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SOME CONSIDERATIONS ABOUT HYERS–ULAM STABILITY OF THE INTERMEDIARY POINT ARISING FROM THE MEAN VALUE THEOREMS

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ABSTRACT. We present some results involving the Hyers–Ulam stability of some intermediary point arising from the mean value theorems. First, we make some considerations about the stability of Lagrange mean point and we prove a stability result involving Cauchy mean point. Also, we establish some conditions to the stability of the intermediary point defined by some recent theorems due to Wachnichi or Tong. We investigate similar problems to the integral version of these theorems. We complete with Hyers–Ulam stability of the intermediary point arising from second integral mean value theorem. Also, we present a new integral mean theorem and its stability.

1. Introduction

The starting point of a wide range of stability notions was the following question:

"*When is it true that the solution of an equation differing slightly from a given one, must of necessity be close to the solution of the given equation?*"

The author of the question was Ulam [**12**]. The first mathematician which gave an important answer to Ulam's question was Hyers [**4**]. In fact, he proved that, for any $\varepsilon > 0$ and for any function $f : U \to V$ between two Banach spaces and satisfying $\|f(x+y) - f(x) - f(y)\| < \varepsilon$, for all $x, y \in U$, there exists $\delta > 0$ and a unique additive function $A\colon U \to V$ such that $\|A(x) - f(x)\| < \delta$, for all $x \in U$.

This result opened a new research direction and a new mathematical concept was born. Now it is said that the Cauchy additive functional equation, $f(x+y) = f(x) + f(y)$, satisfies the Hyers–Ulam stability. This concept has influenced a number of mathematicians studying the stability problems of functional equations (a large collection of results can be found in [**5**]). Today, the terminologies Hyers–Ulam stability is also applicable to the case of other mathematical objects as differential equation, dynamical equations, linear recurrences or convexity.

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Starting with the paper [**1**], a new direction was created. It is about the stability of the intermediary point arising from the mean value theorems. For example, we have papers [**3**, **6**–**9**], where we can find some results involving Lagrange, Flett or Cauchy mean theorems.

Our aim is to complete the work from these references. The first results are reserved to the mean value theorems involving differential functions. We start with some considerations about Lagrange mean point. Găvruţă [**3**] or Pawlikowska [**8**] investigated its stability. The most recent result is represented by [**7**, Theorem 7] and we present an example which shows that some restrictions from the hypothesis of this theorem cannot be removed. Further, we proved a stability result about Cauchy mean point, similar to [**3**, Theorem 2.3]. Also, the stability of the point from Wachnichi theorem [**13**] will be investigated. Recently, Tong [**10**] proved a new extension of Lagrange mean theorem involving two functions. The stability of the intermediary point from Tong's result will be investigated.

The second part of this paper contains results involving stability of the intermediary points arising from integral mean theorems. We establish some conditions to the Hyers–Ulam stability of the point defined by the integral version of Cauchy or Wachnichi theorem. In the same mode, we obtain similar results for the second mean value theorem for integral. We present a new integral mean value theorem and we will investigate its stability. Finally, we study a recent mean integral theorem due to Tong [**11**].

The main tools, which we will use to prove all the results from the following sections are represented by the next two propositions [**7**, Theorem 2 and 4].

Proposition 1.1. *Let* $f\colon [a,b] \to \mathbb{R}$ *be a continuous function that admits a unique minimum (maximum) point* $c \in (a,b)$*. Then, for any* $\varepsilon > 0$*, there exists* $\delta > 0$ *with the following property: any continuous function* $g\colon [a,b] \to \mathbb{R}$*, satisfying the relation* $|g(x) - f(x)| < \delta$*, for any* $x \in [a,b]$*, admits a minimum (maximum) point* $d \in (a,b)$ *such that* $|d - c| < \varepsilon$*.*

Proposition 1.2. *Let* $f\colon [a,b] \to \mathbb{R}$ *be a continuous function with* $f(a)f(b) < 0$*. We suppose that there exists a unique point* $c \in (a,b)$ *such that* $f(c) = 0$*. Then, for any* $\varepsilon > 0$*, there exists* $\delta > 0$ *with the following property: for any continuous function* $g\colon [a,b] \to \mathbb{R}$*, satisfying the relation* $|g(x) - f(x)| < \delta$*, for any* $x \in [a,b]$*, there exists* $d \in (a,b)$ *such that* $g(d) = 0$ *and* $|d - c| < \varepsilon$*.*

We conclude this section with some notations that we will use in this paper. Let $a, b \in \mathbb{R}$ be such that $a < b$. Denote $C[a,b]$ the set of the continuous functions $f\colon [a,b] \to \mathbb{R}$ and $C^*[a,b]$ the set of the functions $f \in C[a,b]$, with property $f(x) \neq 0$, for any $x \in [a,b]$. Also, denote $D[a,b]$ the set of differentiable functions $f\colon [a,b] \to \mathbb{R}$ and $D^*[a,b]$ the set of functions $f \in D[a,b]$, with property $f'(x) \neq 0$, for any $x \in [a,b]$.

2. Mean value theorems and Hyers–Ulam stability

We start this section with Lagrange mean theorem. Recall that for any function $f \in C[a,b]$, differentiable on (a,b), there exists an intermediary point $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \tag{2.1}$$

Further, any point c satisfying (2.1) is called a *Lagrange point*.

From [**7**, Theorem 7] establishes some conditions for the Hyers–Ulam stability of this point. It was proven that, for any function $f \in C[a,b]$, differentiable on (a,b), having a unique Lagrange point $c \in (a,b)$ and for every $\varepsilon > 0$, there exists $\delta > 0$ such that any function $g \in C[a,b]$, differentiable on (a,b), satisfying the relation $|g(x) - f(x)| < \delta$, for every $x \in [a,b]$, admits a Lagrange point $d \in (a,b)$ such that $|d - c| < \varepsilon$.

The following example shows that the uniqueness condition of c cannot be removed, eventually can be replaced with its uniqueness on a neighbourhood. To prove this, we consider the function $f\colon [-2,2] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} (x+1)^2, & x \in [-2,-1] \\ 0, & x \in (-1,1) \\ (x-1)^2, & x \in [1,2] \end{cases}.$$

It is a continuous and differentiable function on $[-2,2]$ and

$$f'(x) = \begin{cases} 2(x+1), & x \in [-2,-1] \\ 0, & x \in (-1,1) \\ 2(x-1), & x \in [1,2] \end{cases}$$

Applying Lagrange theorem, we find $c \in (-2,2)$ such that

$$\frac{f(2) - f(-2)}{2 - (-2)} = f'(c).$$

Since $f(2) = f(-2) = 0$, we have $f'(c) = 0$. We obtain $c \in [-1,1]$. Then, the Lagrange point of f is not unique.

