PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 104(118) (2018), 183–191 DOI: https://doi.org/10.2298/PIM1818183S

REMARKS ON NEIGHBORHOOD STAR-MENGER SPACES

Yan-Kui Song

ABSTRACT. A space X is said to be neighborhood star-Menger if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X one can choose finite $A_n \subseteq X$, $n \in \mathbb{N}$ such that for every open $O_n \supseteq A_n$, $n \in \mathbb{N}$, $\{\mathrm{St}(O_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is an open cover of X. We investigate the relationship between neighborhood star-Menger spaces and related spaces, and study the topological properties of neighborhood star-Menger spaces.

1. Introduction

By a space, we mean a topological space. We give definitions of terms which are used in this paper. Let $\mathbb{N}$ denote the set of positive integers. Let X be a space and $\mathcal{U}$ a collection of subsets of X. For $A \subseteq X$, let $\mathrm{St}(A, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$. As usually, we write $\mathrm{St}(x, \mathcal{U})$ instead of $\mathrm{St}(\{x\}, \mathcal{U})$.

Let $\mathcal{A}$ and $\mathcal{B}$ be collections of open covers of a space X. Then the symbol $S_1(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis so that for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there exists a sequence $(U_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $U_n \in \mathcal{U}_n$ and $\{U_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$. The symbol $S_{\mathrm{fin}}(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis so that for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n \in \mathbb{N}} \mathcal{V}_n$ is an element of $\mathcal{B}$ (see [**6, 11**]).

Kočinac [**7, 8**] introduced a star selection hypothesis similar to the previous ones. Let $\mathcal{A}$ and $\mathcal{B}$ be collections of open covers of a space X. Then:

(A) The symbol $S^*_{\mathrm{fin}}(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis so that for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n \in \mathbb{N}}\{\mathrm{St}(V, \mathcal{U}_n) : V \in \mathcal{V}_n\}$ is an element of $\mathcal{B}$.

2010 *Mathematics Subject Classification:* 54D20; 54D30.

Key words and phrases: selection principles, (strongly) star-Menger, neighborhood star-Menger.

Communicated by Miloš Kurilić.

(B) The symbol $SS^*_{\rm fin}(\mathcal{A},\mathcal{B})$ denotes the selection hypothesis so that for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there exists a sequence $(A_n : n \in N)$ of finite subsets of X such that $\{\mathrm{St}(A_n,\mathcal{U}_n) : n \in \mathbb{N}\} \in \mathcal{B}$.

Bonanzinga et al. [**2**] introduced the following definition.

(C) The symbol $\mathrm{NSM}(\mathcal{A},\mathcal{B})$ denotes the selection hypothesis so that for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ one can choose finite $A_n \subseteq X, n \in \mathbb{N}$, such that for every open $O_n \supseteq A_n, n \in \mathbb{N}$, $\{\mathrm{St}(O_n,\mathcal{U}_n) : n \in \mathbb{N}\} \in \mathcal{B}$.

Let $\mathcal{O}$ denote the collection of all open covers of X.

Definition 1.1. [**7, 8**] A space X is said to be *star-Menger* if it satisfies the selection hypothesis $S^*_{\rm fin}(\mathcal{O},\mathcal{O})$.

Definition 1.2. [**7, 8**] A space X is said to be *strongly star-Menger* if it satisfies the selection hypothesis $SS^*_{\rm fin}(\mathcal{O},\mathcal{O})$.

Definition 1.3. [**2**] A space X is said to be *neighborhood star-Menger* if it satisfies the selection hypothesis $\mathrm{NSM}(\mathcal{O},\mathcal{O})$.

From the above definitions, we have the following diagram

strongly star-Menger $\Rightarrow$ *neighborhood star-Menger* $\Rightarrow$ *star-Menger.*

The purpose of this paper is to investigate the relationships between neighborhood star-Menger spaces and related spaces, and also study topological properties of neighborhood star-Menger spaces.

