

## A NEW APPROACH TO SPACE FRACTIONAL DIFFERENTIAL EQUATIONS BASED ON FRACTIONAL ORDER EULER POLYNOMIALS

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**ABSTRACT.** The fractional order Euler polynomials are introduced to obtain the solution of the class of space fractional diffusion equations. This is an innovative method for solving space fractional differential equations among the fractional calculus. These properties are utilized to transform the partial differential equation to algebraic equations with unknown Euler coefficients. The fractional derivatives are described based on the Caputo sense by using Riemann–Liouville fractional integral operator. A new hybrid function approximation based on fractional Euler polynomials and the algebraic polynomial is initiated. The solution obtained by our method coincides with the solution obtained through other methods mentioned in the literature. Finally, several numerical examples are given to illustrate the accuracy and stability of this method.

### 1. Introduction

In recent years, due to the abundant applications in various fields of science and engineering, considerable interest in fractional differential equations (FDEs) have been stimulated. Important phenomenon in finance, electromagnetics, acoustics, viscoelasticity, electrochemistry and material science [4, 18, 19, 30] are well described by differential equations of fractional order. Recently, Magin et al. [17] have reviewed the FDEs in fractional signals and systems with applications to control theory. The edited volume of Machado [16] possessed various applications of fractional calculus like image processing. The importance and necessity of fractional calculus is very much apparent from these applications of interdisciplinary sciences. Recently, to obtain exact and approximate analytic solutions, several mathematical methods inclusive of the Adomian decomposition method [21], variational iteration method [23], homotopy perturbation method [25] and fractional difference method [26] have been developed. A few of these methods makes use of

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transformations to reduce equations into simpler equations or systems of equations and a few others give the solution in the form of a series which converges to an exact solution. The Euler numbers and polynomials (so-named by Scherk in 1825) appear in Euler's famous book, *Institutiones Calculi Differentialis* (1755, pp. 487–491 and p. 522). Euler polynomials appear in many classical results (see [1, Chapter 23]). Let us recall the background on Euler numbers and polynomials. Let  $\mathbb{N} = \{0, 1, 2, \dots\}$  and  $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$ . In general, Euler polynomials are usually defined by means of the generating function

$$\sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} = \frac{2e^{xt}}{e^t + 1}, \quad |t| < \pi.$$

The Euler numbers are also given by the following recursion

$$\sum_{\substack{k=0 \\ 2|n-k}}^n \binom{n}{k} E_k = 0, \quad \text{i.e.,} \quad - \sum_{\substack{k=0 \\ 2|n-k}}^{n-1} \binom{n}{k} E_k = E_n, \quad (n = 1, 2, 3, \dots) \text{ with } E_0 = 1.$$

The Euler numbers are integers and it is well known that,  $E_{2n+1} = 0$  for  $n \geq 0$ . The first few values of the Euler numbers are  $E_0 = 1$ ,  $E_2 = -1$ ,  $E_4 = 5$ ,  $E_6 = -61$ .

In this paper, we consider the space-fractional diffusion equations [3]

$$(1.1) \quad \frac{\partial u(x, t)}{\partial t} = d(x) \frac{\partial^\alpha u(x, t)}{\partial x^\alpha} + p(x, t) \quad 0 < x < 1, \quad 0 \leq t \leq T, \quad 1 < \alpha \leq 2,$$

subject to the initial and boundary conditions:  $u(x, 0) = f(x)$ ,  $0 < x < 1$ ,  $u(0, t) = g_0(t)$ ,  $0 < t \leq T$ ,  $u(1, t) = g_1(t)$ ,  $0 < t \leq T$ . The function  $p(x, t)$  is a source term and note that for  $\alpha = 2$ , (1.1) is the classical diffusion equation.

Many physical problems [5, 10, 12, 20] and finance [11] are successfully modeled by using some partial differential equations of fractional order like one-dimensional time-fractional diffusion-wave equation and space fractional differential equations. Several authors have tried giving numerical solutions to a type of fractional partial differential equations called the space fractional diffusion equations. As an illustration, Khader [14] discretize space fractional diffusion equations to obtain a linear system of ordinary differential equations by using the Chebyshev collocation method and then used the finite difference method for solving the resulting system whereas space fractional diffusion equations were solved by Saadatmandi and Dehghan [28] using tau approach and Sousa [29] used splines approach.

