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WEAK P_G-PROPERTY AND BEST PROXIMITY POINTS

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Abstract. Abkar and Gbeleh proved the existence of best proximity points for multivalued nonself mappings on a complete metric space. We generalize/ extend their result by introducing the notion of weak P_G-property. We also construct some examples in the support of our results.

1. Introduction and Preliminaries

Nieto and Rodriguez-Lopez [**23**] generalized the Banach contraction principle on partially ordered metric space. Jachymski [**16**] further extended the work of Nieto and Rodriguez-Lopez, by considering a graph on a metric space. Afterwards, Bojor [**10**–**12**], Nicolae et al. [**22**], Aleomraninejad et al. [**4**], Tiammee [**30**], Samreen et al. [**28**, **29**], Kamran et al. [**19**] and several other authors generalized the work of Jachymski [**16**] in various directions. Abkar and Gbeleh [**3**] proved the existence of best proximity points for multivalued nonself mappings on a complete metric space. In this paper, by combining the ideas of Jachymski [**16**], and Abkar and Gbeleh [**3**] we obtain the best proximity point theorems for multivalued mappings on a complete metric space.

The solutions of the equation $Tx = x$ are fixed points of the mapping $T\colon X \to X$. If A, B are nonempty subsets of a metric space (X, d) and $T\colon A \to B$, then for the existence of a fixed point it is necessary that $T(A) \cap A \neq \emptyset$. If this does not hold, $d(x, Tx) > 0$ for each $x \in A$. In this situation our aim is to minimize the term $d(x, Tx)$. The best approximation theory has been developed is this sense. Some important contributions to this theory can be found in [**1**–**32**]. Now, we recollect some basic notions, definitions and results, from references. Let (X, d) be a metric space. For $A, B \subseteq X$, $\operatorname{dist}(A, B) = \inf\{d(a, b) : a \in A,\ b \in B\}$, $d(x, B) = \inf\{d(x, b) : b \in B\}$, $A_0 = \{a \in A : d(a, b) = \operatorname{dist}(A, B) \text{ for some } b \in B\}$, $B_0 = \{b \in B : d(a, b) = \operatorname{dist}(A, B) \text{ for some } a \in A\}$, $CB(X)$ denotes the set of all nonempty closed and bounded subsets of X and $\mathrm{CL}(X)$ denotes the set of all

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nonempty closed subsets of X. For every $A, B \in \mathrm{CL}(X)$, let

$$H(A,B) = \begin{cases} \max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\}, & \text{if the maximum exists;} \\ \infty, & \text{otherwise.} \end{cases}$$

Such a map H is called generalized Hausdorff metric induced by the metric d.

Definition 1.1. [**3**] An element $x^* \in A$ is said to be a best proximity point of a multivalued non-self mapping $T\colon A \to \mathrm{CL}(B)$, if $d(x^*, Tx^*) = \mathrm{dist}(A,B)$.

Definition 1.2. [**32**] Let (A,B) be a pair of nonempty subsets of a metric space (X,d) with $A_0 \neq \emptyset$. Then the pair (A,B) is said to have the weak P-property if and only if for any $x_1, x_2 \in A$ and $y_1, y_2 \in B$, we have

$$\begin{cases} d(x_1,y_1) = \mathrm{dist}(A,B) \\ d(x_2,y_2) = \mathrm{dist}(A,B) \end{cases} \Rightarrow d(x_1,x_2) \leqslant d(y_1,y_2).$$

Remark 1.1. Weak P-property is said to be P-property [**24**], if the inequality is replaced with equality.

Theorem 1.1. [**3**] *Let A and B be nonempty closed subsets of a complete metric space (X,d) such that A_0 is nonempty. Let $T\colon A \to CB(B)$ be a mapping such that for each $x,y \in A$, we have*

$$H(Tx,Ty) \leqslant \alpha d(x,y) \tag{1.1}$$

where $\alpha \in (0,1)$. Assume that the following conditions hold:

(i) *for each $x \in A_0$, we have $Tx \subseteq B_0$;*
(ii) *the pair (A,B) satisfies the P-property;*

Then there exists $x^* \in A_0$ such that $d(x^*,Tx^*) = \mathrm{dist}(A,B)$.