Throughout this paper, let ω denote the first infinite cardinal, ω_1 the first uncountable cardinal, $\mathfrak{c}$ the cardinality of the set of all real numbers. For a cardinal κ, let κ^+ be the smallest cardinal greater than κ. For each pair of ordinals α, β with $\alpha < \beta$, we write $[\alpha,\beta) = \{\gamma : \alpha \leqslant \gamma < \beta\}$, $(\alpha,\beta] = \{\gamma : \alpha < \gamma \leqslant \beta\}$, $(\alpha,\beta) = \{\gamma : \alpha < \gamma < \beta\}$ and $[\alpha,\beta] = \{\gamma : \alpha \leqslant \gamma \leqslant \beta\}$. As usually, a cardinal is an initial ordinal and an ordinal is the set of smaller ordinals. Every cardinal is often viewed as a space with the usual order topology. Other terms and symbols that we do not define follow from [**5**].

2. Neighborhood star-Menger spaces and related spaces

In this section, first we give some examples to clarify the relationships between neighborhood star-Menger spaces and related spaces. Recall that a space is called *Urysohn* if every two distinct points have neighborhoods with disjoint closures. Clearly, the property is between the Hausdorff condition and regularity. First we give a consistent example showing that there exists a neighborhood star-Menger space that is not strongly star-Menger by using the following example from [**2**]. We make use of one of the cardinals defined in [**4**]. Define ${}^\omega\omega$ as the set of all functions from ω to itself. For all $f, g \in {}^\omega\omega$, we say $f \leqslant^* g$ if and only if $f(n) \leqslant g(n)$ for all but finitely many n. The dominating number, denoted by $\mathfrak{d}$, is the smallest cardinality of a cofinal subset of $({}^\omega\omega, \leqslant^*)$ (see [**4**] for details). Recall that a space X is *strongly star-Lindelöf* (see [**3**] or [**9**] under different names) if for every open cover $\mathcal{U}$ of X there exists a countable subset A of X such that $X = \mathrm{St}(A,\mathcal{U})$. Clearly every strongly star-Menger space is strongly star-Lindelöf.

EXAMPLE 2.1. [**2**] ($\omega_1 < \mathfrak{d}$) There is a Urysohn neighborhood star-Menger space that is not strongly star-Lindelöf (hence not strongly star-Menger).

Recall that a space X is *strongly starcompact* (see [**3**] or [**9**] under different names) if for every open cover $\mathcal{U}$ of X there exists a finite subset A of X such that $X = \mathrm{St}(A,\mathcal{U})$. It is well known that strongly starcompactness is equivalent to countable compactness for Hausdorff spaces (see [**3**] or [**9**]). Recall that a space X is *weakly starcompact* [**1**] if for every open cover $\mathcal{U}$ of X there exists a finite subset A of X such that for every open $O \supseteq A$, $X = \mathrm{St}(O,\mathcal{U})$. It is clear that every weakly starcompact space is neighborhood star-Menger. For T_1 spaces, we have the following example.

EXAMPLE 2.2. There exists a T_1 neighborhood star-Menger space that is not strongly star-Menger.

PROOF. Let $X = [0,\omega_1) \cup D$, where $D = \{d_\alpha : \alpha < \omega_1\}$ is a set of cardinality ω_1. We topologize X as follows: $[0,\omega_1)$ has the usual order topology and is an open subspace of X; a basic neighborhood of a point $d_\alpha \in D$ takes the form

$$O_\beta(d_\alpha) = \{d_\alpha\} \cup (\beta,\omega_1), \text{ where } \beta < \omega_1.$$

Then X is a T_1 space.

First we show that X is neighborhood star-Menger. We only show that X is weakly starcompact, since every weakly starcompact space is neighborhood star-Menger. To this end, let $\mathcal{U}$ be an open cover of X. Without loss of generality, we can assume that $\mathcal{U}$ consists of basic open subsets of X. Thus it is sufficient to show that there exists a finite subset A of X such that for every open $O \supseteq A$, $X = \mathrm{St}(O,\mathcal{U})$. Since $[0,\omega_1)$ is countably compact, it is strongly starcompact (see [**3, 9**]), then we can find a finite subset A_1 of $[0,\omega_1)$ such that $[0,\omega_1) \subseteq \mathrm{St}(A_1,\mathcal{U})$. On the other hand, if we pick $\alpha_0 < \omega_1$, then for every open $O \ni d_{\alpha_0}$, $D \subseteq \mathrm{St}(O,\mathcal{U})$. In fact, for each $\alpha < \omega_1$, if $d_\alpha \in U_\alpha \in \mathcal{U}$, then $U_\alpha \cap O \neq \emptyset$ by the construction of the topology of X, thus $d_\alpha \in \mathrm{St}(O,\mathcal{U})$. Therefore $D \subseteq \mathrm{St}(O,\mathcal{U})$. If we put $A = A_1 \cup \{d_{\alpha_0}\}$, then A is a finite subset of X and $X = \mathrm{St}(O,\mathcal{U})$ for every open $O \supseteq A$, which shows that X is weakly starcompact.

