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ON CLASSES OF HARMONIC FUNCTIONS OF CARLEMAN TYPE

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Abstract. Let f be harmonic functions on the unit disk $\mathbb{D}$, of the complex plane $\mathbb{C}$. We show that f can be expanded in a series $f = \sum_n f_n$, where f_n is a harmonic function on $\mathbb{D}_{n,\Gamma,A}$ satisfying $\sup_{z\in\mathbb{D}_{n,\Gamma,A}} |f_n(z)| \leqslant C\rho^n$ for some constants $C > 0$ and $0 < \rho < 1$, and where $(\mathbb{D}_{n,\Gamma,A})_n$ is a suitably chosen sequence of decreasing neighborhoods of the closure of $\mathbb{D}$. Conversely, if f admits such an expansion then f is of Carleman type. The decrease of the sequence $(\mathbb{D}_{n,\Gamma,A})_n$ characterizes the smoothness of f. These constructions are perfectly explicit.

1. Introduction

It was shown for special classes of harmonic functions in [**2**] that Gevrey harmonic functions on the unit disk, $\mathbb{D}$, of the complex plane are, in fact, sums of certain series of harmonic functions on specific neighborhoods of $\bar{\mathbb{D}}$. It seems that a similar result holds in largest classes. This gives a motivation for asking about classes of harmonic functions of Carleman type. The principal difficulty is that we have to control an infinity of derivatives of a function. The purpose of this paper is to extend results of [**2**] to he case of Harmonic functions of Carleman type. Our principal result (Theorem 4.1) gives a more useful characterization. Precisely, we show that the Harmonic functions of Carleman type on $\mathbb{D}$ are exactly those which are sums of certain series in specific neighborhoods of $\bar{\mathbb{D}}$.

2. Notations and Definitions

Let $\mathbb{D} = \{z \in \mathbb{C},\ |z| < 1\}$ the disk of the plane $\mathbb{C}$ and $m(t)$ be a real-valued $\mathcal{C}^\infty$ function defined for $t \gg 0$. We suppose that $m(t), m'(t), m''(t)$ are strictly positive and $\lim_{t\to+\infty} m'(t) = +\infty$; we suppose also that there exists $\delta > 0$ such that $m''(t) \leqslant \delta$. Put $M(t) = \exp(m(t))$. $\mathcal{H}(\bar{\mathbb{D}})$ denotes the spaces of harmonic functions on a neighborhood of $\bar{\mathbb{D}}$. Let us consider $\mathcal{H}_M(\mathbb{D})$ to be the class of

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harmonic functions on $\mathbb{D}$ given by those f for which there are positive constants C and ρ such that

$$|f^{(n)}_{(x,y)}(z)| \leqslant C\rho^n M(n),$$

for any $z = x + iy \in \mathbb{D}$ and any $n = p + q \in \mathbb{N}$ where

$$f^{(n)}_{(x,y)}(z) := \frac{\partial^n f}{\partial x^p \partial y^q}(z).$$

Note that every function f belonging to a class $\mathcal{H}_M(\mathbb{D})$ can be extended in a unique way to a $\mathcal{C}^\infty$ function on $\bar{\mathbb{D}}$: if $\xi \in \partial\mathbb{D}$ and if $z_j \in \mathbb{D}$ converges to ξ, all partial derivatives of f at z_j are uniformly bounded on $\mathbb{D}$ and we can apply the mean value theorem. We denote this extension by the same symbol f. The class does not change if we replace $M(t)$ by $c\rho^t M(t)$, c, ρ being positive constants, so it does not change if we replace $m(t)$ by $m(t) + at + b$ (a, b being constants). $\mathcal{H}_M(\mathbb{D})$ is an algebra (immediate) closed under differentiation. In fact, $m(n+1) - m(n) = m'(\theta_n) \leqslant \delta n + r$, where $\theta_n \in\,]n, n+1[$ and r is a suitable constant, so

