in which}$$

$$\Delta f(x_i, y_j) = f([x_i, x_{i+1}] \times [y_j, y_{j+1}]).$$

Consider a function $f\colon \bar{\mathbb{T}}^2 \to \mathbb{R}$ defined by $f(x, y) = g(x) + h(y)$, where g and h are any two arbitrary functions from $\bar{\mathbb{T}}$ into $\mathbb{R}$. Then $V_{\Lambda_{p,q}}(f, \bar{\mathbb{T}}^2) = 0$ implies $f \in \Lambda BV^{(p,q)}(\bar{\mathbb{T}}^2)$. Here, g, or h, or both g and h need not be bounded (or measurable). Thus a function $f \in \Lambda BV^{(p,q)}(\bar{\mathbb{T}}^2)$ need not be bounded (or measurable).

If $f \in \Lambda BV^{(p,q)}(\bar{\mathbb{T}}^2)$ is such that the marginal functions $f(0, .) \in \Lambda^2 BV^{(q)}(\bar{\mathbb{T}})$ and $f(., 0) \in \Lambda^1 BV^{(p)}(\bar{\mathbb{T}})$ (refer [**4**] for the definition of $\Lambda BV^{(p)}(\bar{\mathbb{T}})$) then f is said to be of p, q-Λ^*-bounded variation (that is, $f \in \Lambda^* BV^{(p,q)}(\bar{\mathbb{T}}^2)$).

Note that, for $q = p$, the classes $\Lambda BV^{(p,q)}(\bar{\mathbb{T}}^2)$ and $\Lambda^* BV^{(p,q)}(\bar{\mathbb{T}}^2)$ reduce to the classes $\Lambda BV^{(p)}(\bar{\mathbb{T}}^2)$ [**6**, Definition 1.2, p.28] and $\Lambda^* BV^{(p)}(\bar{\mathbb{T}}^2)$ respectively; for $q = p = 1$, the classes $\Lambda BV^{(p,q)}(\bar{\mathbb{T}}^2)$ and $\Lambda^* BV^{(p,q)}(\bar{\mathbb{T}}^2)$ reduce to the classes $\Lambda BV(\bar{\mathbb{T}}^2)$ [**1**, Definition 2, p. 8] and $\Lambda^* BV(\bar{\mathbb{T}}^2)$ respectively.

Definition 2.2. We say $f \in L^{(p,q)}(\bar{\mathbb{T}}^2)$ $(p, q \geqslant 1)$ if

$$\|f\|_{(p,q)} = \left(\int_0^{2\pi} \left(\int_0^{2\pi} |f(x, y)|^p dx \right)^{q/p} dy \right)^{1/q} < \infty.$$

Note that, for $q = p$, the class $L^{(p,q)}(\bar{\mathbb{T}}^2)$ reduces to the class $L^p(\bar{\mathbb{T}}^2)$.

Benedek and Panzone [**2**] observed that the space $(L^{(p,q)}(\bar{\mathbb{T}}^2), \|.\|_{(p,q)})$ is a Banach space.

3. New results for functions of two variables

For any $\mathbf{x} = (x, y) \in \bar{\mathbb{T}}^2$ and $\mathbf{k} = (m, n) \in \mathbb{Z}^2$, we denote their scalar product by $\mathbf{k} \cdot \mathbf{x} = mx + ny$.

For any $f \in L^1(\bar{\mathbb{T}}^2)$, where f is 2π-periodic in each variable, its Fourier series is defined as

$$f(\mathbf{x}) \sim \sum_{\mathbf{k} \in \mathbb{Z}^2} \hat{f}(\mathbf{k}) e^{i(\mathbf{k} \cdot \mathbf{x})},$$

where

$$\hat{f}(\mathbf{k}) = \frac{1}{(2\pi)^2} \iint_{\bar{\mathbb{T}}^2} f(\mathbf{x}) e^{-i(\mathbf{k} \cdot \mathbf{x})} d\mathbf{x}$$

denotes the $\mathbf{k}^{\text{th}}$ Fourier coefficient of f.

We prove the following results.

LEMMA 3.1. *If* $f \in \Lambda^* BV^{(p,q)}(\bar{\mathbb{T}}^2)$ $(p, q \geqslant 1)$ *then* f *is bounded on* $\bar{\mathbb{T}}^2$.