We use the following lemma in our results.

Lemma 1.1. [**18**] *Let (X,d) be a metric space and $B \in \mathrm{CL}(X)$. Then for each $x \in X$ and $q > 1$, there exists an element $b \in B$ such that $d(x,b) \leqslant qd(x,B)$.*

The following definition is an extension of the definition introduced by Jachymski [**16**].

Definition 1.3. A mapping $T\colon A \to \mathrm{CL}(B)$ is said to be G-continuous if for each sequence $\{x_n\}$ in A such that $(x_n, x_{n+1}) \in E$ and $x_n \to x$, we have $Tx_n \to Tx$.

2. Main result

Throughout this section, A and B are nonempty closed subsets of a complete metric space (X,d). $G = (V,E)$ is a directed graph without parallel edges such that its vertex set $V = A_0$ and $\Delta = \{(x,x) : x \in A_0\}$ is contained in its edge set E. For basic terminologies of graph theory we refer the reader to the excellent text by Chartrand et al. [**13**].

We start this section by introducing the notion of weak P_G-property.

Definition 2.1. Let (A,B) be a pair of nonempty subsets of a metric space (X,d). Assume that $A_0 \neq \emptyset$ and endowed with the graph G as defined above. Then the pair (A,B) is said to have the weak P_G-property if and only if for any $x_1, x_2 \in V$ and $y_1, y_2 \in B$, we have

$$\begin{cases} (x_1,x_2) \in E \\ d(x_1,y_1) = \mathrm{dist}(A,B) \\ d(x_2,y_2) = \mathrm{dist}(A,B) \end{cases} \Rightarrow d(x_1,x_2) \leqslant d(y_1,y_2).$$

Remark 2.1. Note that if a pair (A,B) satisfies weak P-property then it satisfies weak P_G-property for the graph $G = (A_0, A_0 \times A_0)$. In general, a weak P_G-property is weaker than the weak P-property.

Example 2.1. Let $X = \mathbb{R}^2$ be endowed with metric $d((x_1,y_1),(x_2,y_2)) = |x_1 - x_2| + |y_1 - y_2|$. Suppose that $A = \{(0,0)\} \cup \{(1,x) : 2 \leqslant x \leqslant 3\}$ and $B = \{(0,1)\} \cup \{(0,x) : 2 \leqslant x \leqslant 3\}$. Here, we have $A_0 = A$. Consider a graph G such that $V = A_0$ and $E = \{((1,x),(1,y) : x,y \in [2,3])\} \cup \{(\xi,\xi) : \xi \in V\}$. Clearly, the pair (A,B) satisfies the weak P_G-property. Further note that the pair (A,B) does not satisfy the weak P-property. To see, consider, $x_1 = (0,0), x_2 = (1,2)$ and $y_1 = (0,1)$, $y_2 = (0,2)$. Then we have $d(x_1,y_1) = 1 = \mathrm{dist}(A,B)$ and $d(x_2,y_2) = 1 = \mathrm{dist}(A,B)$, but $d(x_1,x_2) = 3$ and $d(y_1,y_2) = 1$.

Now, we are in the position to state our first result.

Theorem 2.1. *Let A and B be nonempty closed subsets of a complete metric space (X,d). Assume that A_0 is nonempty set endowed with the graph G and $T\colon A \to \mathrm{CL}(B)$ be a mapping such that for each $(x,y) \in E$, we have*

$$H(Tx,Ty) \leqslant \alpha\, d(x,y) \tag{2.1}$$

where $\alpha \in (0,1)$. Further, assume that the following conditions hold:

(i) *for each $x \in A_0$, we have $Tx \subseteq B_0$;*
(ii) *the pair (A,B) satisfies weak P_G-property;*
(iii) *for $a,b \in A$, $u \in Tx$ and $v \in Ty$ such that $d(u,v) \leqslant \sqrt{\alpha} d(x,y)$ and $d(a,u) = d(b,v) = \mathrm{dist}(A,B)$ then $(a,b) \in E$, whenever $(x,y) \in E$;*
(iv) *there exist $x_0, x_1 \in A_0$ with $(x_0,x_1) \in E$ and we have $y_0 \in Tx_0$ such that $d(x_1,y_0) = \mathrm{dist}(A,B)$;*
(v) *T is G-continuous.*

Then there exists $x^* \in A_0$ such that $d(x^*,Tx^*) = \mathrm{dist}(A,B)$.