Next we show that X is not strongly star-Menger. For each $n \in \mathbb{N}$, let

$$\mathcal{U}_n = \{O_\alpha(d_\alpha) : \alpha < \omega_1\} \cup \{[0,\omega_1)\}.$$

Then $\mathcal{U}_n$ is an open cover of X. Let us consider the sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X. It suffices to show that $\bigcup_{n\in\mathbb{N}} \mathrm{St}(A_n,\mathcal{U}_n) \neq X$ for any sequence $(A_n : n \in \mathbb{N})$ of finite subsets of X. Let $(A_n : n \in \mathbb{N})$ be any sequence of finite subsets of X. For each $n \in \mathbb{N}$, the set $A_n \cap \{d_\alpha : \alpha < \omega_1\}$ is finite, since A_n is finite. Then there exists $\alpha_n < \omega_1$ such that $A_n \cap \{d_\alpha : \alpha > \alpha_n\} = \emptyset$. Let $\alpha' = \sup\{\alpha_n : n \in \mathbb{N}\}$. Then $\alpha' < \omega_1$ and $\left(\bigcup_{n\in\mathbb{N}} A_n\right) \cap \{d_\alpha : \alpha > \alpha'\} = \emptyset$. For each $n \in \mathbb{N}$, the set $A_n \cap [0,\omega_1)$ is finite suborder of the linear order $[0,\omega_1)$ and thus has a maximum. Let $\alpha'_n = \max(A_n \cap [0,\omega_1))$. Then $A_n \cap (\alpha'_n,\omega_1) = \emptyset$. Let $\alpha'' = \sup\{\alpha'_n : n \in \mathbb{N}\}$. Then $\alpha'' < \omega_1$ and $\left(\bigcup_{n\in\mathbb{N}} A_n\right) \cap (\alpha'',\omega_1) = \emptyset$. If we pick $\beta > \max\{\alpha',\alpha''\}$. Then $O_\beta(d_\beta) \cap A_n = \emptyset$ for each $n \in \mathbb{N}$. Hence $d_\beta \notin \mathrm{St}(A_n,\mathcal{U}_n)$

for each $n \in \mathbb{N}$, since $O_\beta(d_\beta)$ is the only element of $\mathcal{U}_n$ containing the point d_β for each $n \in \mathbb{N}$, which shows that X is not strongly star-Menger. $\square$

REMARK 2.1. The author does not know if there exists a Hausdorff (Urysohn, regular or Tychonoff) neighborhood star-Menger space that is not strongly star-Menger.

Bonanzinga et al. [**2**] constructed an example showing that there exists a Tychonoff star-Menger space X that is not strongly star-Lindelöf. In fact, the Example also shows that there exists a Tychonoff star-Menger space X that is not neighborhood star-Menger. Here we give the construction roughly for the sake of completeness.

EXAMPLE 2.3. [**2**] There exists a Tychonoff star-Menger space X that is not neighborhood star-Menger.

PROOF. Let $D = \{d_\alpha : \alpha < \kappa\}$ be a discrete space of cardinality κ and let $Y = D \cup \{d^*\}$ be one-point compactification of D.

Let $X = (Y \times [0, \kappa^+)) \cup (D \times \{\kappa^+\})$ be the subspace of the product space $Y \times [0, \kappa^+]$. Then X is star-Menger, but not strongly star-Lindelöf (see [**2**, Example 3.7] for detail). Hence it is not neighborhood star-Menger, since every neighborhood star-Menger space is strongly star-Lindelöf. $\square$

Next we study the topological properties of neighborhood star-Menger spaces. First we give an example from [**1**] that we shall use it in the following text.

