PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 104(118) (2018), 139–148 DOI: https://doi.org/10.2298/PIM1818139K

GENERALIZATIONS OF RAD-SUPPLEMENTED MODULES

Engin Kaynar, Ergül Türkmen, and Yıldız Aydın

Abstract. Let R be an associative ring with identity. We introduce the notion of semi-τ-supplemented modules, which is adapted from srs-modules, for a preradical τ on R-Mod. We provide basic properties of these modules. In particular, we study the objects of R-Mod for $\tau = \mathrm{Rad}$. We show that the class of semi-τ-supplemented modules is closed under finite sums and factor modules. We prove that, for an idempotent preradical τ on R-Mod, a module M is semi-τ-supplemented if and only if it is τ-supplemented. For $\tau = \mathrm{Rad}$, over a local ring every left module is semi-Rad-supplemented. We also prove that a commutative semilocal ring whose semi-Rad-supplemented modules are a direct sum of w-local left modules is an artinian principal ideal ring.

1. Introduction

Throughout this study, R will be an associative ring with identity and all modules are unitary left R-modules, unless otherwise specified. Let M be such a module over the ring R. By R-Mod we denote the category of left R-modules. The notation $N \subseteq M$ means that N is a submodule of M. A functor $\tau\colon R\text{-Mod} \to R\text{-Mod}$ is said to be a *preradical* if $\tau(M) \subseteq M$ for every $M \in R\text{-Mod}$ and for every homomorphism $f\colon M \to N$ in R-Mod, we have $f(\tau(M)) \subseteq \tau(N)$. A preradical τ is called *radical* if $\tau(M/\tau(M)) = 0$ for every left R-module M. A module M is called *τ-torsion* (respectively, *τ-torsion free*) if $\tau(M) = M$ (respectively, $\tau(M) = 0$).

A nonzero submodule N of a module M is called *essential*, written by $N \trianglelefteq M$, if $N \cap K \neq 0$ for every nonzero submodule K of M. Dually, a proper submodule S of M is called *small*, denoted by $S \ll M$, if $S + K = M$ implies that $K = M$, where K is a submodule of M. By $\mathrm{Rad}(M)$ we will denote the Jacobson radical for a module M. If $M = \mathrm{Rad}(M)$, then it is called *radical*. A nonzero module M is said to be *hollow* if every proper submodule is small in M, and it is said to be *local* if it is hollow and finitely generated. M is local if and only if it is finitely generated and $\mathrm{Rad}(M)$ is maximal (see [**6**, 2.12 §2.15]).

2010 *Mathematics Subject Classification:* Primary 16D10; Secondary 16N80.
Key words and phrases: preradical, Jacobson radical, Rad-supplement, τ-supplement.
Communicated by Zoran Petrović.

For two submodules N and K of a module M, K is said to be *supplement* of N in M (or N is said to *have a supplement* K) if $M = N + K$ and $N \cap K \ll K$. M is called *supplemented* if every submodule of M has a supplement in M. Since every direct summand of a module has a supplement, supplemented modules are a proper generalization of semisimple modules. Hollow modules are supplemented.

Al-Takhman, Lomp and Wisbauer [**1**] generalize supplemented modules to τ-supplemented modules for a preradical τ for R-Mod. A module M is called *τ-supplemented* if every submodule N of M has a τ-supplement K in M, that is, $M = N + K$ and $N \cap K \subseteq \tau(K)$ where τ is a preradical for R-Mod. Instead of a preradical τ for R-Mod, we can use the radical Rad on R-Mod. A module M is called Rad-*supplemented* if every submodule N of M has a Rad-supplement K in M. Since the Jacobson radical of any module is the sum of all small submodules, every supplement submodule is Rad-supplement, and so supplemented modules are Rad-supplemented. Also, a noetherian Rad-supplemented module is supplemented.

For the properties and characterizations of (Rad-) supplemented modules and in general τ-supplemented modules we refer to [**1, 4, 10**].