Now, let $\varepsilon = \frac{1}{2}$. For any $\delta > 0$, we choose $\nu \in \left(0, \frac{1}{2}\right)$ such that $\nu < \delta$. We consider the function $g\colon [-2,2] \to \mathbb{R}$, defined by

$$g(x) = \begin{cases} (x+1)^2, & x \in [-2,-1-\nu] \\ \frac{\nu}{1+\nu}x^2 - \nu, & x \in (-1-\nu, 1+\nu) \\ (x-1)^2, & x \in [1+\nu, 2] \end{cases}$$

Hence,

$$\lim_{x \nearrow -1-\nu} g(x) = \lim_{x \searrow -1-\nu} g(x) = g(-1-\nu) = \nu^2,$$
$$\lim_{x \nearrow 1+\nu} g(x) = \lim_{x \searrow 1+\nu} g(x) = g(1+\nu) = \nu^2,$$

we can conclude that the function g is continuous on $[-2,2]$. Also,

$$g'_s(-1-\nu) = \lim_{x \nearrow -1-\nu} 2(x+1) = -2\nu,$$
$$g'_d(-1-\nu) = \lim_{x \searrow -1-\nu} \frac{2\nu x}{1+\nu} = -2\nu.$$

We obtain that g is differentiable at $x=-1-\nu$. A similar argument shows that g is differentiable at $x=1+\nu$. Then, g is satisfying the hypothesis of Lagrange theorem. Then there exists $d\in(-2,2)$ such that $\frac{g(-2)-g(2)}{2-(-2)}=g'(d)$. Hence $g(-2)=g(2)$, we obtain $g'(d)=0$. Since

$$g'(x)=\begin{cases}2(x+1), & x\in[-2,-1-\nu]\\ \frac{2\nu x}{1+\nu}, & x\in(-1-\nu,1+\nu)\\ 2(x-1), & x\in[1+\nu,2]\end{cases}$$

we obtain that $d=0$.

Now, we prove that $|g(x)-f(x)|<\delta$, for any $x\in[-2,2]$. If $x\in[-2,-1-\nu]$ or $x\in[1+\nu,2]$, we have $g(x)=f(x)$. If $x\in(-1-\nu,-1]$, we obtain

$$|g(x)-f(x)|=\Big|\frac{\nu}{1+\nu}x^2-\nu-(x+1)^2\Big|=\Big|-\frac{1}{1+\nu}x^2-2x-1-v\Big|.$$

Let us define the function $u\colon(-1-\nu,-1]\to\mathbb{R}$ by $u(x)=-\frac{1}{1+\nu}x^2-2x-1-v$, for any $x\in(-1-\nu,-1]$. Since $u'(x)=-\frac{2}{1+\nu}x-2=-\frac{2}{1+\nu}(x+1+\nu)<0$, for any $x\in(-1-\nu,-1]$, we obtain $u(-1)=-\frac{\nu^2}{1+\nu}\leqslant u(x)\leqslant u(-1-\nu)=0$. Then

$$|g(x)-f(x)|\leqslant\frac{\nu^2}{1+\nu}\leqslant\nu<\delta.$$

A similar conclusion is obtained if $x\in[1,1+\nu)$. For every $x\in(-1,1)$, we have

$$|g(x)-f(x)|=\Big|\frac{\nu}{1+\nu}x^2-\nu\Big|\leqslant\nu<\delta.$$

Now, $c=-1$ is a Lagrange point of f and the function g has no Lagrange point on the interval $(-1-\varepsilon,-1+\varepsilon)$.

A first extension of Lagrange theorem refers to the fact that to every pair of functions $f,g\in C[a,b]$, differentiable on (a,b), we can associate an intermediary point $c\in(a,b)$ (called *Cauchy point*) such that

$$(f(b)-f(a))g'(c)=(g(b)-g(a))f'(c).$$

The stability of this point was proven in [**9**] and, with more weaker conditions, in [**7**]. Another type of Hyers–Ulam stability is presented in the following theorem.

THEOREM 2.1. *Let $f,g\colon[a,b]\to\mathbb{R}$ be two twice differentiable functions, with $f'',g''\in C[a,b]$, such that $f''(x)(g(b)-g(a))-g''(x)(f(b)-f(a))\neq 0$, for every $x\in[a,b]$. Let $\varepsilon>0$ and $d\in(a,b)$ such that*

$$|(f(b)-f(a))g'(d)-(g(b)-g(a))f'(d)|<\varepsilon.$$

Then, there exists a point $c\in(a,b)$ such that $(f(b)-f(a))g'(c)=(g(b)-g(a))f'(c)$ and

$$|d-c|<\frac{\varepsilon}{\min_{x\in[a,b]}|f''(x)(g(b)-g(a))-g''(x)(f(b)-f(a))|}.$$

PROOF. We consider the function $h\colon[a,b]\to\mathbb{R}$, defined by

$$h(x)=(f(b)-f(a))g(x)-(g(b)-g(a)f(x),$$

for any $x \in [a,b]$. The hypotesis means that $h''(x) \neq 0$, for any $x \in [a,b]$. Then, either $h''(x) > 0$ or $h''(x) < 0$, for any $x \in [a,b]$. Let $c \in (a,b)$ be a Cauchy point of f and g. Then $h'(c) = 0$.

If $c = d$, the proof is ready, else, we suppose $c < d$. The relation

$$|(f(b) - f(a))g'(d) - (g(b) - g(a))f'(d)| < \varepsilon.$$

means $|h'(d)| < \varepsilon$. Then $|h'(d) - h'(c)| < \varepsilon$. Using Lagrange theorem, we find $u \in (c,d)$ such that $h'(d) - h'(c) = h''(u)(d - c)$. Then $|h''(u)(d-c)| < \varepsilon$, also $|d - c| < \frac{\varepsilon}{|h''(u)|}$ and the conclusion follows now. □

Flett [**2**] proved that for any function $f \in D[a,b]$, with $f'(a) = f'(b)$, there exists an intermediary point $c \in (a,b)$ such that $f(c) - f(a) = (c - a)f'(c)$. An improvement by Cauchy type was natural. Wachnicki [**13**] proved that, for any functions $f \in D[a,b]$ and $g \in D^*[a,b]$, with $\frac{f'(a)}{g'(a)} = \frac{f'(b)}{g'(b)}$, there exists an intermediary point $c \in (a,b)$ such that $\frac{f'(c)}{g'(c)} = \frac{f(c)-f(a)}{g(c)-g(a)}$.