$$M(n+1) \leqslant e^r [e^\delta]^n M(n) \tag{2.1}$$

In view of Cauchy's inequalities, Stirling formula and Heine–Borel theorem, if f is harmonic on neighborhood of $\bar{\mathbb{D}}$ and real valued, then f is real analytic on $\bar{\mathbb{D}}$ and the restriction of f to $\mathbb{D}$ belongs to the class $\mathcal{H}_M(\mathbb{D})$. The class $\mathcal{H}(\bar{\mathbb{D}})$ correspond, then, to $M(t) = t^t$, so to $m(t) = t\ln(t)$. In this short paper, we will consider classes that contain strictly $\mathcal{H}(\bar{\mathbb{D}})$; for this end we suppose that $m(t) = t\ln(t) + t\mu(t)$ with $\mu(t)$ be a strictly increasing $\mathcal{C}^\infty$ function for $t \gg 0$ (so $\mu'(t) > 0$) and $\lim_{t\to+\infty} \mu(t) = +\infty$. In order to ensure that the functions $(m^{(i)}(t))_{0\leqslant i\leqslant 2}$ are monotonic at infinity, we suppose that $\mu(t)$ belongs to a Hardy field (a field of germs at infinity in $\mathbb{R}$ which is closed under differentiation). We have $m'(t) = 1 + \ln(t) + \mu(t) + t\mu'(t)$ is strictly positive and $\lim_{t\to+\infty} m'(t) = +\infty$; so $m'(t)$ is strictly increasing (μ belongs to a Hardy field) and $m''(t) > 0$.

It is interesting to note that there exist functions in $\mathcal{H}_M(\mathbb{D})$ which are not harmonic on any neighborhood of $\bar{\mathbb{D}}$. Take, for instance,

$$f(z) = \sum_{p\in\mathbb{Z}} \exp\left(-\sqrt{|p|}\right)|z|^{|p|}\exp(ip\theta).$$

This function belongs to $\mathcal{H}_M(\mathbb{D})$ for $\mu(t) = \ln(t)$ but it cannot be extended to be harmonic on any neighborhood of $\bar{\mathbb{D}}$ as we verify easily [**2**, p. 413].

Finally the condition $m''(t) \leqslant \delta$ means that $m(t)$ has a growth at infinity less than t^2; then we may suppose also that

$$\mu(t) \leqslant at,\ t \gg 0,\ a > 0. \tag{2.2}$$

3. The associated functions $\Omega(s)$ and $\Gamma(u)$

Set $\Omega(s) := \inf_{t\geqslant t_0} s^{-t}M(t)$, $s \gg 0$, where t_0 is fixed. The infinimum is attained when $m'(t) = \ln(s)$. The function $t\mu'(t) + \mu(t)$ tends to infinity as $t \to \infty$ and so it is strictly increasing ($\mu(t)$ belongs to a Hardy field); so we have a unique value of t where the infinimum is attained. Thus, if $\Omega(s) = \exp(-\omega(s))$, then we get the system

$$s = \exp(m'(t)), \quad \omega(s) = tm'(t) - m(t). \tag{3.1}$$

Since $\mu'(t) > 0$, we have $\omega(s) > 0$ and $\lim_{s\to+\infty} \omega(s) = +\infty$. Thus $\Omega(s)$ is strictly decreasing and $\lim_{s\to+\infty} \Omega(s) = 0$. Set $\Gamma(u) := \exp(-\gamma(u))$, where u and $\gamma(u)$ are defined by

$$u = t^2\mu'(t), \quad \gamma(u) = t\mu'(t) + \mu(t). \tag{3.2}$$

as $\mu(t)$ is strictly increasing, and $\lim_{t\to+\infty} \mu(t) = +\infty$, it follows that $\gamma(u)$ is strictly increasing and $\lim_{u\to+\infty} \gamma(u) = +\infty$. Hence $\Gamma(u)$ is strictly decreasing and $\lim_{u\to+\infty} \Gamma(u) = 0$. System (3.2) gives easily

$$t = \frac{1}{\gamma'(u)}, \quad \mu(t) = \gamma(u) - u\gamma'(u), \tag{3.3}$$

which shows that $\gamma'(u)$ is strictly decreasing, positive and $\lim_{u\to+\infty} \gamma'(u) = 0$. Note that $\gamma(u)$, just as $\mu(t)$, is defined modulo an additive constant.

4. Main Result

By harmonic polynomial on $\mathbb{R}^2 \simeq \mathbb{C}$, we mean a complex polynomial P of two variables which satisfies Laplace's equation, $\Delta P = 0$. In other words, P is a finite linear combinations, on the field $\mathbb{C}$, of harmonic polynomials δ_n ($n \geqslant 1$):

$$\delta_1 = 1, \quad \delta_n(r\exp(i\theta)) = \begin{cases} r^k \cos k\theta & \text{if } n = 2k, \\ r^k \sin k\theta & \text{if } n = 2k+1. \end{cases}$$

Consequently a harmonic polynomial is a polynomial in $|z|$ with coefficients in $\mathbb{C}$. Define $\mathbb{D}_{n,\Gamma,A} := \{z \in \mathbb{C};\ d(z,\mathbb{D}) < A\Gamma(n)\}$, where A is a positive real number and $n = 1, 2, \ldots$