PROOF. For any $f \in \Lambda^* BV^{(p,q)}(\bar{\mathbb{T}}^2)$,

$$\begin{aligned}
|f(x,y)| &\leqslant |f([0,x]) \times [0,y])| + |f(x,0) - f(0,0)| + |f(0,y) - f(0,0)| + |f(0,0)| \\
&= (\lambda_1^1)^{\frac{1}{p}} (\lambda_1^2)^{\frac{1}{q}} \Big(\frac{1}{\lambda_1^2} \Big(\frac{|f([0,x]) \times [0,y])|^p}{\lambda_1^1} \Big)^{\frac{q}{p}} \Big)^{\frac{1}{q}} \\
&\quad + (\lambda_1^1)^{\frac{1}{p}} \Big(\frac{|f(x,0) - f(0,0)|^p}{\lambda_1^1} \Big)^{\frac{1}{p}} + (\lambda_1^2)^{\frac{1}{q}} \Big(\frac{|f(0,y) - f(0,0)|^q}{\lambda_1^2} \Big)^{\frac{1}{q}} + |f(0,0)| \\
&\leqslant (\lambda_1^1)^{\frac{1}{p}} (\lambda_1^2)^{\frac{1}{q}} V_{\Lambda_{p,q}}(f, \bar{\mathbb{T}}^2) + (\lambda_1^1)^{\frac{1}{p}} V_{\Lambda_p^1}(f(.,0), \bar{\mathbb{T}}) \\
&\quad + (\lambda_1^2)^{\frac{1}{q}} V_{\Lambda_q^2}(f(0,.), \bar{\mathbb{T}}) + |f(0,0)|
\end{aligned}$$

implies f is bounded on $\bar{\mathbb{T}}^2$. □

THEOREM 3.1. *The class* $\Lambda^* BV^{(p,q)}(\bar{\mathbb{T}}^2)$ *is a Banach space with respect to the pointwise operations and the variation norm:*
$\|f\| = \|f\|_\infty + V_{\Lambda_{p,q}}(f, \bar{\mathbb{T}}^2) + V_{\Lambda_p^1}(f(.,0), \bar{\mathbb{T}}) + V_{\Lambda_q^2}(f(0,.), \bar{\mathbb{T}}),\ f \in \Lambda^* BV^{(p,q)}(\bar{\mathbb{T}}^2)$.

PROOF. Let $\{f_k\}_{k=1}^\infty$ be a Cauchy sequence in $\Lambda^* BV^{(p,q)}(\bar{\mathbb{T}}^2)$. Then it converges uniformly to some function say f. In view of [**5**, Corollary 2.7, p.183], we get

$$\lim_{k\to\infty} V_{\Lambda_p^1}(f_k(.,0) - f(.,0), \bar{\mathbb{T}}) = 0, \tag{3.1}$$

$$\lim_{k\to\infty} V_{\Lambda_q^2}(f_k(0,.) - f(0,.), \bar{\mathbb{T}}) = 0. \tag{3.2}$$

For any $P = P_1 \times P_2$, where $P_1 : 0 = x_0 < x_1 < \cdots < x_m = 2\pi$ and $P_2 : 0 = y_0 < y_1 < \cdots < y_n = 2\pi$, by Minkowski's inequality, we have