EXAMPLE 2.4. [**1**] Let $\mathcal{A}$ be an almost disjoint family of infinite subsets of ω (i.e., the intersection of every two distinct elements of $\mathcal{A}$ is finite) and Let $X = \omega \cup \mathcal{A}$ be the Isbell-Mrówka space constructed from $\mathcal{A}$ [**3, 5**]. Then

(1) X is strongly star-Menger if and only if $|\mathcal{A}| < \mathfrak{d}$;
(2) If $|\mathcal{A}| = \mathfrak{c}$, then X is not star-Menger.

If $\omega_1 < \mathfrak{d}$, the space $X = \omega \cup \mathcal{A}$ with $|\mathcal{A}| = \omega_1$ is strongly star-Menger (hence neighborhood star-Menger) by Example 2.4. It shows that a closed subset of a Tychonoff strongly star-Menger (hence neighborhood star-Menger) space X need not be neighborhood star-Menger, since $\mathcal{A}$ is a discrete closed subset of cardinality ω_1. Now we give a stronger example showing that a regular-closed subset of a Tychonoff neighborhood star-Menger space X need not be neighborhood star-Menger. Here a subset A of a space X is said to be *regular-closed* in X if $\overline{\mathrm{Int}\,A} = A$. For the next example, we need the following Lemma.

LEMMA 2.1. [**2**] *A space X is neighborhood star-Menger if and only if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there exists a sequence $(A_n : n \in \mathbb{N})$ of finite subsets of X such that for every $x \in X$ there exists $n \in \mathbb{N}$ such that $\overline{\mathrm{St}(x, \mathcal{U})} \cap A_n \neq \emptyset$.*

EXAMPLE 2.5. ($\omega_1 < \mathfrak{d}$) There exists a Tychonoff strongly star-Menger (hence neighborhood star-Menger) space having a regular-closed subspace which is not neighborhood star-Menger.

PROOF. Let $S_1 = \omega \cup \mathcal{A}$ be the Isbell-Mrówka space with $|\mathcal{A}| = \omega_1$ in Example 2.4. Then S_1 is Tychonoff strongly star-Menger by Example 2.4. Hence S_1 is neighborhood star-Menger.

Let $D = \{d_\alpha : \alpha < \omega_1\}$ be a discrete space of cardinality ω_1 and $Y = D \cup \{d^*\}$ one-point compactification of D. Let $S_2 = (Y \times [0, \omega)) \cup (D \times \{\omega\})$ be the subspace of the product space $Y \times [0, \omega]$. We show that S_2 is not neighborhood star-Menger. For each $n \in \mathbb{N}$, let

$$\mathcal{U}_n = \{Y \times \{m\} : m \in \omega\} \cup \{\{d_\alpha\} \times [0, \omega] : \alpha < \omega_1\}.$$

Then $\mathcal{U}_n$ is an open cover of S_2. Let us consider the sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of S_2. It suffices to show that for any sequence $(A_n : n \in \mathbb{N})$ of finite subsets of S_2, there exists a point $a \in S_2$ such that $\overline{\mathrm{St}(a, \mathcal{U}_n)} \cap A_n = \emptyset$ for all $n \in \mathbb{N}$ by Lemma 2.1. Let $(A_n : n \in \mathbb{N})$ be any sequence of finite subsets of S_2. For each $n \in \mathbb{N}$, since A_n is finite, there exists $\alpha_n < \omega_1$ such that $A_n \cap (\{d_\alpha\} \times [0, \omega]) = \emptyset$ for each $\alpha > \alpha_n$. Let $\beta = \sup\{\alpha_n : n \in \mathbb{N}\}$. Then $\beta < \omega_1$ and

$$\left(\bigcup_{n \in \mathbb{N}} A_n\right) \cap (\{d_\alpha\} \times [0, \omega]) = \emptyset \text{ for each } \alpha > \beta.$$