In [**11**], Zöschinger studied on modules whose Jacobson radical have a supplement and termed these modules *radical supplemented*. He determined the structure of these modules over local Dedekind domains. Büyükaşık and Türkmen called a module M *strongly radical supplemented* (for shortly srs) if every submodule N of M with $\mathrm{Rad}(M) \subseteq N$ have a supplement K in M (see [**5**]). They gave the various properties of srs-modules in the same paper. In particular, it was shown in [**5**, Proposition 2.3] that every finite sum of srs-modules is srs. By [**5**, Proposition 3.3], over a local Dedekind domain a module is radical supplemented if and only if it is srs.

In this paper, we introduce the notion of semi-τ-supplemented modules, which is adapted from srs-modules, for a preradical τ on R-Mod. We provide basic properties of these modules. In particular, we study on the objects of R-Mod for $\tau = \mathrm{Rad}$. We show that the class of semi-τ-supplemented modules is closed under finite sums and factor modules. We prove that, for an idempotent preradical τ on R-Mod, a module M is semi-τ-supplemented if and only if it is τ-supplemented. Let $\tau = \mathrm{Rad}$. Any direct sum of w-local modules is semi-Rad-supplemented. It follows that over a local ring every left module is semi-Rad-supplemented. We give some counterexamples to separate classes of semi-Rad-supplemented modules, Rad-supplemented modules and srs-modules (see Example 3.3). We have that the following proper implications on modules hold:

$$\begin{array}{ccccc} & & \text{supplemented} & & \\ & \swarrow & & \searrow & \\ \text{srs} & & & & \text{Rad-supplemented} \\ & \searrow & & \swarrow & \\ & & \text{semi-Rad-supplemented} & & \end{array}$$

We also prove that a commutative semilocal ring whose semi-Rad-supplemented modules are a direct sum of w-local left modules is an artinian principal ideal ring.

2. Semi-τ-Supplemented Modules

Let τ be a preradical on R-Mod. We call a module M *semi-τ-supplemented* if every submodule N of M with $\tau(M) \subseteq N$ has a τ-supplement in M. By definitions, every τ-supplemented module is semi-τ-supplemented. In this section, we obtain the various properties of semi-τ-supplemented modules. We prove that, for an idempotent preradical τ on R-Mod, a module M is semi-τ-supplemented if and only if it is τ-supplemented.

Recall from [**2**] that a module M is *τ-local* if it is τ-torsion or $\tau(M)$ is maximal.

LEMMA 2.1. *Every τ-local module is semi-τ-supplemented.*

PROOF. Let M be a τ-local module. If M is τ-torsion, it is clear. Suppose that $\tau(M)$ is the maximal submodule of M. Then $M = \tau(M) + M$ and $\tau(M) \cap M \subseteq \tau(M)$. Thus it is semi-$\tau$-supplemented. □

COROLLARY 2.1. *Let M be a module and N be a maximal submodule of M. Then every τ-supplement of N is semi-τ-supplemented.*

PROOF. It follows from [**2**, Lemma 2.2] and Lemma 2.1. □

Now we show that the finite sum of semi-τ-supplemented modules is semi-τ-supplemented. For this fact, we use the standard lemma (see, [**1**, 2.3(1)]).

LEMMA 2.2. *Let M be an R-module and M_1, U be submodules of M such that M_1 is semi-τ-supplemented, $\tau(M) \subseteq U$ and $M_1 + U$ has a τ-supplement V in M. Then, $M_1 \cap (U + V)$ has a τ-supplement L in M_1 and $V + L$ is a τ-supplement of U in M.*

THEOREM 2.1. *Let M_1 and M_2 be semi-τ-supplemented modules. If $M = M_1 + M_2$, then M is semi-τ-supplemented.*

PROOF. Let $\tau(M) \subseteq U \subseteq M$. Since $M = M_1 + M_2$, $M_1 + (M_2 + U)$ has the trivial τ-supplement 0 in M. So by Lemma 2.2, $M_2 + U$ has a τ-supplement in M. Again applying Lemma 2.2, we obtain a τ-supplement for U in M. Hence M is semi-τ-supplemented. □

COROLLARY 2.2. *A finite direct sum of semi-τ-supplemented modules is semi-τ-supplemented.*

A module M is said to be a *duo module* if every submodule N of M is fully invariant [**8**]. Now we prove that arbitrary direct sums of semi-τ-supplemented modules are semi-τ-supplemented, under a certain condition: namely, when M is a duo module. The proof of the next result is the same as [**7**, Theorem 1].