Under the condition

$$\lim_{t\to\infty} \frac{\ln(t)}{\mu(t)} \neq 0 \tag{4.1}$$

we prove the following results:

Theorem 4.1. (1) *Let $f \in \mathcal{H}_M(\mathbb{D})$. Then there exist constants $C > 0$, and ρ with $0 < \rho < 1$; there exist a sequence $(P_l)_l$ of harmonic polynomials defined on $\mathbb{D}_{l,\Gamma,1}$ such that $f(z) = \sum_{l\geqslant 0} P_l(z)$ and $\|P_l\|_{\mathbb{D}_{l,\Gamma,1}} \leqslant C\rho^l$, for every $l \gg 0$.*

(ii) *Conversely, suppose that there exist constants $A > 0$, $C >$, $0 < \rho < 1$, and a sequence $(f_n)_{n\geqslant 1}$ of harmonic functions on $(\mathbb{D}_{n,\Gamma,A})_n$, such that $\|f_n\|_{\mathbb{D}_{n,\Gamma,A}} \leqslant C\rho^n$ for all $n \gg 1$; then the series $\sum_n f_n := f$ belong to $\mathcal{H}_M(\mathbb{D})$.*

5. Proof of Theorem 4.1

Following [**2**–**4**], we denote by $\mathcal{H}_M([0,2\pi])$ the indefinitely differentiable functions g on the interval $[0,2\pi]$, $g \in \mathcal{C}^\infty([0,2\pi])$, such that the following holds. There exist constants $C>0$ and $\rho>0$ such that $|g^n(x)| \leqslant C\rho^n M(n)$ for all $x \in [0,2\pi]$ and for all $n \in \mathbb{N}$. By remarking a fact that a given function f which is harmonic on $\mathbb{D}$ and $\mathcal{C}^\infty$ on $\bar{\mathbb{D}}$ belongs to the class $\mathcal{H}_M(\mathbb{D})$ if and only if the function $\theta \mapsto g(\theta) := f(\exp(i\theta))$ belongs to the class $\mathcal{H}_M([0,2\pi])$, we have the following:

Proposition 5.1. *Let $f(z) = \sum_{p\in\mathbb{Z}} a_p |z|^{|p|}\exp(ip\theta)$ be a harmonic function on $\mathbb{D}$ and $\mathcal{C}^\infty$ on $\bar{\mathbb{D}}$. Then $f \in \mathcal{H}_M(\mathbb{D})$ if and only if, there exist constants $C>0$ and $\rho>0$ such that $|a_p| \leqslant Ce^{\frac{\delta}{2}}\Omega(\frac{|p|}{\rho})$, $|p| \gg 0$.*

Proof. If $f \in \mathcal{H}_M(\mathbb{D})$ then $g(\theta) := f(\exp(i\theta)) \in \mathcal{H}_M([0,2\pi])$. There exist, then, constants $C>0$ and $\rho>0$ such that $|g^{(n)}(\theta)| \leqslant C\rho^n M(n)$. But $g^{(n)}(\theta) = \sum_{p\in\mathbb{Z}} a_p (ip)^n \exp(ip\theta)$; consequently $|a_p||p|^n \leqslant C\rho^n M(n)$ for every $n \geqslant 1$; and, so, $|a_p| \leqslant C\inf_{n\geqslant 1}(\frac{\rho}{|p|})^n M(n)$. A suitable application of Taylor's formula shows that the last infinimum is bounded by $e^{\frac{\delta}{2}}\inf_{t\geqslant 1}(\frac{\rho}{|p|})^t M(t)$, which is equal to $e^{\frac{\delta}{2}}\Omega(\frac{|p|}{\rho})$. Conversely, if the coefficients a_p satisfy these estimates then $|a_p| \leqslant C\inf_{n\geqslant 1}(\frac{\rho}{|p|})^n M(n)$; or equivalently $|a_p||p|^n \leqslant C\rho^n M(n)$, for every $n \geqslant 1$. Then, we have

$$|g^{(n)}(\theta)| \leqslant \sum_{p\in\mathbb{Z}\setminus\{0\}} |a_p||p|^{n+2}\frac{1}{p^2} \leqslant C\rho^{n+2}M(n+2)\sum_{p\in\mathbb{Z}\setminus\{0\}}\frac{1}{p^2}.$$

We conclude by (2.1) and the remark preceding the proposition (5.1) that $f \in \mathcal{H}_M(\mathbb{D})$. The proof is, then, complete. □

Let us remark that, in general, $f(z) = \sum_{p\in\mathbb{Z}} \Omega(|p|)|z|^{|p|}\exp(ip\theta)$. This function belongs to $\mathcal{H}_M(\mathbb{D})$ but it cannot be extended to be harmonic function on any neighborhood of $\bar{\mathbb{D}}$ as we verify easily.