This point will be called *Wachnicki point* of the function f and g. The following two theorems present some conditions for its Hyers–Ulam stability.

THEOREM 2.2. *Let the functions $f \in D[a,b]$ and $g \in D^*[a,b]$, with $f' \in C[a,b]$ and $g' \in C[a,b]$, such that $\frac{f'(a)}{g'(a)} = \frac{f'(b)}{g'(b)}$. We suppose that these functions admit a unique Wachnicki point $c \in (a,b)$. Then, for any $\varepsilon > 0$, there exists $\gamma > 0$ with the following property: for any functions $F \in D[a,b]$ and $G \in D^*[a,b]$, with $F', G' \in C[a,b]$, satisfying the relation*

$$\max_{x\in[a,b]} \{|F(x) - f(x)|,\ |G'(x) - g'(x)|,\ |G(a) - g(a)|\} < \gamma,$$

there exists a Wachnicki point $d \in (a,b)$ such that $|d - c| < \varepsilon$.

PROOF. Hence $g'(x) \neq 0$, for every $x \in [a,b]$, we can admit that g' is positive. Then, the differentiable function $h\colon [a,b] \to \mathbb{R}$, defined by

$$h(x) = (f(x) + f(a))(g(x) - g(a) - 2\int_a^x f(t)g'(t)dt,$$

has $h'(x) = f'(x)(g(x) - g(a)) - g'(x)(f(x) - f(a))$, for any $x \in [a,b]$. More, $h'(c) = 0$ and $h'(x) \neq 0$ if $x \in (a,b) \setminus \{c\}$.

We claim that c is an extremum point of h. If not, then we can accept that $h'(x) > 0$, for any $x \in (a,b) \setminus \{c\}$. Then, $h'(b) \geqslant 0$. We obtain

$$f'(b)(g(b) - g(a)) - g'(b)(f(b) - f(a)) \geqslant 0 \tag{2.2}$$

Let us define the function $u\colon [a,b] \to \mathbb{R}$, by

$$u(x) = \begin{cases} \frac{f(x)-f(a)}{g(x)-g(a)}, & x \in (a,b] \\ \frac{f'(a)}{g'(a)}, & x = a \end{cases}$$

The function u is continuous on $[a,b]$ and differentiable on (a,b), with

$$u'(x) = \frac{f'(x)(g(x) - g(a)) - g'(x)(f(x) - f(a))}{(g(x) - g(a))^2},$$

for any $x \in (a,b)$. We have $u'(x) > 0$, for every $x \in (a,b) \setminus \{c\}$, so u is increasing. From $u(a) < u(b)$, we obtain $\frac{f'(a)}{g'(a)} < \frac{f(b)-f(a)}{g(b)-g(a)}$. From the hypothesis, we deduce that $\frac{f'(b)}{g'(b)} < \frac{f(b)-f(a)}{g(b)-g(a)}$ or $f'(b)(g(b)-g(a)) - g'(b)(f(x)-f(a)) < 0$. This relation contradicts (2.2) and proves our claim.

Further, for some $\gamma > 0$, we consider two functions $F \in D[a,b]$ and $G \in D^*[a,b]$, with $F', G' \in C[a,b]$, satisfying the relation

$$\max_{x\in[a,b]} \{|F(x) - f(x)|,\ |G'(x) - g'(x)|,\ |G(a) - g(a)|\} < \delta,$$

for every $x \in [a,b]$. We consider the differentiable function $H\colon [a,b] \to \mathbb{R}$, defined by

$$H(x) = (F(x) + F(a))(G(x) - G(a)) - 2\int_a^x F(t)G'(t)dt,$$

for every $x \in [a,b]$. We have $H'(x) = F'(x)(G(x) - G(a) - G'(x)(F(x) - F(a))$, for every $x \in [a,b]$.

Further, we have

$$\begin{aligned}
\left|\int_a^x (F(t)G'(t) - f(t)g'(t))dt\right| &= \left|\int_a^x ((F(t) - f(t))G'(t) + f(t)(G'(t) - g'(t)))dt\right| \\
&\leqslant \gamma \int_a^x |G'(t)|dt + \gamma \int_a^x |f(t)|dt \\
&= \gamma \int_a^x |G'(t) - g'(t) + g'(t)|dt + \gamma \int_a^x |f(t)|dt \\
&\leqslant \gamma \int_a^x |G'(t) - g'(t)|dt + \gamma \int_a^x g'(t)dt + \gamma \int_a^x |f(t)|dt \\
&\leqslant \gamma^2(b-a) + 2\gamma \max_{x\in[a,b]} |g(x)| + \gamma \max_{x\in[a,b]} |f(x)|(b-a).
\end{aligned}$$

For any $x \in [a,b]$, Lagrange theorem gives us a point $s \in (a,x)$ such that

$$(G(x) - g(x)) - (G(a) - g(a)) = (G'(s) - g'(s))(x-a).$$

Then, for every $x \in [a,b]$, we have

$$|(G(x) - g(x)) - (G(a) - g(a))| = |G'(s) - g'(s)|(x-a) < \gamma(b-a).$$

More,

$$\begin{aligned}
&|(F(x) + F(a))(G(x) - G(a)) - (f(x) + f(a))(g(x) - g(a))| \\
&\quad \leqslant |(F(x) + F(a))(G(x) - G(a)) - (f(x) + f(a))(G(x) - G(a))| \\
&\qquad + |(f(x) + f(a))(G(x) - G(a)) - (f(x) + f(a))(g(x) - g(a))| \\
&\quad \leqslant 2\gamma|G(x) - G(a)| + 2\gamma(b-a) \max_{x\in[a,b]} |f(x)| \\
&\quad \leqslant 2\gamma|G(x) - G(a) - g(x) + g(a)| + 2\gamma|g(x) - g(a)| + 2\gamma(b-a) \max_{x\in[a,b]} |f(x)| \\
&\quad \leqslant 2\gamma^2(b-a) + 4\gamma \max_{x\in[a,b]} |g(x)| + 2\gamma(b-a) \max_{x\in[a,b]} |f(x)|,
\end{aligned}$$

which goes to the relation

$$|H(x)-h(x)| \leqslant 3\gamma^2(b-a)+6\gamma \max_{x\in[a,b]}|g(x)|+3\gamma \max_{x\in[a,b]}|f(x)|(b-a),$$

for every $x \in [a,b]$.