THEOREM 2.2. *Let M_i $(i \in I)$ be any collection of semi-τ-supplemented modules in R-Mod and $M = \bigoplus_{i\in I} M_i$. If M is a duo module, then it is a semi-τ-supplemented module.*

PROPOSITION 2.1. *If M is a semi-τ-supplemented module, then every factor module of M is semi-τ-supplemented.*

Proof. For any submodule N of M, let $U/N \subseteq M/N$ with $\tau(M/N) \subseteq U/N$. Since $(\tau(M) + N)/N \subseteq \tau(M/N)$, we can write $\tau(M) \subseteq U$. By the hypothesis, U has a τ-supplement V in M, that is, $M = U + V$ and $U \cap V \subseteq \tau(V)$ for some submodule V of M. So $M/N = U/N + (V + N)/N$. Therefore,

$$\begin{aligned} U/N \cap (V+N)/N &= [U \cap (V+N)]/N = (U \cap V + N)/N \\ &\subseteq (\tau(V) + N)/N \subseteq \tau(V+N)/N \end{aligned}$$

and so $(V + N)/N$ is a τ-supplement of U/N in M/N. Hence M/N is semi-τ-supplemented. □

Recall that a module M is *weakly supplemented* if every submodule N of M has a weak supplement K in M, that is, $M = N + K$ and $N \cap K \ll M$ [**6**, 17.8].

Lemma 2.3. *Let M be a semi-τ-supplemented module. Suppose that $\tau(M)$ is a small submodule of M. Then M is τ-supplemented. In particular, M is weakly supplemented.*

Proof. Let U be any submodule of M. Then, $\tau(M) \subseteq \tau(M) + U$. It follows from the hypothesis that $\tau(M)+U$ has a τ-supplement V in M. So $M = (\tau(M)+U) + V$ and $(\tau(M) + U) \cap V \subseteq \tau(V)$. Since $\tau(M) \ll M$, we get $M = U + V$. Therefore $U \cap V \subseteq (\tau(M) + U) \cap V \subseteq \tau(V)$, we obtain that $U \cap V \subseteq \tau(V)$. Hence V is a τ-supplement of U in M.

Since $U \cap V \subseteq \tau(V) \subseteq \tau(M)$, it follows from [**10**, 19.3(4)] that $U \cap V$ is small in M. Hence V is a weak supplement of U in M. This means that M is a weakly supplemented module. □

Corollary 2.3. *Let M be a τ-torsion free module. Then the following statements are equivalent:*

(1) *M is (semi) τ-supplemented.*
(2) *M is semisimple.*

Proof. Clearly, we have the implications (2) $\Rightarrow$ (1); since M is τ-torsion free, $\tau(M) = 0$. It follows from Lemma 2.3 that M is (semi-)τ-supplemented.

(1) $\Rightarrow$ (2) is obvious. □

Corollary 2.4. *Let M be a semi-τ-supplemented module. Suppose that τ is radical. Then $M/\tau(M)$ is semisimple and* $\mathrm{Rad}(M) \subseteq \tau(M)$.

Proof. By Proposition 2.1, we get that $M/\tau(M)$ is semi-τ-supplemented. Since τ is radical, $M/\tau(M)$ is a τ-torsion free module. Hence $M/\tau(M)$ is semisimple by Corollary 2.3. It follows that $\mathrm{Rad}(M/\tau(M)) = 0$. Thus $\mathrm{Rad}(M) \subseteq \tau(M)$. □

Proposition 2.2. *Let M be a semi-τ-supplemented module. If $\tau(M)$ is τ-supplemented, then M is τ-supplemented.*

Proof. Let $U \subseteq M$. By the hypothesis, $\tau(M) + U$ has a τ-supplement in M. Since $\tau(M)$ is τ-supplemented, by Lemma 2.2, U has a τ-supplement in M. □

A preradical τ is said to be *idempotent* if $\tau(\tau(M)) = \tau(M)$ for every left R-module M. For an example of an idempotent preradical on R-Mod, we consider an idempotent ideal I of a ring R and put $\tau^I(M) = IM$ each $M \in R$-Mod. Then, τ^I is an idempotent preradical for R-Mod.