Let us pick $\alpha > \beta$. Since $\{d_\alpha\} \times [0, \omega]$ is the only element of $\mathcal{U}_n$ containing the point $\langle d_\alpha, \omega\rangle$ for each $n \in \mathbb{N}$, $\mathrm{St}(\langle d_\alpha, \omega\rangle, \mathcal{U}_n) = \{d_\alpha\} \times [0, \omega]$ for each $n \in \mathbb{N}$. Thus

$$\left(\bigcup_{n \in \mathbb{N}} A_n\right) \cap \mathrm{St}(\langle d_\alpha, \omega\rangle, \mathcal{U}_n) = \emptyset.$$

By the constructions of the topology of S_2 and the open cover $\mathcal{U}_n$, we have

$$\overline{\mathrm{St}(\langle d_\alpha, \omega\rangle, \mathcal{U}_n)} = \{d_\alpha\} \times [0, \omega].$$

Hence $\overline{\mathrm{St}(\langle d_\alpha, \omega\rangle, \mathcal{U}_n)} \cap A_n = \emptyset$ for all $n \in \mathbb{N}$.

We assume $S_1 \cap S_2 = \emptyset$. Let $\pi : D \times \{\omega\} \to \mathcal{A}$ be a bijection. Let X be the quotient image of the disjoint sum $S_1 \oplus S_2$ obtained by identifying $\langle d_\alpha, \omega\rangle$ of S_2 with $\pi(\langle d_\alpha, \omega\rangle)$ of S_1 for every $\alpha < \mathfrak{c}$. Let $\varphi : S_1 \oplus S_2 \to X$ be the quotient map. It is clear that $\varphi(S_2)$ is a regular-closed subspace of X which is not neighborhood star-Menger, since it is homeomorphic to S_2.

Finally we show that X is strongly star-Menger. To this end, let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. Since $\varphi(S_1)$ is homeomorphic to S_1, then $\varphi(S_1)$ is strongly star-Menger, there exists a sequence $(A'_n : n \in \mathbb{N})$ of finite subsets of $\varphi(S_1)$ such that $\varphi(S_1) \subseteq \bigcup_{n \in \mathbb{N}} \mathrm{St}(A'_n, \mathcal{U}_n)$. On the other hand, for each $n \in \omega$, since $\varphi(Y \times \{n\})$ is homeomorphic to $Y \times \{n\}$, then $\varphi(Y \times \{n\})$ is compact, we can find a finite subset $A''_n \subseteq \varphi(Y \times \{n\})$ such that $\varphi(Y \times \{n\}) \subseteq \mathrm{St}(A''_n, \mathcal{U}_{n+1})$. For each $n \in \mathbb{N}$, we put $A_n = A'_n \cup A''_{n-1}$. Then the sequence $(A_n : n \in \mathbb{N})$ of finite subsets of X witnesses that X is strongly star-Menger, which completes the proof. □

In the following, we give a positive result, which can be easily proved.

THEOREM 2.1. *If X is a neighborhood star-Menger space, then every open and closed subset of X is neighborhood star-Menger.*

It is known that a continuous image of a strongly star-Menger space is strongly star-Menger. Similarly, we have the following result.

THEOREM 2.2. *A continuous image of a neighborhood star-Menger space is neighborhood star-Menger.*

PROOF. Let $f\colon X \to Y$ be a continuous mapping from a neighborhood star-Menger space X onto a space Y. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of Y. Then $(\{f^{-1}(U) : U \in \mathcal{U}_n\} : n \in \mathbb{N})$ is a sequence of open covers of X. Since X is neighborhood star-Menger, there exists a sequence $(A_n : n \in \mathbb{N})$ of finite subsets of X such that for every open $O_n \supseteq A_n$, $n \in \mathbb{N}$, $\{\mathrm{St}(O_n, \{f^{-1}(U) : U \in \mathcal{U}_n\}) : n \in \mathbb{N}\}$ is an open cover of X. Thus $(f(A_n) : n \in \mathbb{N})$ is a sequence of finite subsets of Y. We only show that for every open $W_n \supseteq f(A_n)$, $n \in \mathbb{N}$, $\{\mathrm{St}(W_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is an open cover of Y. In fact, let $y \in Y$. Then there is $x \in X$ such that $f(x) = y$. Let W_n be an open subset of Y such that $f(A_n) \subseteq W_n$ for $n \in \mathbb{N}$. Then $f^{-1}(W_n)$ is an open subset of X, $A_n \subseteq f^{-1}(W_n)$ for each $n \in \mathbb{N}$ and