Now, for any $\varepsilon > 0$, we consider $\delta > 0$ defined by Proposition 1.1 for f and ε. We choose $\gamma > 0$ such that

$$3\gamma^2(b-a)+6\gamma \max_{x\in[a,b]}|g(x)|+3\gamma \max_{x\in[a,b]}|f(x)|(b-a) < \delta.$$

Then $|H(x)h(x)| < \delta$, for every $x \in [a,b]$. Proposition 1.1 gives us an extremum point $d \in (a,b)$ of H such that $|d-c| < \varepsilon$. Using Fermat theorem, we obtain $H'(d) = 0$ and the conclusion follows now. □

THEOREM 2.3. *Let the functions $f, g \in D^*[a,b]$, with $f', g' \in C[a,b]$, such that $\frac{f'(a)}{g'(a)} = \frac{f'(b)}{g'(b)}$. We suppose that these functions admit a unique Wachnicki point $c \in (a,b)$. Then, for any $\varepsilon > 0$, there exists $\gamma > 0$ with the following property: for any functions $F, G \in D^*[a,b]$, with $F', G' \in C[a,b]$, satisfying the relation*

$$\max_{x\in[a,b]}\{|F'(x)-f'(x)|,\ |F(a)-f(a)|,\ |G(x)-g(x)|\} < \gamma,$$

there exists a Wachnicki point $d \in (a,b)$ such that $|d-c| < \varepsilon$.

PROOF. Hence $f'(x) \neq 0$ and $g'(x) \neq 0$, for every $x \in [a,b]$, we can admit that f' and g' are positive. Then, g is continuous and increasing. We sonsider the differentiable function $h\colon [a,b] \to \mathbb{R}$, defined by

$$h(x) = (f(x)-f(a))(g(x)+g(a)) - 2\int_a^x f'(t)g(t)dt,$$

for any $x \in [a,b]$. We have $h'(x) = g'(x)(f(x)-f(a)) - f'(x)(g(x)-g(a))$, for every $x \in [a,b]$, $h'(c) = 0$ and $h'(x) \neq 0$, for any $x \in (a,b) \setminus \{c\}$.

We claim that c is an extremum point of h. If not, then we can accept that $h'(x) > 0$, for any $x \in (a,b) \setminus \{c\}$. Then, $h'(b) \geqslant 0$. We obtain

$$g'(b)(f(b)-f(a)) - f'(b)(g(b)-g(a)) \geqslant 0 \tag{2.3}$$

Let us define the function $u\colon [a,b] \to \mathbb{R}$ by

$$u(x) = \begin{cases} \frac{f(x)-f(a)}{g(x)-g(a)}, & x \in (a,b] \\ \frac{f''(a)}{g'(a)}, & x = a \end{cases}.$$

Then, u is continuous on $[a,b]$ and differentiable on (a,b), with

$$u'(x) = \frac{f'(x)(g(x)-g(a)) - g'(x)(f(x)-f(a))}{(g(x)-g(a))^2},$$

for any $x \in (a,b)$. We obtain $u'(x) < 0$, for every $x \in (a,b) \setminus \{c\}$, so u is decreasing. From $u(a) > u(b)$, we have

$$\frac{f'(a)}{g'(a)} > \frac{f(b)-f(a)}{g(b)-g(a)}.$$

From the hypothesis, we deduce that

$$\frac{f'(b)}{g'(b)} > \frac{f(b)-f(a)}{g(b)-g(a)}$$

or

$$f'(b)(g(b)-g(a)) - g'(b)(f(x)-f(a)) > 0.$$

This relation contradicts (2.3) and proves our claim.

Further, for some $\gamma > 0$, we consider two functions $F, G \in D^*[a,b]$, with $F', G' \in C[a,b]$, satisfying the relation

$$\max_{x\in[a,b]}\{|F'(x)-f'(x)|,\ |F(a)-f(a)|,\ |G(x)-g(x)|\} < \delta,$$

for every $x \in [a,b]$. We consider the differentiable function $H\colon [a,b] \to \mathbb{R}$, defined by

$$H(x) = (F(x)-F(a))(G(x)+G(a)) - 2\int_a^x F'(t)G(t)dt,$$

for every $x \in [a,b]$. We have $H'(x) = G'(x)(F(x)-F(a)) - F'(x)(G(x)-G(a))$, for every $x \in [a,b]$. Then

$$\begin{aligned}
\left|\int_a^x (F'(t)G(t)-f'(t)g(t))dt\right| &= \left|\int_a^x ((F'(t)-f'(t))G(t)+f'(t)(G(t)-g(t)))dt\right| \\
&\leqslant \gamma\int_a^x |G(t)|dt + \gamma\int_a^x |f'(t)|dt \\
&= \gamma\int_a^x |G(t)-g(t)+g(t)|dt + \gamma\int_a^x f'(t)dt \\
&\leqslant \gamma\int_a^x |G(t)-g(t)|dt + \gamma\int_a^x |g(t)|dt + \gamma\int_a^x f'(t)dt \\
&\leqslant \gamma^2(b-a)+\gamma(b-a)\max_{x\in[a,b]}|g(x)| + 2\gamma\max_{x\in[a,b]}|f(x)|,
\end{aligned}$$

for any $x \in [a,b]$.

Lagrange theorem gives us a point $s \in (a,x)$ such that

$$(F(x)-f(x)) - (F(a)-f(a)) = (F'(s)-f'(s))(x-a).$$

Then, we have

$$\begin{aligned}
&|(F(x)-F(a))(G(x)+G(a)) - (f(x)-f(a))(g(x)+g(a))| \\
&\leqslant |(F(x)-F(a))(G(x)+G(a)) - (F(x)-F(a))(g(x)+g(a))| \\
&\qquad + |(F(x)-F(a))(g(x)+g(a)) - (f(x)-f(a))(g(x)+g(a))| \\
&\leqslant 2\gamma|F(x)-F(a)| + 2\gamma(b-a)\max_{x\in[a,b]}|g(x)| \\
&\leqslant 2\gamma|F(x)-F(a)-f(x)+f(a)| + 2\gamma|f(x)-f(a)| + 2\gamma(b-a)\max_{x\in[a,b]}|g(x)| \\
&\leqslant 2\gamma^2(b-a) + 4\gamma\max_{x\in[a,b]}|f(x)| + 2\gamma(b-a)\max_{x\in[a,b]}|g(x)|,
\end{aligned}$$

which goes to the relation

$$|H(x)-h(x)| \leqslant 3\gamma^2(b-a)+6\gamma \max_{x\in[a,b]}|f(x)|+3\gamma \max_{x\in[a,b]}|g(x)|(b-a),$$

for any $x \in [a,b]$.