$$\begin{aligned}
\left(\sum_j \frac{\Big(\sum_i \frac{|\Delta f_k(x_i,y_j)|^p}{\lambda_i^1} \Big)^{\frac{q}{p}}}{\lambda_j^2} \right)^{\frac{1}{q}} &= \left(\sum_j \frac{\Big(\sum_i \frac{|\Delta(f_k - f_l)(x_i,y_j) + \Delta f_l(x_i,y_j)|^p}{\lambda_i^1} \Big)^{\frac{q}{p}}}{\lambda_j^2} \right)^{\frac{1}{q}} \\
&\leqslant \left(\sum_j \frac{\Big(\sum_i \frac{(|\Delta(f_k - f_l)(x_i,y_j)| + |\Delta f_l(x_i,y_j)|)^p}{\lambda_i^1} \Big)^{\frac{q}{p}}}{\lambda_j^2} \right)^{\frac{1}{q}} \\
&= \left(\sum_j \frac{\Big(\Big(\sum_i \Big(\frac{|\Delta(f_k - f_l)(x_i,y_j)|}{(\lambda_i^1)^{\frac{1}{p}}} + \frac{|\Delta f_l(x_i,y_j)|}{(\lambda_i^1)^{\frac{1}{p}}} \Big)^p \Big)^{\frac{1}{p}} \Big)^q}{\lambda_j^2} \right)^{\frac{1}{q}} \\
&\leqslant \left(\sum_j \frac{\Big(\Big(\sum_i \frac{|\Delta(f_k - f_l)(x_i,y_j)|^p}{\lambda_i^1} \Big)^{\frac{1}{p}} + \Big(\sum_i \frac{|\Delta f_l(x_i,y_j)|^p}{\lambda_i^1} \Big)^{\frac{1}{p}} \Big)^q}{\lambda_j^2} \right)^{\frac{1}{q}} \\
&= \left(\sum_j \Big(\frac{\Big(\sum_i \frac{|\Delta(f_k - f_l)(x_i,y_j)|^p}{\lambda_i^1} \Big)^{\frac{1}{p}}}{(\lambda_j^2)^{\frac{1}{q}}} + \frac{\Big(\sum_i \frac{|\Delta f_l(x_i,y_j)|^p}{\lambda_i^1} \Big)^{\frac{1}{p}}}{(\lambda_j^2)^{\frac{1}{q}}} \Big)^q \right)^{\frac{1}{q}}
\end{aligned}$$

$$\leqslant \left(\sum_j \frac{\left(\sum_i \frac{|\Delta(f_k - f_l)(x_i,y_j)|^p}{\lambda_i^1} \right)^{\frac{q}{p}}}{\lambda_j^2} \right)^{\frac{1}{q}} + \left(\sum_j \frac{\left(\sum_i \frac{|\Delta f_l(x_i,y_j)|^p}{\lambda_i^1} \right)^{\frac{q}{p}}}{\lambda_j^2} \right)^{\frac{1}{q}}$$
$$= V_{\Lambda_{p,q}}(f_k - f_l, \bar{\mathbb{T}}^2, P) + V_{\Lambda_{p,q}}(f_l, \bar{\mathbb{T}}^2, P).$$

Thus,

$$\begin{aligned} V_{\Lambda_{p,q}}(f_k, \bar{\mathbb{T}}^2, P) &\leqslant V_{\Lambda_{p,q}}(f_k - f_l, \bar{\mathbb{T}}^2, P) + V_{\Lambda_{p,q}}(f_l, \bar{\mathbb{T}}^2, P) \\ &\leqslant V_{\Lambda_{p,q}}(f_k - f_l, \bar{\mathbb{T}}^2) + V_{\Lambda_{p,q}}(f_l, \bar{\mathbb{T}}^2). \end{aligned}$$

This implies,

$$V_{\Lambda_{p,q}}(f_k, \bar{\mathbb{T}}^2) \leqslant V_{\Lambda_{p,q}}(f_k - f_l, \bar{\mathbb{T}}^2) + V_{\Lambda_{p,q}}(f_l, \bar{\mathbb{T}}^2)$$

and

$$|V_{\Lambda_{p,q}}(f_k, \bar{\mathbb{T}}^2) - V_{\Lambda_{p,q}}(f_l, \bar{\mathbb{T}}^2)| \leqslant V_{\Lambda_{p,q}}(f_k - f_l, \bar{\mathbb{T}}^2) \to 0 \ \text{ as } \ k, l \to \infty.$$

Hence, $\{V_{\Lambda_{p,q}}(f_k, \bar{\mathbb{T}}^2)\}_{k=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$ and it is bounded by some constant say $M > 0$. Therefore,

$$\begin{aligned} V_{\Lambda_{p,q}}(f, \bar{\mathbb{T}}^2, P) &= \left(\sum_j \frac{\left(\sum_i \frac{|\Delta f(x_i,y_j)|^p}{\lambda_i^1} \right)^{\frac{q}{p}}}{\lambda_j^2} \right)^{\frac{1}{q}} \\ &= \lim_{k\to\infty} \left(\sum_j \frac{\left(\sum_i \frac{|\Delta f_k(x_i,y_j)|^p}{\lambda_i^1} \right)^{\frac{q}{p}}}{\lambda_j^2} \right)^{\frac{1}{q}} \\ &\leqslant \lim_{k\to\infty} V_{\Lambda_{p,q}}(f_k, \bar{\mathbb{T}}^2) \leqslant M < \infty. \end{aligned}$$