Here we propose a new fractional Euler polynomial method (FEPM) for solving space fractional partial differential equations by using fractional order Euler function. Moreover, the main characteristic of this technique is that it reduces these problems to those of solving the fractional partial differential equation to algebraic equations, thereby simplifying them.

This paper is organized as follows. In Section 2, we present some necessary definitions and mathematics preliminaries of the fractional calculus theory. In Section 3, fractional order Euler functions and its properties are discussed. In Section 4, we demonstrate the accuracy of the proposed scheme by considering some numerical examples. The conclusion is given in Section 5.

**2. Preliminaries and notations**

In this section, we give some basic definitions and properties of fractional calculus theory which are further used in this article [13].

DEFINITION 2.1. A real function  $f(x)$ ,  $x > 0$ , is said to be in the space  $C_\mu$ ,  $\mu \in R$  if there exists a real number  $p > \mu$ , such that  $f(x) = x^p f_1(x)$ , where  $f_1(x) \in C[0, \infty]$ . Clearly  $C_\mu < C_\beta$  if  $\beta < \mu$ .

DEFINITION 2.2. A function  $f(x)$ ,  $x > 0$ , is said to be in the space  $C_\mu^m$ ,  $m \in \mathbb{N} \cup \{0\}$  if  $f^{(m)} \in C_\mu$ .

DEFINITION 2.3. The fractional derivative of  $f(x)$  in the Caputo sense is defined as

$$D_*^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t) dt,$$

for  $m-1 < \alpha \leq m$ ,  $m \in \mathbb{N}$ ,  $x > 0$ ,  $f \in C_{-1}^m$ .

Caputo fractional derivative first computes an ordinary derivative followed by a fractional integral to achieve the desired order of fractional derivative. Some properties of the operator  $D^\alpha$ , which are needed here, are as follows

$$D^\alpha D^\beta f(x) = D^{\alpha+\beta} f(x), \quad D^\alpha C = 0, \quad (C \text{ is a constant})$$

$$D^\alpha x^\beta = \begin{cases} 0, & \text{for } \beta \in \mathbb{N}_0 \text{ and } \beta < [\alpha], \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha} & \text{for } \beta \in \mathbb{N}_0 \text{ and } \beta \geq [\alpha] \text{ or } \beta \notin \mathbb{N} \text{ and } \beta > [\alpha], \end{cases}$$

We use the ceiling function  $[\alpha]$  to denote the smallest integer greater than or equal to  $\alpha$ , and the floor function  $[\alpha]$  to denote the largest integer less than or equal to  $\alpha$ . Also  $\mathbb{N} = \{1, 2, \dots\}$  and  $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ . Similar to the integer-order derivative, the Caputo fractional derivative is a linear operation:

$$D^\alpha \left( \sum_{i=1}^n c_i f_i(t) \right) = \sum_{i=1}^n c_i D^\alpha f_i(t),$$

where  $\{c_i\}_{i=1}^n$  are constants.

LEMMA 2.1. If  $m-1 < \alpha \leq m$ ,  $m \in \mathbb{N}$ , and  $f \in C_\mu^m$ ,  $m \geq -1$ , then

$$D_*^\alpha J^\alpha f(x) = f(x), \quad J^\alpha D_*^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0.$$

To obtain a numerical scheme for the approximation of the Caputo derivative, we can use a representation introduced by Elliott [8];

$$D_*^q f(x) = \frac{1}{\Gamma(-q)} \int_0^x \frac{f(s) - f(0)}{(x-s)^{1+q}} ds, \quad 0 < q < 1,$$

where the integral in the above equation is a Hadamard finite-part integral.

### 3. Fractional order Euler functions

In this section, we discuss the properties of fractional order Euler functions and the function approximation based on Euler polynomials and its properties.