Now, for any $\varepsilon > 0$, we consider $\delta > 0$ defined by Proposition 1.1 for h and ε. We choose $\gamma > 0$ such that

$$\gamma^2(b-a)+6\gamma \max_{x\in[a,b]}|f(x)|+3\gamma \max_{x\in[a,b]}|g(x)|(b-a) < \delta.$$

Then $|H(x)-h(x)| < \delta$, for every $x \in [a,b]$. Proposition 1.1 gives us an extremum point $d \in (a,b)$ of H such that $|d-c| < \varepsilon$. Fermat theorem shows that $H'(d) = 0$ and the conclusion follows now. □

Another type of Lagrange extension is due to Tong [**10**]. He posed a problem to obtain a mean value theorem, involving a pair of differentiable functions, but by other type than Cauhy theorem. He proposed a new result, such that for any $f, g \in D[a,b]$, there exists an intermediary point $c \in (a,b)$ such that

$$f'(c)(b-c)+g'(c)(c-a) = (g(b)-g(c))+(f(c)-f(a)). \tag{2.4}$$

We propose the following stability theorem that concludes this section.

Theorem 2.4. *Let the functions $f, g \in D[a,b]$. We suppose that there exists a unique point $c \in (a,b)$ that satisfies relation* (2.4)*. Then, for any $\varepsilon > 0$, there exists $\gamma > 0$ with the following property: for any functions $F, G \in D[a,b]$, satisfying the relations $|F(x)-f(x)| < \gamma$ and $|G(x)-g(x)| < \gamma$, for any $x \in [a,b]$, there exists a point $d \in (a,b)$ such that*

$$F'(d)(b-d)+G'(d)(d-a) = (G(b)-G(d))+(F(c)-F(a)) \text{ and } |d-c| < \varepsilon.$$

Proof. The differentiable function $h\colon [a,b] \to \mathbb{R}$, defined by relation

$$h(x) = (f(x)-f(a))(x-b)-(g(x)-g(b))(x-a),$$

has $h'(x) = f'(x)(x-b)+(f(x)-f(a))-g'(x)(x-a)-(g(x)-g(b))$, for any $x \in [a,b]$. Hence $h(a) = h(b)$, there exists an extremum point α, of h, on the interval (a,b). Fermat theorem gives us $h'(\alpha) = 0$. Then α is satisfying the relation (2.4). The hypothesis goes to $\alpha = c$.

Furthermore, let $\gamma > 0$ and $F, G \in D[a,b]$, such that $|F(x)-f(x)| < \gamma$ and $|G(x)-g(x)| < \gamma$, for any $x \in [a,b]$. We consider the function $H\colon [a,b] \to \mathbb{R}$, defined by

$$H(x) = (F(x)-F(a))(x-b)-(G(x)-G(b))(x-a),$$

for any $x \in [a,b]$. Hence

$$H'(x) = F'(x)(x-b)+(F(x)-F(a))-G'(x)(x-a)-(G(x)-G(b));$$

we obtain

$$\begin{aligned}|H(x)-h(x)| &\leqslant \gamma|(F(x)-F(a))-(f(x)-f(a))(x-b)| \\ &\quad +|(G(x)-G(b))-(g(x)-g(b))(x-a)| \\ &\leqslant 2\gamma(b-a)+2\gamma(b-a) = 4\gamma(b-a),\end{aligned}$$

for any $x \in [a,b]$.

Now, we consider $\delta > 0$ defined by Proposition 1.1 for the function h and $\varepsilon > 0$. We choose $\gamma < \frac{\delta}{4(b-a)}$. Then $|H(x) - h(x)| < \delta$, for any $x \in [a,b]$. Proposition 1.1 gives us an extremum point $d \in (a,b)$ of H such that $|d - c| < \varepsilon$. Fermat theorem goes to $H'(d) = 0$ and the conclusion follows now. □

3. The Hyers–Ulam stability of some integral mean points

In this section we investigate the Hyers–Ulam stability of the point arising from some integral mean theorems. First, let $f \in C[a,b]$ and $g \in C^*[a,b]$. Then, the functions $F, G\colon [a,b] \to \mathbb{R}$, defined by $F(x) = \int_a^x f(t)dt$ and $G(x) = \int_a^x g(t)dt$, for any $x \in [a,b]$, are differentiable on (a,b). If $g(x) \neq 0$, for any $x \in (a,b)$, then G is a monotone function and $G(b) \neq 0 = G(a)$. Under these assumptions, the integral form of Cauchy theorem said there exists $c \in (a,b)$ such that

$$\frac{\int_a^b f(t)dt}{\int_a^b g(t)dt} = \frac{f(c)}{g(c)}. \tag{3.1}$$

The following theorem establishes some conditions to the stability of the point c from the relation (3.1).

Theorem 3.1. *Let the functions $f \in C[a,b]$ and $g \in C^*[a,b]$. We suppose that there exist a unique point $c \in (a,b)$ satisfying relation* (3.1)*. Then, for any $\varepsilon > 0$, there exists $\gamma > 0$ with the following property: any functions $F \in C[a,b]$ and $G \in C^*[a,b]$, satisfying the relations $|F(x) - f(x)| < \gamma$ and $|G(x) - g(x)| < \gamma$, for any $x \in [a,b]$, admit a point $d \in (a,b)$ such that*

$$\frac{\int_a^b F(t)dt}{\int_a^b G(t)dt} = \frac{F(d)}{G(d)} \text{ and } |d - c| < \varepsilon.$$

Proof. We consider the continuous function $h\colon [a,b] \to \mathbb{R}$, defined by

$$h(x) = f(x)\int_a^b g(t)dt - g(x)\int_a^b f(t)dt,$$

for any $x \in [a,b]$. We claim that there exists $\alpha \in (a,b)$ such that $h(\alpha) = 0$. We assume by contradiction that $h(x) \neq 0$, for any $x \in (a,b)$. Then, the function $u\colon [a,b] \to \mathbb{R}$ defined by $u(x) = \int_a^x h(t)dt$, for every $x \in [a,b]$, is monotone. Hence, $u(a) = 0$ and $u(b) = 0$, then u is constant. Then $h \equiv 0$ and we obtain a contradiction, also our claim is true. Previous reasoning proved that the function h changes its sign in α.

Hence $h(\alpha) = 0$, the point α satisfies relation (3.1). Due to uniqueness, we obtain $\alpha = c$.