This together with (3.1) and (3.2) imply $f \in \Lambda^* BV^{(p,q)}(\bar{\mathbb{T}}^2)$. Moreover,

$$\begin{aligned} V_{\Lambda_{p,q}}(f_k - f, \bar{\mathbb{T}}^2, P) &= \left(\sum_j \frac{\left(\sum_i \frac{|\Delta(f_k - f)(x_i,y_j)|^p}{\lambda_i^1} \right)^{\frac{q}{p}}}{\lambda_j^2} \right)^{\frac{1}{q}} \\ &= \lim_{l\to\infty} \left(\sum_j \frac{\left(\sum_i \frac{|\Delta(f_k - f_l)(x_i,y_j)|^p}{\lambda_i^1} \right)^{\frac{q}{p}}}{\lambda_j^2} \right)^{\frac{1}{q}} \\ &\leqslant \lim_{l\to\infty} V_{\Lambda_{p,q}}(f_k - f_l, \bar{\mathbb{T}}^2) \to 0 \ as \ k \to \infty. \end{aligned}$$

This together with (3.1) and (3.2) imply $(\Lambda^* BV^{(p,q)}(\bar{\mathbb{T}}^2), \|.\|)$ is a Banach space. □

Theorem 3.2. *If $f \in \Lambda BV^{(p,q)}(\bar{\mathbb{T}}^2) \cap L^{(p,q)}(\bar{\mathbb{T}}^2)$ $(p, q \geqslant 1)$ and $\mathbf{k} = (m, n) \in \mathbb{Z}^2$ is such that $mn \neq 0$, then*

$$\hat{f}(\mathbf{k}) = O\left(\frac{1}{\left(\sum_{j=1}^{|m|} \frac{1}{\lambda_j^1} \right)^{\frac{1}{p}} \left(\sum_{k=1}^{|n|} \frac{1}{\lambda_k^2} \right)^{\frac{1}{q}}} \right). \tag{3.3}$$

PROOF. Since $\hat{f}(m,n) = \frac{1}{4\pi^2}\iint_{\mathbb{T}^2} f(x,y)e^{-imx}\ e^{-iny}dx\,dy$, we have

$$4|\hat{f}(m,n)| = \frac{1}{4\pi^2}\left|\iint_{\mathbb{T}^2} f\Big(\Big[x, x+\frac{\pi}{m}\Big]\times\Big[y, y+\frac{\pi}{n}\Big]\Big)e^{-imx}\ e^{-iny}dx\,dy\right|.$$

Because of the periodicity of f in each variable, we get

$$\iint_{\mathbb{T}^2}|\Delta f_{jk}(x,y)|dx\,dy = \iint_{\mathbb{T}^2}\Big|f\Big(\Big[x, x+\frac{\pi}{m}\Big]\times\Big[y, y+\frac{\pi}{n}\Big]\Big)\Big|dx\,dy,$$

where $\Delta f_{jk}(x,y) = f\big(\big[x+\frac{(j-1)\pi}{m}, x+\frac{j\pi}{m}\big]\times\big[y+\frac{(k-1)\pi}{n}, y+\frac{k\pi}{n}\big]\big)$, for any $j,k\in\mathbb{Z}$. Hence,

$$|\hat{f}(m,n)| \leqslant \frac{1}{16\pi^2}\iint_{\mathbb{T}^2}|\Delta f_{jk}(x,y)|dx\,dy.$$

Dividing both sides of the above inequality by λ_j^1 and then summing over $j=1$ to $|m|$, we have

$$|\hat{f}(m,n)|\left(\sum_{j=1}^{|m|}\frac{1}{\lambda_j^1}\right) \leqslant \frac{1}{16\pi^2}\iint_{\mathbb{T}^2}\left(\sum_{j=1}^{|m|}\frac{|\Delta f_{jk}(x,y)|}{(\lambda_j^1)^{\frac{1}{p}+\frac{1}{r}}}\right)dx\,dy,$$

where r is the index conjugate to p. Applying Hölder's inequality on the right-hand side of the above inequality, we get

$$|\hat{f}(m,n)|\left(\sum_{j=1}^{|m|}\frac{1}{\lambda_j^1}\right) \leqslant \frac{1}{16\pi^2}\iint_{\mathbb{T}^2}\left(\sum_{j=1}^{|m|}\frac{|\Delta f_{jk}(x,y)|^p}{\lambda_j^1}\right)^{\frac{1}{p}}\left(\sum_{j=1}^{|m|}\frac{1}{\lambda_j^1}\right)^{\frac{1}{r}}dx\,dy.$$