**3.1. Euler polynomials.** The Euler polynomials  $E_n(x)$  [15] satisfy the following equality for a product of two polynomials in the interval  $(0, 1)$  for  $m+n \geq 2$ ,

$$\int_0^1 E_m(x)E_n(x)dx = 2(-1)^{m+1} \frac{m!n!E_{m+n+1}}{(m+n)!m+n+1}.$$

and are given by  $E_i(x) = \sum_{k=0}^i \binom{i}{k} \frac{E_k}{2^k} (x - \frac{1}{2})^{i-k}$ , where  $E_k = -\sum_{m=0}^{n-1} \binom{n}{m} E_m$ ,  $n = 1, 2, \dots$ ,  $E_i(0) = 1$ ,  $E_i(1) = 0$ . A function  $y(x)$ , square integrable in  $(0, 1]$ , may be expressed in terms of Euler polynomials as  $y(x) = \sum_{j=0}^{\infty} c_j E_j(x)$ , where the coefficients  $c_j$  are given by

$$c_j \int_0^1 (E_j(x))^2 dx = \int_0^1 y(x)E_j(x)dx, \quad j = 1, 2, \dots$$

**3.2. Some properties of Euler polynomials.** The well known properties of Euler polynomials are  $E_n(x) + E_n(x+1) = 2x^n$ ,  $E_n(x+y) = \sum_{k=0}^n \binom{n}{k} E_k(x)y^{n-k}$ , and  $m^k \sum_{a=0}^{m-1} (-1)^a E_k(\frac{x+a}{m}) = E_k(x)$ .

**3.3. Fractional order Euler functions.** The series expansion of the efficient method is of the form  $\sum_{i=0}^N c_i x^{i\alpha}$  for solving the fractional differential equations of order  $\alpha$ , such as Adomian's decomposition method (ADM) [9], homotopy perturbation method (HPM) [25] and He's variational iteration method [22].

Recently, Rida and Yousef [27] generated a fractional extension of the classical Legendre polynomials by replacing the integer order derivative in Rodrigues formula by fractional order derivatives. Subsequently, Kazem [13] generated the orthogonal fractional order Legendre functions based on shifted Legendre polynomials to obtain the solution of FDEs more simply and efficiently. In this paper, we generate a fractional extension of Euler polynomials to solve FDEs effectively.

The fractional Euler polynomial  $FE_i^\alpha(x)$  of degree  $i\alpha$  is defined as

$$FE_i^\alpha(x) = \sum_{k=0}^i \binom{i}{k} \frac{E_k}{2^k} (x - \frac{1}{2})^{(i-k)\alpha},$$

where  $FE_0^\alpha(x) = 1$ ,  $FE_1^\alpha(x) = x^\alpha - 1/2$ .

The fractional Euler polynomials satisfy the following integral for a product of two polynomials with respect to the weight function  $w(x) = x^{\alpha-1}$  in the interval  $(0, 1]$  for  $m+n \geq 2$ ,

$$\int_0^1 FE_m^\alpha(x)FE_n^\alpha(x)w(x)dx = 2(-1)^{m+1} \frac{m!n!E_{m+n+1}}{\alpha(m+n)!m+n+1}.$$

For solving FPDE, we define  $u(x, t)$  and  $D_x^\alpha u(x, t)$  over the intervals  $(0, 1]$  as

$$(3.1) \quad u(x, t) = \sum_{j=0}^{\infty} \left( \sum_{i=0}^{\infty} c_{ij} FE_i^\alpha(x) \right) \phi_j(t),$$

where

$$c_{ij} = \frac{\alpha(m+n)!m+n+1(j+1)}{(-1)^{m+1}m!n!E_{m+n+1}} \int_0^1 \int_0^1 FE_i^\alpha(x) w(x) u(x, t) \phi_j(t) dt dx \quad [13].$$