Before giving the proof of our result we have the following remark.

Remark 2.2. Note that contractive condition (2.1) holds on a subset of $A_0 \times A_0$ whereas contractive condition (1.1) holds on $A \times A$. Therefore, contractive condition (2.1) is weaker then corresponding condition (1.1).

It is evident from Example 2.1 that condition (ii) of our Theorem 2.1 is weaker than corresponding condition (ii) of Theorem 1.1.

If we consider the graph $G = (A, A \times A)$, we see that the contractive condition (2.1) holds on $A \times A$. In this case conditions (iii) and (iv) of our Theorem 2.1 are automatically satisfied. Thus, Theorem 1.1 is a special case of Theorem 2.1.

Proof. By hypothesis (iv) there exist $x_0, x_1 \in A_0$ with $(x_0, x_1) \in E$ and $y_0 \in Tx_0$ such that $d(x_1, y_0) = \operatorname{dist}(A, B)$. From (2.1), we have

$$d(y_0, Tx_1) \leqslant H(Tx_0, Tx_1) \leqslant \alpha\, d(x_0, x_1).$$

As $\alpha > 0$, then by Lemma 1.1, we have $y_1 \in Tx_1$ such that

$$d(y_0, y_1) \leqslant \frac{1}{\sqrt{\alpha}}\, d(y_0, Tx_1) \leqslant \sqrt{\alpha}\, d(x_0, x_1). \tag{2.2}$$

Since $Tx_1 \subseteq B_0$, then for $y_1 \in Tx_1 \subset B_0$, we have $x_2 \in A_0$ satisfying $d(x_2, y_1) = \operatorname{dist}(A, B)$. Thus, by hypothesis (iii), we have $(x_1, x_2) \in E$. As (A, B) satisfies the weak P_G-property, we have

$$d(x_1, x_2) \leqslant d(y_0, y_1). \tag{2.3}$$

From (2.2) and (2.3), we have

$$d(x_1, x_2) \leqslant \sqrt{\alpha}\, d(x_0, x_1). \tag{2.4}$$

Again from (2.1), we have $d(y_1, Tx_2) \leqslant H(Tx_1, Tx_2) \leqslant \alpha\, d(x_1, x_2)$. By using Lemma 1.1, we have $y_2 \in Tx_2$ such that

$$d(y_1, y_2) \leqslant \frac{1}{\sqrt{\alpha}}\, d(y_1, Tx_2) \leqslant \sqrt{\alpha}\, d(x_1, x_2). \tag{2.5}$$

Since $y_2 \in Tx_2 \subseteq B_0$, there exists $x_3 \in A_0$ such that $d(x_3, y_2) = \operatorname{dist}(A, B)$. Thus, by hypothesis (iii), we have $(x_2, x_3) \in E$. As (A, B) satisfies the weak P_G-property, we have

$$d(x_2, x_3) \leqslant d(y_1, y_2). \tag{2.6}$$

From (2.6), (2.5) and (2.4), we have

$$d(x_2, x_3) \leqslant \sqrt{\alpha}\, d(x_1, x_2) \leqslant \alpha\, d(x_0, x_1).$$

Continuing in this way, we get two sequences $\{x_n\} \subseteq A_0$ with $(x_n, x_{n+1}) \in E$ and $\{y_n\} \subseteq B_0$ with $y_n \in Tx_n$, such that

$$d(x_n, y_{n-1}) = \operatorname{dist}(A, B) \text{ for each } n \in \mathbb{N}. \tag{2.7}$$

Moreover,

$$d(x_n, x_{n+1}) \leqslant d(y_{n-1}, y_n) \leqslant (\sqrt{\alpha})^n d(x_0, x_1) \text{ for each } n \in \mathbb{N}.$$

For $n > m$, we have

$$d(x_n, x_m) \leqslant \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leqslant \sum_{i=n}^{m-1} (\sqrt{\alpha})^i d(x_0, x_1) < \sum_{i=n}^{\infty} (\sqrt{\alpha})^i d(x_0, x_1) < \infty.$$