Proof of part 1 of Theorem 4.1. Let

$$f(z) = \sum_{p\in\mathbb{Z}} a_p|z|^{|p|}\exp(ip\theta) \in \mathcal{H}_M(\mathbb{D}).$$

Without loss of generality, by proposition (5.1) we can suppose that we have $|a_p| \leqslant Ce^{\frac{\delta}{2}-\omega(|p|)}$, $|p| \gg 0$. Let $n \geqslant 0$ and consider all indexes $p \in \mathbb{Z}$ satisfying

$$n \leqslant u(|p|) < n+1. \tag{5.1}$$

For p satisfying (5.1) and $z \in \mathbb{D}_{n+1,\Gamma,1}$ we have with $C_1 = C_0 e^{\frac{\delta}{2}}$,

$$\begin{aligned}
|a_p||z|^{|p|}|\exp(ip\theta)| &\leqslant C_1 e^{-\omega(|p|)+|p|\ln(1+\Gamma(n+1))} \\
&\leqslant C_1 e^{-\omega(|p|)+|p|\Gamma(n+1)} \\
&\leqslant C_1 e^{-\omega(|p|)+|p|\Gamma(u(|p|))} \\
&\leqslant C_1 e^{-\omega(|p|)+|p|\omega'(|p|)}.
\end{aligned}$$

Using (3.1) and (3.2) and taking into account (5.1), we obtain

$$|a_p||z|^{|p|}|\exp(ip\theta)| \leqslant C_1 e^{-u(|p|)} \leqslant C_1 e^{-n}.$$

Otherwise the number δ_n of indexes p satisfying (5.1) is bounded by

$$2(u^{-1}(n+1) - u^{-1}(n)) = 2\Big(\frac{1}{\Gamma'(n)} - \frac{1}{\Gamma'(n+1)}\Big) = 2\frac{\Gamma''(\theta)}{(\Gamma'(\theta))^2}$$

where $n < \theta < n+1$. Condition (2.2) implies that the function Γ has subexponential decay when $u \longrightarrow \infty$. Then the function $\frac{1}{\Gamma'}$ and its derivative have also subexponential decay. Thus $\delta_n \leqslant 2e^{\frac{1}{2}n}$ for $n \gg 0$. Now, set

$$P_n(z) = \sum_{u^{-1}(n) \leqslant |p| < u^{-1}(n+1)} a_p |z|^{|p|} \exp(ip\theta).$$

Clearly P_n is a harmonic polynomial and $f(z) = \sum_{n \geqslant 0} P_n(z)$, $z \in \mathbb{D}$. Furthermore, by the preceding estimates, we have $\|P_n\|_{\mathbb{D}_{n+1,\Gamma,1}} \leqslant 2C_1 e^{-\frac{1}{2}n}$. This completes the proof of part one of Theorem 4.1, by setting $C = 2C_1$ and $\rho = e^{-\frac{1}{2}}$. $\square$

Proof of part 2 of Theorem 4.1. Without loss of generality, we can suppose that all f_n are real valued. Consequently f_n is the real part of holomorphic function g_n on $\mathbb{D}_{n,\Gamma,A}$. We use Borel–Caratheodory's inequality [**5**, p. 21], to get

$$\begin{aligned}\|g_n\|_{\mathbb{D}_{n,\Gamma,\frac{A}{2}}} \leqslant 2\Big(\frac{A}{2}\Gamma(n)\Big)^{-1}\Big(1 + \frac{A}{2}\Gamma(n)\Big)\|f_n\|_{\mathbb{D}_{n,\Gamma,A}}\\ + \Big(\frac{A}{2}\Gamma(n)\Big)^{-1}\Big(2 + \frac{3A}{2}\Gamma(n)\Big)|f_n(0)|.\end{aligned}$$

This implies

$$\|g_n\|_{\mathbb{D}_{n,\Gamma,\frac{A}{2}}} \leqslant \frac{2C}{A}\Big(4 + \frac{5A}{2}\Gamma(n)\Big)\frac{\rho^n}{\Gamma(n)} \quad \text{for} \quad n \gg 0.$$

We obtain, with $\lambda := -\ln(\rho)$,

$$\|g_n\|_{\mathbb{D}_{n,\Gamma,\frac{A}{2}}} \leqslant \frac{2C}{A}\Big(4 + \frac{5A}{2}\Gamma(n)\Big)\exp(-\lambda n + \gamma(n)) \quad \text{for} \quad n \gg 0.$$