**3.4. Function approximation.** In practice, only the first  $nm$ -terms of (3.1) are considered. Then we have

$$u(x, t) \simeq \sum_{j=0}^{n-1} \left( \sum_{i=0}^{m-1} c_{ij} FE_i^\alpha(x) \right) \phi_j(t) = C^T FE^\alpha(x) \Phi(t),$$

where the Euler polynomials coefficient vector  $C$  and the Euler polynomials vector  $FE^\alpha(x)$  are given by  $C = [c_{00}, c_{10}c_{20}, \dots, c_{m-10}, c_{01}, c_{11}c_{21}, \dots, c_{m-11}, \dots, c_{0n-1}, c_{1n-1}c_{2n-1}, \dots, c_{m-1n-1}]^T$ ,  $FE^\alpha(x) = [FE_0^\alpha(x), FE_1^\alpha(x), \dots, FE_{m-1}^\alpha(x)]^T$ ,  $\Phi(t) = [\phi_0(t), \phi_1(t), \dots, \phi_{n-1}(t)]^T = [1, t, t^2, \dots, t^{n-1}]^T$ .

Also  $FE^\alpha(x) \otimes \Phi(t)$  is  $nm \times 1$  matrix displayed as

$$(3.2) \quad FE^\alpha(x) \otimes \Phi(t) = \begin{bmatrix} \Phi_0(t) FE_0^\alpha(x) \\ \Phi_0(t) FE_1^\alpha(x) \\ \vdots \\ \Phi_{n-1}(t) FE_{m-1}^\alpha(x) \end{bmatrix}$$

Now apply the following property of the Kronecker product  $(A_1 \otimes A_2)(B_1 \otimes B_2) = A_1 B_1 \otimes A_2 B_2$  together with the operational matrix of FEFs, the partial derivatives of  $FE^\alpha(x) \otimes \Phi(t)$  can be obtained as

$$(3.3) \quad \frac{\partial^\alpha}{\partial x^\alpha} FE^\alpha(x) \otimes \Phi(t) \simeq (D_x^\alpha \otimes I_n) FE^\alpha(x) \otimes \Phi(t)$$

$$(3.4) \quad \frac{\partial}{\partial t} FE^\alpha(x) \otimes \Phi(t) \simeq (I_m \otimes D_t) FE^\alpha(x) \otimes \Phi(t)$$

**3.5. Description of the method.** To solve (1.1),  $u(x, t)$  is replaced as given in (3.2) also using (3.3) and (3.4)

$$(3.5) \quad \frac{\partial u}{\partial t} \simeq (I_m \otimes D_t) FE^\alpha(x) \otimes \Phi(t)$$

$$(3.6) \quad d(x) \frac{\partial^\alpha u}{\partial x^\alpha} \simeq C^T K_1 (D_x^\alpha \otimes I_n) FE^\alpha(x) \otimes \Phi(t)$$

$$(3.7) \quad p(x, t) \simeq \sum_{j=0}^{n-1} \left( \sum_{i=0}^{m-1} g_{ij} FE_i^\alpha(x) \right) \phi_j(t) = G^T FE^\alpha(x) \otimes \Phi(t),$$

Substituting (3.5), (3.6) and (3.7) in (1.1), one gets

$$(3.8) \quad [C^T (I_m \otimes D_t) + C^T K (D_x^\alpha \otimes I_n) - G^T] FE^\alpha(x) \otimes \Phi(t) = 0$$

Now to satisfy the initial and boundary conditions of (1.1), we can write

$$(3.9) \quad u(0, t) = g_0(t) = \sum_{j=0}^{n-1} \sum_{i=0}^{m-1} c_{ij}(FE_i^\alpha(0))\Phi_j(t)$$

$$(3.10) \quad u(1, t) = g_1(t) = \sum_{j=0}^{n-1} \sum_{i=0}^{m-1} c_{ij}(FE_i^\alpha(1))\Phi_j(t)$$

$$(3.11) \quad u(x, 0) = f(x) = \sum_{j=0}^{n-1} \sum_{i=0}^{m-1} c_{ij}(FE_i^\alpha(x))\Phi_j(0)$$