Corollary 2.5. *Let τ be an idempotent preradical on R-*Mod*. Then an R-module M is semi-τ-supplemented if and only if it is τ-supplemented.*

Proof. We only need to show that M is semi-τ-supplemented, then it is τ-supplemented.

Let $N \subseteq M$. Let us look at the submodule $N + \tau(M)$. By the assumption, there exists a submodule K such that

$$(N + \tau(M)) + K = M \quad \text{and} \quad (N + \tau(M)) \cap K \subseteq \tau(K).$$

Let $K_1 = \tau(M) + K$. We have $N + K_1 = M$ and we only need to prove that $N \cap K_1 \subseteq \tau(K_1)$, or, more explicitly, that $N \cap (\tau(M) + K) \subseteq \tau(\tau(M) + K)$. So, let $x \in N \cap (\tau(M) + K)$. This means that $x \in N$ and there exist elements $m' \in \tau(M)$ and $k \in K$ such that $x = m' + k$. From this, we get that $k = x - m'$. Since $x \in N$ and $m' \in \tau(M)$, we get $k \in (N + \tau(M)) \cap K$. Since $(N + \tau(M)) \cap K \subseteq \tau(K)$, it follows that $k \in \tau(K)$. So, $x \in \tau(M) + \tau(K) = \tau(\tau(M)) + \tau(K) \subseteq \tau(\tau(M) + K)$, which concludes our proof. The last inclusion follows from the fact that $\tau(A) + \tau(B) \subseteq \tau(A + B)$, since τ is a preradical. □

By $P_\tau(M)$ we denote the sum of all τ-torsion submodules of an R-module M. It is clear that $P_\tau(M)$ is the largest τ-torsion submodule of M. Note that $P_\tau(M) \subseteq \tau(M)$ and P_τ is an idempotent preradical for R-Mod, whenever τ is a radical on R-Mod.

Theorem 2.3. *Let M be a module. Suppose that τ is a radical on R-*Mod*. Then it is semi-τ-supplemented if and only if $M/P_\tau(M)$ is semi-τ-supplemented.*

Proof. Let M be a semi-τ-supplemented module. It follows from Proposition 2.1 that $M/P_\tau(M)$ is semi-τ-supplemented as a factor module of M. Conversely, suppose that U is any submodule of M with $\tau(M) \subseteq U$. Then $P_\tau(M) \subseteq U$. By properties of a radical, we have $\tau(M/P_\tau(M)) = \tau(M)/P_\tau(M) \subseteq U/P_\tau(M)$. Since $M/P_\tau(M)$ is a semi-τ-supplemented module, $U/P_\tau(M)$ has a τ-supplement, say $V/P_\tau(M)$, in $M/P_\tau(M)$. So

$$M/P_\tau(M) = U/P_\tau(M) + V/P_\tau(M),$$
$$U/P_\tau(M) \cap V/P_\tau(M) \subseteq \tau(V/P_\tau(M)).$$

Therefore, $M = U + V$. Note that

$$(U \cap V)/P_\tau(M) = (U/P_\tau(M)) \cap (V/P_\tau(M)) \subseteq \tau(V/P_\tau(M)) = \tau(V)/P_\tau(M)$$

and this implies $U \cap V \subseteq \tau(V)$. Consequently, M is semi-τ-supplemented. □

Let M be an R-module. M is said to be *τ-reduced* if $P_\tau(M) = 0$. If τ is radical, by [**4**, Theorem 3.1 (vii)], then $P_\tau(M/P_\tau(M)) = 0$ and so $M/P_\tau(M)$ is τ-reduced. Using Theorem 2.3, we obtain the following fact.