$$\{\mathrm{St}(f^{-1}(W_n), \{f^{-1}(U) : U \in \mathcal{U}_n\}) : n \in \mathbb{N}\}$$

is an open cover of X. Hence there exist $n \in \mathbb{N}$ and $U \in \mathcal{U}_n$ such that $x \in f^{-1}(U)$ and $f^{-1}(U) \cap f^{-1}(W_n) \neq \emptyset$. Thus $y = f(x) \in f(f^{-1}(U)) = U$ and $U \cap W_n \neq \emptyset$. This means that $y \in \mathrm{St}(W_n, \mathcal{U}_n)$, which completes the proof. $\square$

Next we turn to considering preimages. To show that the preimage of a neighborhood star-Menger space under a closed 2-to-1 continuous map need not be neighborhood star-Menger, we use the Alexandorff duplicate $A(X)$ of a space X. The underlying set $A(X)$ is $X \times \{0, 1\}$; each point of $X \times \{1\}$ is isolated and a basic neighborhood of $\langle x, 0\rangle \in X \times \{0\}$ is a set of the form $(U \times \{0\}) \cup ((U \times \{1\}) \setminus \{\langle x, 0\rangle\})$, where U is a neighborhood of x in X.

EXAMPLE 2.6. ($\omega_1 < \mathfrak{d}$) There exists a closed 2-to-1 continuous map $f\colon X \to Y$ such that Y is a neighborhood star-Menger space, but X is not neighborhood star-Menger.

PROOF. Let Y be the space S_1 in the proof of Example 2.5. Then Y is neighborhood star-Menger. Let X be the Alexandorff duplicate $A(Y)$ of the space Y. Then X is not neighborhood star-Menger. In fact, let $A = \{\langle a, 1\rangle : a \in \mathcal{A}\}$. Then A is an open and closed subset of X with $|A| = \omega_1$, and each point $\langle a, 1\rangle$ is isolated. Hence $A(X)$ is not neighborhood star-Menger by Theorem 2.1, since A is not neighborhood star-Lindelöf. Let $f : X \to Y$ be the projection. Then f is a closed 2-to-1 continuous map, which completes the proof. $\square$

Now we give a positive result:

THEOREM 2.3. *Let f be an open and closed, finite-to-one continuous map from a space X onto a neighborhood star-Menger space Y. Then X is neighborhood star-Menger.*

PROOF. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X and let $y \in Y$. For each $n \in \mathbb{N}$, since $f^{-1}(y)$ is finite, there exists a finite subcollection $\mathcal{U}_{n,y}$ of $\mathcal{U}_n$ such that $f^{-1}(y) \subseteq \bigcup \mathcal{U}_{n,y}$ and $U \cap f^{-1}(y) \neq \emptyset$ for each $U \in \mathcal{U}_{n,y}$. Since f is closed, there exists an open neighborhood $V_{n,y}$ of y in Y such that $f^{-1}(V_{n,y}) \subseteq \bigcup\{U : U \in \mathcal{U}_{n,y}\}$. Since f is open, we can assume that

$$V_{n,y} \subseteq \bigcap\{f(U) : U \in \mathcal{U}_{n,y}\}. \tag{2.1}$$

For each $n \in \mathbb{N}$, taking such open set $V_{n,y}$ for each $y \in Y$, we have an open cover $\mathcal{V}_n = \{V_{n,y} : y \in Y\}$ of Y. Thus $(\mathcal{V}_n : n \in \mathbb{N})$ is a sequence of open covers of Y, so that there exists a sequence $(A_n : n \in \mathbb{N})$ of finite subsets of Y such that for very open $O_n \supseteq A_n, n \in \mathbb{N}$, $\{\mathrm{St}(O_n, \mathcal{V}_n) : n \in \mathbb{N}\}$ is an open cover of Y, since Y is neighborhood star-Menger. Since f is finite-to-one, $(f^{-1}(A_n) : n \in N)$ is a sequence of finite subsets of X. We show that for very open $O'_n \supseteq f^{-1}(A_n)$, $n \in \mathbb{N}$, $\{\mathrm{St}(O'_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is an open cover of X. Since f is closed and A_n is finite, there exists an open subset O_n such that $A_n \subseteq O_n$ and $f^{-1}(O_n) \subseteq O'_n$ for each $n \in \mathbb{N}$, thus $\{\mathrm{St}(O_n, \mathcal{V}_n) : n \in \mathbb{N}\}$ is an open cover of Y. Let $x \in X$. Then there exist $n \in \mathbb{N}$ and $y \in Y$ such that $f(x) \in V_{n,y}$ and $V_{n,y} \cap O_n \neq \emptyset$. Since