Above (3.9), (3.10) and (3.11) generate  $2n + m - 2$  set of equations. Now we should generate  $(m - 1)(n - 2)$  equations from (3.8) as follows

$$E^T \int_0^1 \int_0^1 (FE^\alpha(x) \otimes \Phi(t))FE_i^\alpha(x)\Phi_j(t)w_\alpha(x)w(t)dt dx = 0,$$

where

$$E^T = [C^T(I_m \otimes D_t) - C^T K(D_x^\alpha \otimes I_n) - G^T], \quad i = 0, 1, \dots, m-2, \quad j = 0, 1, \dots, n-3.$$

In case the exact solution to a problem is known, the accuracy and efficiency of the proposed method based on maximum absolute error  $e_{m,n}$  defined as  $e_{m,n} = \max\{|u_{\text{exact}}(x, t) - u_{m,n}(x, t)|\}$ ,  $a \leq x \leq b$ ,  $0 < t < \tau$ .

#### 4. Numerical results and discussion

In order to check the efficiency and reliability of the proposed method, we present some numerical examples. In the example, we compute the space fractional diffusion equation with variable coefficients to check the accuracy.

EXAMPLE 4.1. Consider the following space fractional differential equation [3]

$$\frac{\partial u(x, t)}{\partial t} = d(x) \frac{\partial^\alpha u(x, t)}{\partial x^\alpha} + p(x, t)$$

on a finite domain  $0 < x < 1$ ,  $0 \leq t \leq 1$ , with the diffusion coefficient  $d(x) = \frac{1}{24}\Gamma(5 - \alpha)x^\alpha$ , the source function  $p(x, t) = -2e^{-t}x^4$ , the initial condition  $u(x, 0) = x^4$ ,  $0 < x < 1$ , and the boundary conditions  $u(0, t) = 0$ ,  $u(1, t) = e^{-t}$ . By the fractional polynomial method when  $m = 9$ ,  $n = 3$  and  $\alpha = 1.5$  with initial and boundary conditions, we get approximate solution of  $u(x, t)$  is  $x^4(1 - t + \frac{t^2}{2})$ . The approximate solution and absolute error values are shown in Figures 1(a) and 1(b) respectively with  $m = 9$ ,  $n = 10$  for Example 4.1.

EXAMPLE 4.2. Consider the following space fractional differential equation [7]

$$\frac{\partial u(x, t)}{\partial t} = d(x) \frac{\partial^\alpha u(x, t)}{\partial x^\alpha} + p(x, t)$$

on a finite domain  $0 < x < 1$ ,  $1 < \alpha \leq 2$ , with the diffusion coefficient  $d(x) = \frac{1}{2}\Gamma(3 - \alpha)x^\alpha$ , the source function  $p(x, t) = \sin(-t)x^2 - \cos(-t)x^2$ , with the initial condition  $u(x, 0) = x^2$ ,  $0 < x < 1$ , and the boundary conditions  $u(0, t) = 0$ ,

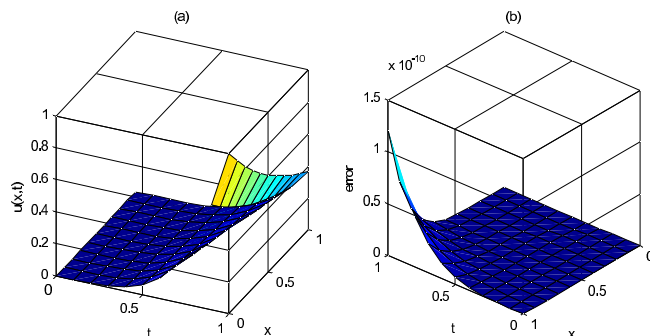


FIGURE 1. (a) Solution  $u(x, t)$  with  $m = 9, n = 10$ , for Example 4.1(b) Plot of error function  $|u_{\text{exact}}(x, t) - u(x, t)|$ , for Example 4.1.

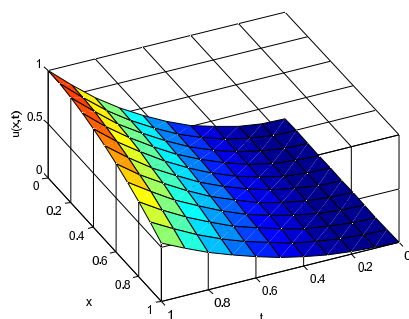


FIGURE 2. Solution  $u(x, t)$  with  $m = 12, n = 10$  for Example 4.2.