Now, let $\gamma > 0$. For any function $F \in C[a,b]$ and $G \in C^*[a,b]$, satisfying $|F(x) - f(x)| < \gamma$ and $|G(x) - g(x)| < \gamma$, we consider $H\colon [a,b] \to \mathbb{R}$, defined by

$$H(x) = F(x)\int_a^b G(t)dt - G(x)\int_a^b F(t)dt,$$

for any $x \in [a,b]$. We have

$$\begin{aligned}|H(x)-h(x)| &= \left|F(x)\int_a^b G(t)dt - G(x)\int_a^b F(t)dt - f(x)\int_a^b g(t)dt + g(x)\int_a^b f(t)dt\right| \\ &\leqslant |F(x)-f(x)|\int_a^b |G(t)|dt + |f(x)|\int_a^b |G(t)-g(t)|dt \\ &\quad + |G(x)-g(x)|\int_a^b |F(t)|dt + |g(x)|\int_a^b |F(t)-f(t)|dt,\end{aligned}$$

for any $x \in [a,b]$. We put $A = \max_{x\in[a,b]} f(x)$ and $B = \max_{x\in[a,b]} g(x)$. Then

$$\int_a^b |F(t)|dt \leqslant \int_a^b |F(t)-f(t)|dt + \int_a^b |f(t)|dt \leqslant \gamma(b-a) + A(b-a).$$

In the same mode, we have $\int_a^b |G(t)|dt \leqslant \gamma(b-a) + B(b-a)$. Then

$$\begin{aligned}|H(x)-h(x)| &\leqslant \gamma(B(b-a)+\gamma(b-a)) + B\gamma(b-a) \\ &\quad + A\gamma(b-a) + \gamma(A(b-a)+\gamma(b-a) \\ &= 2\gamma^2(b-a) + 2\gamma(A+B)(b-a),\end{aligned}$$

for any $x \in [a,b]$.

Now, we consider $\delta > 0$ defined by Proposition 1.2 for h and $\varepsilon > 0$. We choose $\gamma > 0$ such that $2\gamma^2(b-a) + 2\gamma(A+B)(b-a) < \delta$, Then, for every $x \in [a,b]$, we have $|H(x)-h(x)| < \delta$. Then, there exists $d \in (a,b)$ such that $|d-c| < \varepsilon$ and $H(d) = 0$. The last equality concludes the proof. □

The second integral mean value theorem said that, for any $f \in C[a,b]$ and $g \in C^*[a,b]$, there exists an intermediary point $c \in (a,b)$ such that

$$\int_a^b f(t)g(t)dt = f(c)\int_a^b g(t)dt. \tag{3.2}$$

Some conditions for the Hyers–Ulam stability of point c, defined by the previous equality, will be presented in the following theorem.

THEOREM 3.2. *Let the functions $f \in C[a,b]$ and $g \in C^*[a,b]$. We assume that there exists a unique point $c \in (a,b)$ satisfying the relation* (3.2)*. Then, for any $\varepsilon > 0$, there exists $\nu > 0$ with the following property: any functions $F \in C[a,b]$ and $G \in C^*[a,b]$, satisfying the relations $|F(x)-f(x)| < \nu$ and $|G(x)-g(x)| < \nu$, for any $x \in [a,b]$, admit a point $d \in (a,b)$ such that*

$$\int_a^b F(t)G(t)dt = F(d)\int_a^b G(t)dt \text{ and } |d-c| < \varepsilon.$$

PROOF. We consider the function $h\colon [a,b] \to \mathbb{R}$, defined by $h(x) = f(x)\cdot g(x)$, for any $x \in [a,b]$. Cauchy mean integral theorem gives us a point $\alpha \in (a,b)$ such that

$$\frac{\int_a^b h(t)dt}{\int_a^b g(t)dt} = \frac{h(\alpha)}{g(\alpha)}.$$

Then $g(\alpha)\int_a^b f(t)g(t)dt = g(\alpha)f(\alpha)\int_a^b g(t)dt$, Hence $g(\alpha) \neq 0$, we obtain that the point α is satisfying the relation (3.2), so $\alpha = c$ due to uniqueness.

Now, let $\nu > 0$ and $F \in C[a,b], G \in C^*[a,b]$, such that $|F(x) - f(x)| < \nu$ and $|G(x) - g(x)| < \nu$, for any $x \in [a,b]$. We define the function $H\colon [a,b] \to \mathbb{R}$, by relation $H(x) = F(x) \cdot G(x)$, for any $x \in [a,b]$. We have

$$\begin{aligned}
|H(x) - h(x)| &= |F(x)G(x) - f(x)g(x)| \\
&\leqslant |F(x)||G(x) - g(x)| + |g(x)||F(x) - f(x)| \\
&< \nu(|F(x)| + |g(x)|) \\
&\leqslant \nu(|F(x) - f(x)| + |f(x)| + |g(x)|) \\
&< \nu^2 + \nu(|f(x)| + |g(x)|),
\end{aligned}$$

for any $x \in [a,b]$.

Further, we consider $\gamma > 0$ defined by Theorem 3.1 for the functions h and g. We choose $\nu > 0$ such that $\nu < \gamma$ and

$$\nu^2 + \nu(\max_{x\in[a,b]} |f(x)| + \max_{x\in[a,b]} |g(x)|) < \gamma.$$

Then $|H(x) - h(x)| < \gamma$ and $|G(x) - g(x)| < \gamma$, for any $x \in [a,b]$. Theorem 3.1 give us a point $d \in (a,b)$ such that $|d - c| < \varepsilon$ and

$$\frac{\int_a^b H(t)dt}{\int_a^b G(t)dt} = \frac{H(d)}{G(d)}.$$

The proof is complete if we replace the function H with the product $F \cdot G$. $\square$

Similar to the integral version of Cauchy mean theorem, we can obtain an integral version of Wachnicki theorem. It said that, for any $f \in C[a.b]$ and $g \in C^*[a,b]$, with $\frac{f(a)}{g(a)} = \frac{f(b)}{g(b)}$, there exists an intermediary point $c \in (a,b)$ such that

$$\frac{f(c)}{g(c)} = \frac{\int_a^c f(t)dt}{\int_a^c g(t)dt}. \tag{3.3}$$

We investigate the stability of the point c from (3.3) and we propose the following result.