$$x \in f^{-1}(V_{n,y}) \subseteq \bigcup\{U : U \in \mathcal{U}_{n,y}\},$$

we can choose $U \in \mathcal{U}_{n,y}$ with $x \in U$. Then $V_{n,y} \subseteq f(U)$ by (2.1), and hence $U \cap f^{-1}(O_n) \neq \emptyset$. Since $f^{-1}(O_n) \subseteq O'_n$ for each $n \in \mathbb{N}$, $U \cap O'_n \neq \emptyset$. Thus $x \in \mathrm{St}(O'_n, \mathcal{U}_n)$. Hence $\{\mathrm{St}(O'_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is an open cover of X, which shows that X is neighborhood star-Menger. □

Example 2.7. $(\omega_1 < \mathfrak{d})$ There exist a neighborhood star-Menger space X and a compact space Y such that $X \times Y$ is not neighborhood star-Menger.

Proof. Let X be the space S_1 in the proof of Example 2.5. Then X is neighborhood star-Menger. Let $D = \{d_\alpha : \alpha < \omega_1\}$ be a discrete space of cardinality ω_1 and $Y = D \cup \{d^*\}$ the one-point compactification of D. We show that $X \times Y$ is not neighborhood star-Menger. Since $|\mathcal{A}| = \omega_1$, we can enumerate $\mathcal{A}$ as $\{a_\alpha : \alpha < \omega_1\}$. For each $n \in \mathbb{N}$, let

$$\mathcal{U}_n = \{(\{a_\alpha\} \cup a_\alpha) \times (Y \setminus \{d_\alpha\}) : \alpha < \omega_1\} \cup \{X \times \{d_\alpha\} : \alpha < \omega_1\} \cup \{\omega \times Y\}.$$

Then $\mathcal{U}_n$ is an open cover of $X \times Y$. Let us consider the sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of $X \times Y$. It suffices to show that for any sequence $(A_n : n \in \mathbb{N})$ of finite subsets of $X \times Y$ there exists a point $a \in X \times Y$ such that $\overline{\mathrm{St}(a, \mathcal{U}_n)} \cap A_n = \emptyset$ for all $n \in \mathbb{N}$ by Lemma 2.1. Let $(A_n : n \in \mathbb{N})$ be any sequence of finite subsets of $X \times Y$. For each $n \in \mathbb{N}$, since A_n is finite, there exists $\alpha_n < \omega_1$ such that $A_n \cap (X \times \{d_\alpha\}) = \emptyset$ for each $\alpha > \alpha_n$. Let $\beta = \sup\{\alpha_n : n \in \mathbb{N}\}$. Then $\beta < \omega_1$ and $\left(\bigcup_{n \in \mathbb{N}} A_n\right) \cap (X \times \{d_\alpha\}) = \emptyset$ for each $\alpha > \beta$. Pick $\alpha > \beta$, since $X \times \{d_\alpha\}$ is the only element of $\mathcal{U}_n$ containing the point $\langle a_\alpha, d_\alpha \rangle$ for each $n \in \mathbb{N}$, $\mathrm{St}(\langle a_\alpha, d_\alpha \rangle, \mathcal{U}_n) = X \times \{d_\alpha\}$ for each $n \in \mathbb{N}$. Thus

$$\left(\bigcup_{n \in \mathbb{N}} A_n\right) \cap \mathrm{St}(\langle a_\alpha, d_\alpha \rangle, \mathcal{U}_n) = \emptyset.$$

By the constructions of the topology of $X \times Y$ and the open cover $\mathcal{U}_n$, we have $\overline{\mathrm{St}(\langle a_\alpha, d_\alpha \rangle, \mathcal{U}_n)} = X \times \{d_\alpha\}$. Hence $\overline{\mathrm{St}(\langle a_\alpha, d_\alpha \rangle, \mathcal{U}_n)} \cap A_n = \emptyset$ for all $n \in \mathbb{N}$. □