Thus,

$$|\hat{f}(m,n)|\left(\sum_{j=1}^{|m|}\frac{1}{\lambda_j^1}\right)^{\frac{1}{p}} \leqslant \frac{1}{16\pi^2}\iint_{\mathbb{T}^2}\left(\sum_{j=1}^{|m|}\frac{|\Delta f_{jk}(x,y)|^p}{\lambda_j^1}\right)^{\frac{1}{p}}dx\,dy.$$

Dividing both sides of the above inequality by λ_k^2 and then summing over $k=1$ to $|n|$, we have

$$|\hat{f}(m,n)|\left(\sum_{j=1}^{|m|}\frac{1}{\lambda_j^1}\right)^{\frac{1}{p}}\left(\sum_{k=1}^{|n|}\frac{1}{\lambda_k^2}\right) \leqslant \frac{1}{16\pi^2}\iint_{\mathbb{T}^2}\left(\sum_{k=1}^{|n|}\frac{\left(\sum_{j=1}^{|m|}\frac{|\Delta f_{jk}(x,y)|^p}{\lambda_j^1}\right)^{\frac{1}{p}}}{(\lambda_k^2)^{\frac{1}{q}+\frac{1}{s}}}\right)dx\,dy,$$

where s is the index conjugate to q. Applying Hölder's inequality on the right-hand side of the above inequality, we get

$$|\hat{f}(m,n)|\left(\sum_{j=1}^{|m|}\frac{1}{\lambda_j^1}\right)^{\frac{1}{p}}\left(\sum_{k=1}^{|n|}\frac{1}{\lambda_k^2}\right)$$
$$\leqslant \frac{1}{16\pi^2}\iint_{\mathbb{T}^2}\left(\sum_{k=1}^{|n|}\frac{\left(\sum_{j=1}^{|m|}\frac{|\Delta f_{jk}(x,y)|^p}{\lambda_j^1}\right)^{\frac{q}{p}}}{\lambda_k^2}\right)^{\frac{1}{q}}\left(\sum_{k=1}^{|n|}\frac{1}{\lambda_k^2}\right)^{\frac{1}{s}}dx\,dy.$$

Therefore,

$$|\hat{f}(m,n)|\left(\sum_{j=1}^{|m|}\frac{1}{\lambda_j^1}\right)^{\frac{1}{p}}\left(\sum_{k=1}^{|n|}\frac{1}{\lambda_k^2}\right)^{\frac{1}{q}}$$
$$\leqslant \frac{1}{16\pi^2}\iint_{\bar{\mathbb{T}}^2}\left(\sum_{k=1}^{|n|}\frac{\left(\sum_{j=1}^{|m|}\frac{|\Delta f_{jk}(x,y)|^p}{\lambda_j^1}\right)^{\frac{q}{p}}}{\lambda_k^2}\right)^{\frac{1}{q}}dx\,dy \leqslant V_{\Lambda_{p,q}}(f,\bar{\mathbb{T}}^2). \quad \square$$

Theorem 3.2, with $p = q$, reduces to [**6**, Theorem 2.1, p. 30] as a particular case.