$u(1, t) = \cos(-t)$  for  $t > 0$ . The proposed method solution with  $m = 12, n = 5$  and  $\alpha = 1.5$  with initial and boundary condition is  $u(x, t) = x^2(1 + \frac{t^2}{2} + \frac{t^4}{24})$  for larger values for  $m$  and  $n$ , the exact solution is  $u(x, t) = x^2 \cos(-t)$  [7]. The solution  $u(x, t)$  obtained for example 4.2 by our method is depicted in Fig. 2. Note that from Fig. 2 it can be seen that our method achieves a good approximation for the above equation.

EXAMPLE 4.3. Consider the following space fractional differential equation [2]

$$\frac{\partial u(x, t)}{\partial t} = d(x) \frac{\partial^{1.8} u(x, t)}{\partial x^{1.8}} + p(x, t)$$

on a finite domain  $0 < x < 1$ , with the diffusion coefficient  $d(x) = \Gamma(1.2)x^{1.8}$ , the source function

$$p(x, t) = e^{-t} \left\{ x^2(2 - x)^2 + 8(x^2 - \frac{5}{2}x^3 + \frac{25}{22}x^4) \right\},$$

with the initial condition  $u(x, 0) = (2 - x)^2 x^2, 0 < x < 1$ , and the boundary conditions  $u(0, t) = u(1, t) = 0$ .

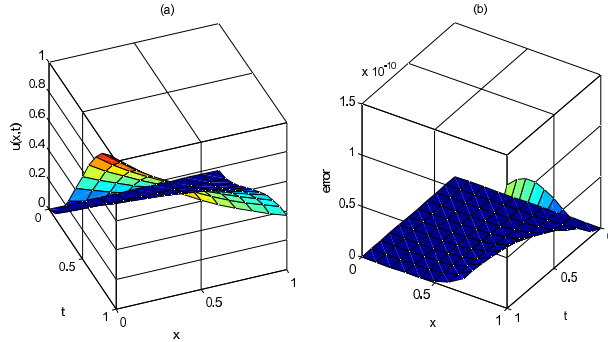


FIGURE 3. (a) Solution  $u(x, t)$  with  $m = 25, n = 10$ , for example 4.3(b) Plot of error function  $|u_{\text{exact}}(x, t) - u(x, t)|$ , for Example 4.3.

We apply the presented technique with initial and boundary conditions, the approximate solution of  $u(x, t)$  is  $(4x^2 + x^4 - 4x^3)(1 - t + \frac{t^2}{2})$  when  $m = 25, n = 3$  and  $\alpha = 0.2$ . The closed form of the above solution is  $u(x, t) = (4x^2 + x^4 - 4x^3)e^{-t}$  which is the exact solution [2]. The solution  $u(x, t)$  and the error obtained for example 4.3 by our method are depicted in Figures 3a and 3b. Note that from figures 3a and 3b, one can see that our method achieves a good approximation for the above equation.

EXAMPLE 4.4. Consider the following space fractional differential equation [6]

$$\frac{\partial u(x, t)}{\partial t} = d(x) \frac{\partial^{1.8} u(x, t)}{\partial x^{1.8}} + p(x, t)$$

on a finite domain  $0 < x < 1$ , with the diffusion coefficient  $d(x) = \frac{\Gamma(2.2)}{6} x^{2.8}$ , the source function  $p(x, t) = -x^3(x + 1)e^{-t}$ , with the initial condition  $u(x, 0) = x^3, 0 < x < 1$  and the boundary conditions  $u(0, t) = 0, u(1, t) = e^{-t}$ . According to the presented method with  $m = 16, n = 3, \alpha = 0.2$  and applying initial, boundary conditions we get that  $u(x, t) = x^3(1 - t + \frac{t^2}{2})$  is the approximate solution and  $u(x, t) = x^3e^{-t}$  for larger values for  $m$  and  $n$ , is the exact solution [6]. The solution  $u(x, t)$  and the error obtained for example 4.4 by our method are depicted in figures 4a and 4b. Note that from figures 4a and 4b can be seen our method achieve a good approximation for the above equation.