Remark 2.2. Example 2.7 also shows that Theorem 2.3 fails to be true if "open and closed, finite-to-one" is replaced by "open perfect". The author does not know if there exists a ZFC example showing that the product of a neighborhood star-Menger space X and a compact space Y is not neighborhood star-Menger

The following well-known example shows that the product of two countably compact (and hence neighborhood star-Menger) spaces need not be neighborhood star-Menger. Here we give the proof roughly for the sake of completeness. For a Tychonoff space X, let βX denote the Čech-Stone compactification of X.

Example 2.8. There exist two countably compact (hence neighborhood star-Menger) spaces X and Y such that $X \times Y$ is not neighborhood star-Menger.

Proof. Let D be a discrete space of cardinality $\mathfrak{c}$. We can define $X = \bigcup_{\alpha<\omega_1} E_\alpha$ and $Y = \bigcup_{\alpha<\omega_1} F_\alpha$, where E_α and F_α are the subsets of βD which are defined inductively so as to satisfy the following conditions (1), (2) and (3):

(1) $E_\alpha \cap F_\beta = D$ if $\alpha \neq \beta$;
(2) $|E_\alpha| \leqslant \mathfrak{c}$ and $|F_\beta| \leqslant \mathfrak{c}$;
(3) every infinite subset of E_α (resp., F_α) has an accumulation point in $E_{\alpha+1}$ (resp., $F_{\alpha+1}$).

These sets E_α and F_α are well-defined since every infinite closed set in βD has cardinality $2^{\mathfrak{c}}$ (see [**10**]). Then $X \times Y$ is not neighborhood star-Menger, because the diagonal $\{\langle d, d\rangle : d \in D\}$ is a discrete open and closed subset of $X \times Y$ with cardinality $\mathfrak{c}$ and the open and closed subsets of neighborhood star-Menger spaces are neighborhood star-Menger. □

Acknowledgements. The author would like to thank Prof. Rui Li for his kind help and valuable suggestions. He would also like to thank the referee for reading the paper carefully and making a number of valuable suggestions and comments which led to improvements on several places.

Supported by the National Natural Science Foundation (grant 11771029) of China.

References

1. M. Bonanzinga, M. V. Matveev, *Some covering properties for Ψ-spaces*, Mat. Vesn. **61** (2009), 3–11.
2. M. Bonanzinga, F. Cammaroto, Lj. D. R. Kočinac, M. V. Matveev, *On weaker forms of Menger, Rothberger and Hurewicz properties*, Mat. Vesn. **61** (2009), 13–23.
3. E. K. van Douwen, G. K. Reed, A. W. Roscoe, I. J. Tree, *Star covering properties*, Topology Appl. **39** (1991), 71–103.
4. E. K. van Douwen, *The integers and topology*; in: K. Kunen, J. E. Vaughan (eds.), *Handbook of Set-theoretic Topology*, North-Holland, Amsterdam, 1984, 111–167.
5. R. Engelking, *General Topology*, Revised and completed edition, Heldermann Verlag, Berlin, 1989.
6. W. Just, A. W. Miller, M. Scheepers, P. J. Szeptycki, *Combinatorics of open covers (II)*, Topology Appl. **158** (2011), 1732-1737.
7. Lj. D. R. Kočinac, *Star-Menger and related spaces*, Publ. Math. **55** (1999), 421–431.
8. ———, *Star-Menger and related spaces II*, Filomat **13** (1999), 129–140.
9. M. V. Matveev, *A survey on star-covering properties*, Topology Atlas, preprint **330**, 1998.

10. R. C. Walker, *The Stone–Čech Compactification*, Springer, Berlin, 1974.
11. M. Scheepers, *Combinatorics of open covers I: Ramsey theory*, Topology Appl. **69** (1996), 31–62.

Institute of Mathematics
School of Mathematical Science
Nanjing Normal University
Nanjing
P. R. China
songyankui@njnu.edu.cn

(Received 21 07 2015)
(Revised 20 11 2017)