Theorem 3.3. *Let the functions $f \in C[a,b]$ and $g \in C^*[a,b]$ with $\frac{f(a)}{g(a)} = \frac{f(b)}{g(b)}$. We assume that there exists a unique point $c \in (a,b)$ satisfying relation* (3.3). *Then, for any $\varepsilon > 0$, there exists $\gamma > 0$ with the following property: for any functions $F \in C[a,b]$ and $G \in C^*[a,b]$, satisfying the relations $|F(x) - f(x)| < \gamma$ and $|G(x) - g(x)| < \gamma$, for every $x \in [a,b]$, there exists a point $d \in (a,b)$ such that*

$$\frac{F(d)}{G(d)} = \frac{\int_a^d F(t)dt}{\int_a^d G(t)dt} \text{ and } |d - c| < \varepsilon.$$

Proof. From the hypotesis, we can assume that g is positive. We consider the functions $u, h\colon [a,b] \to \mathbb{R}$, defined by

$$u(x) = \frac{f(x)}{g(x)} \quad \text{and} \quad h(x) = f(x)\int_a^x g(t)dt - g(x)\int_a^x f(t)dt,$$

for every $x \in [a,b]$.

If u is constant, then the point c is not unique. Then

$$\min_{x\in[a,b]} u(x) < \max_{x\in[a,b]} u(x).$$

Hence $u(a) = u(b)$, then at least one of $\min_{x\in[a,b]} u(x)$ or $\max_{x\in[a,b]} u(x)$ is on the interval (a,b). Suppose that $\min_{x\in[a,b]} u(x) \in (a,b)$. Denote

$$p = \inf\left\{y \in [a,b] \mid u(y) = \min_{x\in[a,b]} u(x)\right\}$$

From $u(t) \geqslant u(p)$, we have $\frac{f(t)}{g(t)} \geqslant \frac{f(p)}{g(p)}$, equivalent with $f(t)g(p) \geqslant f(p)g(t)$, for any $t \in [a,b]$. Then $g(p)\int_a^p f(t)dt > f(p)\int_a^p g(t)dt$, the inequality being strictly due to the choosing of p. We obtain $h(p) < 0$.

Further, denote $q = \inf\left\{y \in [p,b] \mid u(y) = \max_{x\in[a,b]} u(x)\right\}$. From $u(t) \leqslant u(q)$, we obtain $\frac{f(t)}{g(t)} \leqslant \frac{f(q)}{g(q)}$, equivalent with $f(t)g(q) \leqslant f(q)g(t)$, for any $t \in [a,b]$. Then $g(q)\int_a^q f(t)dt < f(q)\int_a^q g(t)dt$, the inequality being strictly due to the choosing of q. We obtain $h(q) > 0$. Then, there exists $\alpha \in (p,q)$ such that $h(\alpha) = 0$. From the hypothesis, we have $\alpha = c$. More, the function h changes its sign in c.

Now, let $\gamma > 0$ and the functions $F \in C[a,b]$ and $G \in C^*[a,b]$ such that $|F(x) - f(x)| < \gamma$ and $|G(x) - g(x)| < \gamma$, for any $x \in [a,b]$. We consider the function $H\colon [a,b] \to \mathbb{R}$, defined by $H(x) = F(x)\int_a^x G(t)dt - G(x)\int_a^x F(t)dt$, for any $x \in [a,b]$. Then, we have

$$\begin{aligned}
&|H(x) - h(x)\\
&= \left|F(x)\int_a^x G(t)dt - G(x)\int_a^x F(t)dt - f(x)\int_a^x g(t)dt + g(x)\int_a^x f(t)dt\right|\\
&\leqslant \left|F(x)\int_a^x G(t)dt - F(x)\int_a^x g(t)dt\right| + \left|F(x)\int_a^x g(t)dt - f(x)\int_a^x g(t)dt\right|\\
&\quad + \left|G(x)\int_a^x F(t)dt - G(x)\int_a^x f(t)dt\right| + \left|G(x)\int_a^x f(t)dt - g(x)\int_a^x f(t)dt\right|\\
&< \gamma(b-a)|F(x)| + \gamma\left|\int_a^x g(t)dt\right| + \gamma(b-a)|G(x)| + \gamma\left|\int_a^x f(t)dt\right|.
\end{aligned}$$

Further, we have $|F(x)| \leqslant |F(x) - f(x)| + |f(x)| < \gamma + \max_{x\in[a,b]} |f(x)|$ and similar for g. Then

$$\begin{aligned}
|H(x) - h(x)| &< \gamma(b-a)(\gamma + \max_{x\in[a,b]} |f(x)|) + \gamma(b-a)\max_{x\in[a,b]} |f(x)|\\
&\quad + \gamma(b-a)(\gamma + \max_{x\in[a,b]} |g(x)| + \gamma(b-a)\max_{x\in[a,b]} |g(x)|\\
&= \gamma^2(b-a) + 2\gamma(b-a)(\max_{x\in[a,b]} |f(x)| + \max_{x\in[a,b]} |g(x)|).
\end{aligned}$$

For $\varepsilon > 0$, we choose $\delta > 0$ defined by Proposition 1.2 for h and ε. Now, we choose $\gamma > 0$ such that

$$\gamma^2(b-a) + 2\gamma(b-a)(\max_{x\in[a,b]} |f(x)| + \max_{x\in[a,b]} |g(x)|) < \delta.$$

Then $|H(x)-h(x)| < \gamma$, for any $x \in [a,b]$. Proposition 1.2 gives us a point $d \in (a,b)$ such that $|d-c| < \varepsilon$ and $H(d) = 0$. The last equality concludes our proof. $\square$

Further, we obtain a new integral mean value theorem by applying the integral version of Wachnichi theorem. This result is a combination between this theorem and the second integral mean value theorem. It is following.

THEOREM 3.4. *Let the functions $f \in C[a,b]$, with $f(a) = f(b)$, and $g \in C^*[a,b]$. Then there exists a point $c \in (a,b)$ such that*

$$\int_a^c f(t)g(t)dt = f(c)\int_a^c g(t)dt. \tag{3.4}$$

PROOF. The point $c \in (a,b)$ from equality (3.4) is obtained from relation (3.3) applied to the functions $f \cdot g$ and g. $\square$

In some conditions, the point $c \in (a,b)$ from equality (3.4) is stable in the sense of Hyers–Ulam. The following result shows this.