EXAMPLE 4.5. Consider the following space fractional differential equation [2]

$$\frac{\partial u(x, t)}{\partial t} = d(x) \frac{\partial^{1.8} u(x, t)}{\partial x^{1.8}} + p(x, t)$$

on a finite domain  $0 < x < 1, 0 \leq t \leq 1$ , with the diffusion coefficient  $d(x) = \Gamma(0.2)x^{1.8}$ , the source function  $p(x, t) = -(2x - 11x^2)e^{-t}$ , with the initial condition  $u(x, 0) = x(1 - x), 0 < x < 1$ , and the boundary conditions  $u(0, t) = 0, u(1, t) = 0$ .

By applying the fractional polynomial technique with initial and boundary conditions with  $m = 12, n = 3$ , and  $\alpha = 0.2$ , we get that  $u(x, t) = (x - x^2)(1 - t + \frac{t^2}{2})$



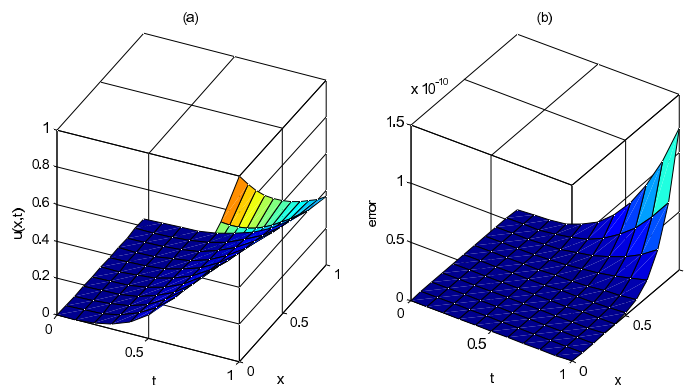


FIGURE 4. (a) Solution  $u(x, t)$  with  $m = 16, n = 10$ , for example 4.4(b) Plot of error function  $|u_{\text{exact}}(x, t) - u(x, t)|$ , for Example 4.4.

is the approximate solution. Thus  $u(x, t) = (x - x^2)e^{-t}$  is the exact solution, which is in full agreement with [2]. The solution  $u(x, t)$  and the error obtained for example 4.5 by our method are depicted in figures 5a and 5b. Note that from figures 5a and 5b it can be seen that our method achieves a good approximation for the above equation.

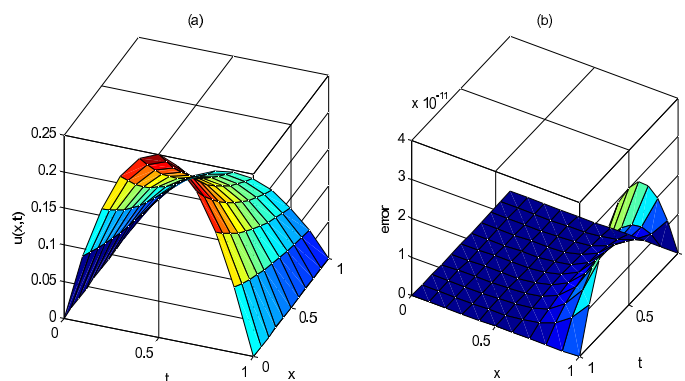


FIGURE 5. (a) Solution  $u(x, t)$  with  $m = 12, n = 10$ , for Example 4.5(b) Plot of error function  $|u_{\text{exact}}(x, t) - u(x, t)|$ , for Example 4.5.