COROLLARY 2.6. *Let R be a ring and τ be a radical on R-Mod. The following statements are equivalent:*

(1) *Every left R-module is semi-τ-supplemented.*
(2) *Every left τ-reduced R-module is semi-τ-supplemented.*

3. Semi-Rad-Supplemented Modules

In this section, we shall consider $\tau = \mathrm{Rad}$. Recall that a module M is *semi-*Rad*-supplemented* if every submodule N of M with $\mathrm{Rad}(M) \subseteq N$ has a Rad-supplement K in M, that is, $M = N + K$ and $N \cap K \subseteq \mathrm{Rad}(K)$. It is clear that Rad-supplemented modules and srs-modules are semi-Rad-supplemented. For modules with zero Jacobson radical the notions of semi-Rad-supplemented, Rad-supplemented and being srs-module coincide by Corollary 2.3. In general, semi-Rad-supplemented modules need not be Rad-supplemented and srs. Later we shall give an example of such modules (see Example 3.3).

Recall that a module M is *semilocal* if $M/\mathrm{Rad}(M)$ is semisimple. A ring R is called *semilocal* if $_RR$ (or R_R) is a semilocal module. It is known that a commutative ring R is semilocal if R has only finitely many maximal ideals. Since the preradical Rad is a radical on R-Mod, we obtain the following fact by Corollary 2.4.

COROLLARY 3.1. *Semi-*Rad*-supplemented modules are semilocal.*

PROOF. It follows from Corollary 2.4. □

In the following example, we show that semilocal modules need not be semi-Rad-supplemented, in general. Firstly, we need this simple lemma.

LEMMA 3.1. *Finitely generated semi-*Rad*-supplemented modules are* Rad*-supplemented.*

PROOF. Let M be a finitely generated module. Then $\mathrm{Rad}(M)$ is a small submodule of M. If M is semi-Rad-supplemented, then it is Rad-supplemented by Lemma 2.3. □

EXAMPLE 3.1. Consider the localization ring $\mathbb{Z}_{(2,3)}$ containing all rational numbers of the form $\frac{a}{b}$ with $2 \nmid b$ and $3 \nmid b$ for prime integers 2, 3 in $\mathbb{Z}$. Let M be the left $\mathbb{Z}_{(2,3)}$-module $\mathbb{Z}_{(2,3)}$. Then M is a semilocal noetherian module, but not Rad-supplemented. By Lemma 3.1, it is not semi-Rad-supplemented.

As a proper generalization of local modules, one calls a module M *w-local* if $\mathrm{Rad}(M)$ is a maximal submodule of M as in [**3**].

PROPOSITION 3.1. *Every w-local module is semi-*Rad*-supplemented.*

PROOF. Let M be any w-local module and $\mathrm{Rad}(M) \subseteq U \subsetneq M$. Since M is w-local, we have $U = \mathrm{Rad}(M)$. Then, $M = U + M$ and $U \cap M \subseteq \mathrm{Rad}(M)$ and so M is a Rad-supplement of U in M. Hence M is semi-Rad-supplemented. □

THEOREM 3.1. *Let M be any direct sum of w-local modules. Then it is semi-*Rad*-supplemented.*

PROOF. Let $M = \bigoplus_{i\in I} M_i$ and each M_i be w-local. Let $\mathrm{Rad}(M) \subseteq U \subseteq M$. For $i \in I$, we have that $(M_i + \mathrm{Rad}(M))/\mathrm{Rad}(M) \cong M_i/\mathrm{Rad}(M_i)$ is simple because M_i is w-local. Note that $M/\mathrm{Rad}(M) = \bigoplus_{i\in I}(M_i + \mathrm{Rad}(M))/\mathrm{Rad}(M)$. So $M/\mathrm{Rad}(M)$ is semisimple by [**6**, 2.8(5)]. It follows that

$$M/\mathrm{Rad}(M) = U/\mathrm{Rad}(M) \oplus \left(\bigoplus_{i\in J}(M_i + \mathrm{Rad}(M))/\mathrm{Rad}(M)\right)$$

for some $J \subseteq I$ by [**10**, 20.1]. Let $V = \bigoplus_{i\in J} M_i$. Therefore, $M = U + V$ and $U \cap V \subseteq \mathrm{Rad}(M)$. Since V is a direct summand of M, $\mathrm{Rad}(V) = V \cap \mathrm{Rad}(M)$ and so $U \cap V \subseteq \mathrm{Rad}(V)$. Hence U has a Rad-supplement in M as required. □