THEOREM 3.5. *Let the functions $f \in C[a,b]$ with $f(a) = f(b)$, and $g \in C^*[a,b]$. We assume that there exists a unique point $c \in (a,b)$ satisfying relation (3.4). Then, for any $\varepsilon > 0$, there exists $\nu > 0$ with the following property: for any functions $F \in C[a,b]$ and $G \in C^*[a,b]$, satisfying the relations $|F(x) - f(x)| < \nu$ and $|G(x) - g(x)| < \nu$, for every $x \in [a,b]$, there exists a point $d \in (a,b)$ such that*

$$\int_a^d F(t)G(t)dt = F(d)\int_a^d G(t)dt \text{ and } |d-c| < \varepsilon.$$

PROOF. We consider the function $h\colon [a,b] \to \mathbb{R}$, defined by $h(x) = f(x) \cdot g(x)$, for any $x \in [a,b]$. The integral version of Wachnichi theorem gives us a point $\beta \in (a,b)$ such that

$$\frac{\int_a^\beta h(t)dt}{\int_a^\beta g(t)dt} = \frac{h(\beta)}{g(\beta)}.$$

Then $g(\beta)\int_a^\beta f(t)g(t)dt = g(\beta)f(\beta)\int_a^\beta g(t)dt$. Hence $g(\beta) \neq 0$, we obtain that the point β is satisfying the relation (3.4), so $\beta = c$.

Now, let $\nu > 0$ and the functions $F \in C[a,b]$ and $G \in C^*[a,b]$, such that $|F(x) - f(x)| < \nu$ and $|G(x) - g(x)| < \nu$, for any $x \in [a,b]$. We consider the function $H\colon [a,b] \to \mathbb{R}$, defined by $H(x) = F(x) \cdot G(x)$, for any $x \in [a,b]$. We have

$$\begin{aligned} |H(x) - h(x)| &= |F(x)G(x) - f(x)g(x)| \\ &\leqslant |F(x)||G(x) - g(x)| + |g(x)||F(x) - f(x)| \\ &< \nu(|F(x)| + |g(x)|) \\ &\leqslant \nu(|F(x) - f(x)| + |f(x)| + |g(x)|) \end{aligned}$$

$$< \nu^2 + \nu(|f(x)| + |g(x)|),$$

for any $x \in [a, b]$. Now, we consider $\gamma > 0$ defined by Theorem 3.3 for h and g. We choose $\nu > 0$ such that $\nu < \gamma$ and $\nu^2 + \nu\big(\max_{x\in[a,b]} |f(x)| + \max_{x\in[a,b]} |g(x)|\big) < \gamma$. Then $|H(x) - h(x)| < \gamma$ and $|G(x) - g(x)| < \gamma$, for any $x \in [a, b]$. Theorem 3.3 gives a point $d \in (a, b)$ such that $|d - c| < \varepsilon$ and

$$\frac{\int_a^d H(t)dt}{\int_a^d G(t)dt} = \frac{H(d)}{G(d)}.$$

Since $H(x) = F(x) \cdot G(x)$, we obtain the conclusion. □

We conclude this paper with a result involving an integral mean value theorem due to Tong [**11**]. It is an integral version of the result described in the previous section. It said that, for any functions $f, g \in C[a, b]$, there exists an intermediary point $c \in (a, b)$ such that

$$\int_a^c f(t)dt + \int_c^b g(t)dt = f(c)(b - c) + g(c)(c - a). \tag{3.5}$$

Then the following stability result holds.

Theorem 3.6. *Let the functions $f, g \in C[a, b]$. We suppose that there exists a unique point $c \in (a, b)$ satisfying relation* (3.5)*. Then, for any $\varepsilon > 0$, there exists $\gamma > 0$ with the following property: for any functions $F, G \in C[a, b]$, satisfying the relations $|F(x) - f(x)| < \nu$ and $|G(x) - g(x)| < \nu$, for every $x \in [a, b]$, there exists a point $d \in (a, b)$ such that*

$$\int_a^d F(t)dt + \int_d^b G(t)dt = F(d)(b - d) + G(d)(d - a) \text{ and } |d - c| < \varepsilon.$$

Proof. We consider the continuous function $h\colon [a, b] \to \mathbb{R}$, defined by

$$h(x) = \int_a^x f(t)dt + \int_x^b g(t)dt - f(x)(b - x) - g(x)(x - a),$$

for any $x \in [a, b]$. Then $h(c) = 0$ and $h(x) \neq 0$ if $x \in (a, b) \setminus \{c\}$.

We claim that h changes its sign in c. We assume by contradiction that $h(x) > 0$, for any $x \in (a, b) \setminus \{c\}$. Then, the function $u\colon [a, b] \to \mathbb{R}$, defined by $u(x) = \int_a^x h(y)dy$, for any $x \in [a, b]$, is increasing.

Further, we have

$$u(b) = \int_a^b h(y)dy = \int_a^b \Big(\int_a^y f(t)dt\Big)dy + \int_a^b \Big(\int_x^b g(t)dt\Big)dy$$
$$- \int_a^b f(y)(b - y)dy - \int_a^b g(y)(y - a)dy.$$

Since

$$\int_a^b f(y)(b - y)dy = (b - y) \int_a^y f(t)dt\Big|_a^b + \int_a^b\Big(\int_a^y f(t)dt\Big)dy = \int_a^b \Big(\int_a^y f(t)dt\Big)dy,$$

$$\int_a^b g(y)(y-a)dy = -(y-a)\int_y^b g(t)dt\Big|_a^b + \int_a^b\Big(\int_y^b g(t)dt\Big)dy = \int_a^b\Big(\int_y^b g(t)dt\Big)dy,$$

then $u(b) = 0$ and u is a constant function. This contradicts the monotonicity of u.

Now, let $\gamma > 0$ and the functions $F, G \in C[a,b]$ such that $|F(x) - f(x)| < \delta$ and $|G(x) - g(x)| < \delta$, for any $x \in [a,b]$. We define $H\colon [a,b] \to \mathbb{R}$, by relation

$$H(x) = \int_a^x F(t)dt + \int_x^b G(t)dt - F(x)(b-x) - G(x)(x-a),$$

for any $x \in [a,b]$. Then

$$|H(x) - h(x)| \leqslant \int_a^x |F(t) - f(t)|dt + \int_x^b |G(t) - g(t)|dt \\ + |F(x) - f(x)|(b-x) + |G(x) - g(x)|(x-a) < 4\gamma(b-a),$$

for any $x \in [a,b]$.

Further, we consider $\delta > 0$ defined by Proposition 1.2 for h and ε. We choose $\gamma < \frac{\delta}{4(b-a)}$. Then $|H(x) - h(x)| < \delta$, for any $x \in [a,b]$. The same proposition gives a point $d \in (a,b)$ such that $|d - c| < \varepsilon$ and $H(d) = 0$. The last equality completes the proof. □

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