EXAMPLE 4.6. Consider the following Fisher’s nonlinear space-fractional equation [24]

$$\frac{\partial u}{\partial t} - \frac{\partial^{1.5} u}{\partial x^{1.5}} - u(x, t)(1 - u(x, t)) = x^2, \quad x > 0$$

subject to the initial condition  $u(x, 0) = x$ . Now apply the fractional Euler polynomial technique for  $m = 7, n = 5$  and  $\alpha = 0.5$ , then we have the series form of

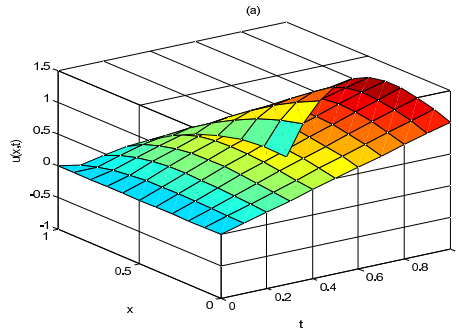


FIGURE 6. Solution  $u(x, t)$  with  $m = 15, n = 10$ , for Example 4.6.

Euler polynomials by substituting initial condition

$$u(x, t) = x + xt + (x - 2x^2)\frac{t^2}{2} + \left(\frac{1}{2}x - 3x^2 + 2x^3 - \frac{2}{\Gamma(3/2)}x^{1/2}\right)\frac{t^3}{3} - \left(\frac{4}{3}x^2 - \frac{8}{3}x^3 + \frac{2}{3\Gamma(3/2)}x^{1/2}\right)\frac{t^4}{4}$$

is the approximate solution [24]. The solution  $u(x, t)$  obtained for example 4.6 by our method is depicted in fig. 6. Note that from fig. 6 it can be seen that our method achieves a good approximation for the above equation. The numerical solutions for different values of  $x$  and  $t$  of Example 4.6 are presented in Table 1 and 2

TABLE 1. Numerical values for different values of  $x$  and  $t = 0.1, 0.2$  of Example 4.6.

$x$	$t = 0.1$			$t = 0.2$		
	GDTM	VIM	FEPM	GDTM	VIM	FEPM
0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.1	0.110233	0.110401	0.110401	0.120145	0.121559	0.121559
0.2	0.220348	0.220589	0.220589	0.240212	0.242240	0.242240
0.3	0.330259	0.330565	0.330565	0.359453	0.362026	0.362026
0.4	0.439957	0.440329	0.440329	0.477796	0.480928	0.480928
0.5	0.549441	0.549880	0.549880	0.595214	0.598938	0.598938
0.6	0.658707	0.659214	0.659214	0.711692	0.716018	0.716018
0.7	0.767755	0.768320	0.768320	0.827221	0.832097	0.832097
0.8	0.876585	0.877185	0.877185	0.941796	0.947058	0.947058
0.9	0.985196	0.985786	0.985786	1.055412	1.060735	1.060735
1.0	1.093587	1.094096	1.094096	1.168067	1.172904	1.172904

TABLE 2. Numerical values for different values of  $x$  and  $t = 0.3, 0.4$  of Example 4.6.

$x$	$t = 0.3$			$t = 0.4$		
	GDTM	VIM	FEPM	GDTM	VIM	FEPM
0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.1	0.128281	0.133294	0.133294	0.132819	0.145276	0.145276
0.2	0.257431	0.264612	0.264612	0.269327	0.287146	0.287146
0.3	0.384716	0.393833	0.393833	0.402530	0.425163	0.425163
0.4	0.509872	0.520995	0.520995	0.531749	0.559433	0.559433
0.5	0.632794	0.646078	0.646078	0.656718	0.689927	0.689927
0.6	0.753428	0.768969	0.768969	0.777297	0.816392	0.816392
0.7	0.871744	0.889430	0.889430	0.893402	0.938285	0.938285
0.8	0.987718	1.007079	1.007079	1.004978	1.054714	1.054714
0.9	1.101336	1.121360	1.121360	1.111987	1.164378	1.164378
1.0	1.212587	1.231524	1.231524	1.214400	1.265508	1.265508

## 5. Conclusion

In this article, fractional order Euler polynomials techniques are used to reduce the space fractional diffusion equation with variable coefficients to the solution of algebraic equations. The fractional derivatives are described in the Caputo sense. The solution quality and accuracy are verified through tables and figures. The solution obtained by our method coincides with the solution obtained through other methods mentioned in the literature. Also a numerical solution was obtained on the basis of the fractional Euler polynomials for fractional order partial differential equations.

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