By the assumption (4.1) and by [**1**, p. 224], we can choose $\gamma = \mu$ and hence there exists $B > 0$ such that $\gamma(n) \leqslant B\ln(n)$ $n \gg 0$. Consequently,

$$\|g_n\|_{\mathbb{D}_{n,\Gamma,\frac{A}{2}}} \leqslant C_1 \exp\Big(-\frac{\lambda n}{2}\Big) \quad \text{for} \quad n \gg 0$$

where $C_1 = C(f,\Gamma,A) > 0$ does not depend on n. By Cauchy's inequalities, for each $p = 0, 1, \dots$, we get

$$\begin{aligned}\|g_n^{(p)}\|_{\mathbb{D}} &\leqslant C_1 p!\Big(\frac{2}{A}\Big)^p(\Gamma(n))^{-p}\exp\Big(-\frac{\lambda n}{2}\Big)\\ &\leqslant C_1 p!\Big(\frac{2}{A}\Big)^p\Big(\max_{t \geqslant 1}\exp(-\frac{\lambda t}{4} + p\gamma(t))\Big)\exp\Big(-\frac{\lambda n}{4}\Big).\end{aligned}$$

That is because the closed disk $\bar{\mathbb{D}}\big(z, \frac{A\Gamma(n)}{2}\big)$ is contained in $\mathbb{D}_{n,\Gamma,\frac{A}{2}}$ for every $z \in \mathbb{D}$. On the other hand the maximum of the function $u \longrightarrow \exp\big(-\frac{\lambda}{4}u + p\gamma(u)\big)$ is obtained at u_0 such that $\frac{1}{\gamma'(u_0)} = \frac{4p}{\lambda}$ and equal, by the system (3.3), to $\exp\big(p\mu(\frac{4}{\lambda}p)\big)$. By [**1**], Lemma 3, we can replace $\mu\big(\frac{4}{\lambda}p\big)$ by $\mu(p)$. Then, we obtain

$$\|g_n^{(p)}\|_{\mathbb{D}} \leqslant C_1 p!\Big(\frac{2}{A}\Big)^p \exp(p\mu(p)) \exp\Big(-\frac{\lambda n}{4}\Big). \tag{5.2}$$

Adding the inequalities (5.2) over $n \gg 1$ and put $g := \sum_n g_n$, it follows that

$$\|g^{(p)}\|_{\mathbb{D}} \leqslant C_1 p!\Big(\frac{2}{A}\Big)^p \exp(p\mu(p)) \sum_n \exp\Big(-\frac{\lambda n}{4}\Big);$$

and, then, the derivatives of f have similar estimates, i.e.,

$$\|f^{(p)}\|_{\mathbb{D}} \leqslant C_1 p!\Big(\frac{2}{A}\Big)^p \exp(p\mu(p)) \sum_n \exp\Big(-\frac{\lambda n}{4}\Big).$$

We conclude that $f \in \mathcal{H}_M(\mathbb{D})$. This finishes the proof of Theorem 4.1. $\square$

Example 5.1. $\mu(t) = \frac{1}{k}\ln(t)$, $k > 0$, which correspond to Gevrey class of order k. From (3.2) we obtain $u = \frac{1}{k}t$ and $\gamma(u) = \frac{1}{k}\ln(t) + \frac{1}{k} = \frac{1}{k}(\ln(t) + \ln(k))$; so we can choose $\gamma(u) = \frac{1}{k}\ln(u)$. In this situation (that is, if $\mu(t) = \frac{1}{k}\ln(t)$, $k > 0$) Theorem 4.1 is exactly the result of [**2**].

Example 5.2. $\mu(t) = \beta\ln(\ln(t))$ $\beta > 0$. We obtain $u = \frac{\beta t}{\ln(t)}$, so $\ln(u) \sim \ln(t)$, and $\gamma(u) = \beta\ln(\ln(t)) + \frac{\beta}{\ln(t)}$; so we can choose $\gamma(u) = \beta\ln(\ln(u))$.

We can construct other examples by taking $\mu(t) = a_1\ln_1(t) + \dots + a_p\ln_p(t)$ where $\ln_1(t) = \ln(t)$ and $\ln_{p+1}(t) = \ln(\ln_p(t)), a_i$ are positive constants $i = 1, \dots, p$ and $p \in \mathbb{N}^*$.

Example 5.3. $\mu(t) = at$, $a > 0$; (extremal case), we obtain $\gamma(u) = 2\sqrt{au}$.

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