In the next Theorem, we characterize commutative semilocal rings in terms of semi-Rad-supplemented modules. A ring R is called *a left max ring* if every nonzero left R-module has a maximal submodule, and it is called *left perfect* if R is semilocal and a left max ring. Note that over a left max ring every nonzero left module has a small Jacobson radical. Left V-rings (i.e., every left simple module is injective) are left max rings.

LEMMA 3.2. *Let R be a left max ring and M be a module over this ring. Then the following statements are equivalent.*

(1) *M is semi-*Rad*-supplemented.*
(2) *M is* Rad*-supplemented.*
(3) *M is supplemented.*
(4) *M is an srs-module.*

PROOF. (1) ⇒ (2). By Lemma 2.3.
(2) ⇒ (3). It follows from [**6**, 20.7(3)].
(3) ⇒ (4) and (4) ⇒ (1) are clear. □

THEOREM 3.2. *Let R be a ring whose semi-*Rad*-supplemented modules are the direct sum of w-local R-modules. Then R is a left max ring and every semi-*Rad*-supplemented R-module is supplemented. If R is a commutative semilocal ring, then R is an artinian principal ideal ring.*

PROOF. Since radical modules are semi-Rad-supplemented, it is enough to prove that R has no radical modules. Let $N = \mathrm{Rad}(N)$ be an R-module. By the assumption, we can write $N = \bigoplus_{i\in I} N_i$, where each N_i is a w-local R-module. It follows that $N_i = \mathrm{Rad}(N_i)$. Therefore, $N_i = 0$ for every $i \in I$. Thus $N = 0$. This means that R is a left max ring. Applying Lemma 3.2, every semi-Rad-supplemented R-module is supplemented.

Let R be a commutative semilocal ring. Then, R is perfect. Let M be any R-module. By [**10**, 43.9], M is supplemented. Since w-local modules over left max rings are local, M is the direct sum of cyclic submodules. Hence R is an artinian principal ideal ring by [**9**, Theorem 6.7]. □

Let $n > 1$ be a positive integer. Then the ring $\mathbb{Z}_n$ is a commutative semilocal ring which satisfies the above theorem.

A ring R is called *local* if R has a unique left maximal ideal.

Theorem 3.3. *Let R be a local ring. Then every left R-module is semi-*Rad*-supplemented.*

Proof. Let M be any left R-module. Then there exists an epimorphism $\Psi\colon R^{(I)} \to M$, where I is an index set. Since R is local, by Theorem 3.1, the free left R-module $R^{(I)}$ is semi-Rad-supplemented. Hence M is semi-Rad-supplemented as a factor module of $R^{(I)}$ by Proposition 2.1. □

The following example shows that the converse of Theorem 3.3 is not true.

Example 3.2. Let R be the factor ring $\mathbb{Z}/6\mathbb{Z}$ of the ring $\mathbb{Z}$. Therefore, R is an artinian principal ideal ring. Let M be any left R-module. Then M is a direct sum of local R-modules. It follows from Theorem 3.3 that it is semi-Rad-supplemented. However, R is not a local ring because R has two maximal ideals.

Now we give examples of a module, which is semi-Rad-supplemented but not Rad-supplemented.

Example 3.3. (1) For a prime integer $p \in \mathbb{Z}$, given $\mathbb{Z}_{(p)} = \{\frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0, p \nmid n\}$. Then $\mathbb{Z}_{(p)}$ is a local Dedekind domain. Let F be the direct sum $\mathbb{Z}_{(p)}^{(\mathbb{N})}$ of countably many copies of $\mathbb{Z}_{(p)}$. By Theorem 3.3, we get that F is semi-Rad-supplemented. On the other hand, F is not Rad-supplemented according to [**4**, Theorem 7.1(i)].