Hence $\{x_n\}$ is a Cauchy sequence in A. Similarly, $\{y_n\}$ is a Cauchy sequence in B. Since A and B are closed subsets of a complete metric space, there exist $x^* \in A$ and $y^* \in B$ such that $x_n \to x^*$ and $y_n \to y^*$. By (2.7), we conclude that $d(x^*, y^*) = \operatorname{dist}(A, B)$ as $n \to \infty$. Since T is G-continuous, clearly, $y^* \in Tx^*$. Hence $\operatorname{dist}(A, B) \leqslant d(x^*, Tx^*) \leqslant d(x^*, y^*) = \operatorname{dist}(A, B)$. Therefore x^* is a best proximity point of the mapping T. $\square$

THEOREM 2.2. *Let A and B be nonempty closed subsets of a complete metric space (X, d). Assume that A_0 is nonempty set endowed with the graph G and $T\colon A \to \mathrm{CL}(B)$ be a mapping such that for each $(x, y) \in E$, we have*

$$H(Tx, Ty) \leqslant \alpha\, d(x, y)$$

where $\alpha \in (0, 1)$. Further, assume that the following conditions hold:

(i) *for each $x \in A_0$, we have $Tx \subseteq B_0$;*
(ii) *the pair (A, B) satisfies weak P_G-property;*
(iii) *for $a, b \in A$, $u \in Tx$ and $v \in Ty$ such that $d(u, v) \leqslant \sqrt{\alpha}\, d(x, y)$ and $d(a, u) = d(b, v) = \mathrm{dist}(A, B)$ then $(a, b) \in E$, whenever $(x, y) \in E$;*
(iv) *there exist $x_0, x_1 \in A_0$ with $(x_0, x_1) \in E$ and we have $y_0 \in Tx_0$ such that $d(x_1, y_0) = \mathrm{dist}(A, B)$;*
(v) *for each sequence $\{x_n\}$ in A_0 such that $(x_n, x_{n+1}) \in E$ for each $n \in \mathbb{N}$ and $x_n \to x$, we have $(x_n, x) \in E$.*

Then there exists $x^* \in A_0$ such that $d(x^*, Tx^*) = \mathrm{dist}(A, B)$.

PROOF. Following the proof of Theorem 2.1, we get Cauchy sequences $\{x_n\} \subseteq A_0$ with $(x_n, x_{n+1}) \in E$ and $\{y_n\} \subseteq B_0$ with $y_n \in Tx_n$, such that

$$d(x_n, y_{n-1}) = \mathrm{dist}(A, B) \text{ for each } n \in \mathbb{N} \tag{2.8}$$

and $x_n \to x^* \in A$ and $y_n \to y^* \in B$. By hypothesis (v), we have $(x_n, x^*) \in E$. From (2.1), we have

$$d(y_n, Tx^*) \leqslant H(Tx_n, Tx^*) \leqslant \alpha\, d(x_n, x^*)$$

Letting $n \to \infty$, in the above inequality, we get

$$d(y^*, Tx^*) = \lim_{n\to\infty} d(y_n, Tx^*) = 0.$$

This implies that, $y^* \in Tx^*$. By (2.8), we conclude that $d(x^*, y^*) = \mathrm{dist}(A, B)$ as $n \to \infty$. Hence $\mathrm{dist}(A, B) \leqslant d(x^*, Tx^*) \leqslant d(x^*, y^*) = \mathrm{dist}(A, B)$. Therefore x^* is a best proximity point of the mapping T. □

EXAMPLE 2.2. Let $X = \mathbb{R}^2$ be endowed with metric $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$. Suppose that $A = \{(1, x) : x \in \mathbb{R}\}$ and $B = \{(0, x) : x \in \mathbb{R}\}$. For given A and B, we have $A_0 = A$ and $B_0 = B$. The graph $G = (V, E)$ is defined as $V = A$ and $E = \{((1, x), (1, y)) : 0 \leqslant x, y \leqslant 1\} \cup \{((1, x), (1, x)) : x \in \mathbb{R}\}$. Define $T\colon A \to \mathrm{CL}(B)$ by

$$T(1, x) = \begin{cases} \{(0, 0)\} & \text{if } x < 0 \\ \{(0, 0), (0, x/4)\} & \text{if } 0 \leqslant x \leqslant 1 \\ \{(0, 0), (0, x^2)\} & \text{if } x > 1. \end{cases}$$