Corollary 3.1. *If a measurable function* $f \in \Lambda^*BV^{(p,q)}(\bar{\mathbb{T}}^2)$ $(p,q \geqslant 1)$ *and* $\mathbf{k} = (m,n) \in \mathbb{Z}^2$ *is such that* $mn \neq 0$, *then* (3.3) *holds true.*

Proof. In view of Lemma 3.1, $f \in \Lambda^*BV^{(p,q)}(\bar{\mathbb{T}}^2)$ implies f is bounded on $\bar{\mathbb{T}}^2$. Since $\Lambda^*BV^{(p,q)}(\bar{\mathbb{T}}^2) \subset \Lambda BV^{(p,q)}(\bar{\mathbb{T}}^2)$, the corollary follows from Theorem 3.2. $\square$

Corollary 3.2. *If a measurable function* $f \in \Lambda^*BV^{(p,q)}(\bar{\mathbb{T}}^2)$ $(p,q \geqslant 1)$ *and* $\mathbf{k} = (m,0) \in \mathbb{Z}^2$ *is such that* $m \neq 0$, *then*

$$\hat{f}(\mathbf{k}) = O\left(\frac{1}{\left(\sum_{j=1}^{|m|}\frac{1}{\lambda_j^1}\right)^{\frac{1}{p}}}\right).$$

Proof. Since $\hat{f}(m,0) = \frac{1}{4\pi^2}\iint_{\bar{\mathbb{T}}^2} f(x,y)e^{-imx}dx\,dy$, we have

$$2|\hat{f}(m,0)| = \frac{1}{4\pi^2}\left|\iint_{\bar{\mathbb{T}}^2}\left(f\left(x+\frac{\pi}{m},y\right)-f(x,y)\right)e^{-imx}dx\,dy\right|.$$

Because of the periodicity of f in each variable, we get

$$\iint_{\bar{\mathbb{T}}^2}|\Delta f_j(x,y)|dx\,dy = \iint_{\bar{\mathbb{T}}^2}\left|f\left(x+\frac{\pi}{m},y\right)-f(x,y)\right|dx\,dy,$$

where $\Delta f_j(x,y) = f\big(x+\frac{j\pi}{m},y\big) - f\big(x+\frac{(j-1)\pi}{m},y\big)$, for any $j \in \mathbb{Z}$. Hence,

$$|\hat{f}(m,0)| \leqslant \frac{1}{8\pi^2}\iint_{\bar{\mathbb{T}}^2}|\Delta f_j(x,y)|\ dx\ dy.$$

Dividing both sides of the above inequality by λ_j^1 and then summing over $j = 1$ to $|m|$, we have

$$|\hat{f}(m,0)|\left(\sum_{j=1}^{|m|}\frac{1}{\lambda_j^1}\right) \leqslant \frac{1}{8\pi^2}\iint_{\bar{\mathbb{T}}^2}\left(\sum_{j=1}^{|m|}\frac{|\Delta f_j(x,y)|}{(\lambda_j^1)^{\frac{1}{p}+\frac{1}{r}}}\right)dx\,dy,$$

where r is the index conjugate to p.

Applying Hölder's inequality on the right-hand side of the above inequality, we get

$$|\hat{f}(m,0)|\left(\sum_{j=1}^{|m|}\frac{1}{\lambda_j^1}\right) \leqslant \frac{1}{8\pi^2}\iint_{\bar{\mathbb{T}}^2}\left(\sum_{j=1}^{|m|}\frac{|\Delta f_j(x,y)|^p}{\lambda_j^1}\right)^{\frac{1}{p}}\left(\sum_{j=1}^{|m|}\frac{1}{\lambda_j^1}\right)^{\frac{1}{r}}dx\,dy.$$

Thus,

$$\begin{aligned}|\hat{f}(m,0)|\left(\sum_{j=1}^{|m|}\frac{1}{\lambda_j^1}\right)^{\frac{1}{p}} &\leqslant \frac{1}{8\pi^2}\iint_{\bar{\mathbb{T}}^2}\left(\sum_{j=1}^{|m|}\frac{|\Delta f_j(x,y)|^p}{\lambda_j^1}\right)^{\frac{1}{p}}dx\,dy\\ &\leqslant V_{\Lambda_p^1}(f(.,y),\bar{\mathbb{T}})\\ &\leqslant 2\big((\lambda_1^2)^{\frac{1}{q}}V_{\Lambda_{p,q}}(f,\bar{\mathbb{T}}^2)+V_{\Lambda_p^1}(f(.,0),\bar{\mathbb{T}})\big),\end{aligned}$$

follows from the inequalities $|x+y|^p \leqslant 2^p(|x|^p+|y|^p)$ and $|x+y|^{\frac{1}{p}} \leqslant |x|^{\frac{1}{p}}+|y|^{\frac{1}{p}}$, for all $x,y\in\mathbb{R}$ and $p\geqslant 1$. □

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