It is easy to see that for each $(x, y) \in E$, the inequality (2.1) holds with $\alpha = \frac{1}{4}$. Furthermore,

- for each $x \in A_0$ we have $Tx \subseteq B_0$ and the pair (A, B) satisfies the weak P_G-property;
- for each $a, b \in A$, $u \in Tx$ and $v \in Ty$ such that $d(u, v) \leqslant \sqrt{\alpha}\, d(x, y)$ and $d(a, u) = d(b, v) = \mathrm{dist}(A, B)$, we have $(a, b) \in E$, whenever $(x, y) \in E$;

- Hypothesis (iv) trivially holds. To see, take $x_0 = (1,1)$, $x_1 = (1,0)$ and $y_0 = (0,0) \in Tx_0$;
- for each sequence $\{x_n\}$ in A_0 such that $(x_n, x_{n+1}) \in E$ for each $n \in \mathbb{N}$ and $x_n \to x$, we have $(x_n, x) \in E$.

Therefore, all the conditions of Theorem 2.2 hold and T has a best proximity point.

If we assume that T is a singlevalued nonself mapping, then the above theorems reduce to the following results:

Corollary 2.1. *Let A and B be nonempty closed subsets of a complete metric space (X,d). Assume that A_0 is nonempty set endowed with the graph G and $T\colon A \to B$ be a mapping such that for each $(x,y) \in E$, we have*

$$d(Tx,Ty) \leqslant \alpha\, d(x,y)$$

where $\alpha \in (0,1)$. Further, assume that the following conditions hold:

(i) *for each $x \in A_0$, we have $Tx \in B_0$;*
(ii) *the pair (A,B) satisfies weak P_G-property;*
(iii) *for $a,b \in A$ and $u = Tx$, $v = Ty \in B$ such that $d(u,v) \leqslant \sqrt{\alpha}\, d(x,y)$ and $d(a,u) = d(b,v) = \operatorname{dist}(A,B)$ then $(a,b) \in E$, whenever $(x,y) \in E$;*
(iv) *there exist $x_0, x_1 \in A_0$ with $(x_0,x_1) \in E$ and we have $y_0 = Tx_0$ such that $d(x_1,y_0) = \operatorname{dist}(A,B)$;*
(v) *T is G-continuous.*

Then there exists $x^* \in A_0$ such that $d(x^*, Tx^*) = \operatorname{dist}(A,B)$.

Corollary 2.2. *Let A and B be nonempty closed subsets of a complete metric space (X,d). Assume that A_0 is a nonempty set endowed with the graph G and $T\colon A \to B$ be a mapping such that for each $(x,y) \in E$, we have*

$$d(Tx,Ty) \leqslant \alpha d(x,y)$$

where $\alpha \in (0,1)$. Further, assume that the following conditions hold:

(i) *for each $x \in A_0$, we have $Tx \in B_0$;*
(ii) *the pair (A,B) satisfies weak P_G-property;*
(iii) *for $a,b \in A$ and $u = Tx, v = Ty \in B$ such that $d(u,v) \leqslant \sqrt{\alpha}\, d(x,y)$ and $d(a,u) = d(b,v) = \operatorname{dist}(A,B)$ then $(a,b) \in E$, whenever $(x,y) \in E$;*
(iv) *there exist $x_0, x_1 \in A_0$ with $(x_0,x_1) \in E$ and we have $y_0 = Tx_0 \in B$ such that $d(x_1,y_0) = \operatorname{dist}(A,B)$;*
(v) *for each sequence $\{x_n\}$ in A_0 such that $(x_n,x_{n+1}) \in E$ for each $n \in \mathbb{N}$ and $x_n \to x$, we have $(x_n,x) \in E$.*

Then there exists $x^* \in A_0$ such that $d(x^*, Tx^*) = \operatorname{dist}(A,B)$.

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