(2) Let p be a prime in $\mathbb{Z}$ and consider the left $\mathbb{Z}$-module $M = \bigoplus_{i\geqslant 1} \mathbb{Z}_{p^i}$ which is the sum of local $\mathbb{Z}$-modules $\mathbb{Z}_{p^i}$. Since local modules are w-local, we obtain that M is a semi-Rad-supplemented module by Theorem 3.1. Suppose that M is Rad-supplemented. Note that M is reduced. By [**4**, Theorem 4.6 and Proposition 3.5], we get that $\mathrm{Rad}(M)$ is small in M. This is a contradiction. Consequently, M is not Rad-supplemented.

The Example 3.3 (2) also shows that the class of semi-Rad-supplemented modules contains properly the class of srs-modules by [**5**, Example 2.2].

In [**3**], M is called *cofinitely* Rad-*supplemented* if every submodule N of M with M/N which is finitely generated has a Rad-supplement in M. It was shown in [**3**, Theoerem 3.7] that a module M is cofinitely Rad-supplemented if and only if every maximal submodule has a Rad-supplement in M. Using the characterization we get the result:

Corollary 3.2. *Semi-*Rad*-supplemented modules are cofinitely* Rad*-supplemented.*

Proof. Let M be a semi-Rad-supplemented module. Therefore, every maximal submodule of M has a Rad-supplement in M. It follows from [**3**, Theorem 3.7] that it is cofinitely Rad-supplemented. □

The following example shows that a cofinitely Rad-supplemented module need not be semi-Rad-supplemented. Let R be a Dedekind domain and M be an R-module. We denote by $T(M)$ the set of all elements m of M for which there exists a nonzero element r of R such that $rm = 0$, i.e., $\mathrm{Ann}(m) \neq 0$. Then $T(M)$, which is a submodule of M, is called *the torsion submodule* of M.

Example 3.4. Consider the left $\mathbb{Z}$-module $M = \prod_{p\in\Omega} \mathbb{Z}/p\mathbb{Z}$, where Ω is an infinite collection of distinct prime elements of $\mathbb{Z}$. Then the torsion submodule $T(M)$ of M is the submodule $\bigoplus_{p\in\Omega} \mathbb{Z}/p\mathbb{Z}$ of M. Therefore, $\operatorname{Rad}(M) = 0$ and there exists a submodule N of M such that $N/T(M) \cong \mathbb{Q}$. Since $\mathbb{Q}$ is injective and $\mathbb{Z}$ is a Dedekind domain, we get $\operatorname{Rad}(\mathbb{Q}) = \mathbb{Q}$. So any maximal submodule of N does not contain $T(M)$. Thus every maximal submodule of N is direct summand. This means that every maximal submodule of N has a Rad-supplement in N. By [**3**, Theorem 3.7], N is cofinitely Rad-supplemented.

If N is Rad-supplemented, then it is semisimple according to Corollary 2.3. Hence $\mathbb{Q}$ is semisimple as a factor module of N, a contradiction.

We prove an analogue of [**5**, Proposition 2.14] in the following.

Proposition 3.2. *Let M be an R-module. Suppose that $M/\operatorname{Rad}(M)$ is finitely generated. If M is cofinitely* Rad-*supplemented, then it is semi*-Rad-*supplemented.*

Proof. Let $\operatorname{Rad}(M) \subseteq U \subseteq M$. Note that

$$(M/\operatorname{Rad}(M)/(U/\operatorname{Rad}(M)) \cong M/U$$

is finitely generated. Since M is cofinitely Rad-supplemented, U has a Rad-supplement in M. Hence M is a semi-Rad-supplemented module. $\square$

Corollary 3.3. *The following statements are equivalent for a finitely generated module M.*

(1) *M is* Rad-*supplemented.*
(2) *M is semi*-Rad-*supplemented.*
(3) *M is cofinitely* Rad-*supplemented.*

Proof. (1) $\Rightarrow$ (3) is clear.
(3) $\Rightarrow$ (2) By Proposition 3.2
(2) $\Rightarrow$ (1) It follows from Lemma 2.3 $\square$

Acknowledgement. The authors sincerely thank Engin Büyükaşık for his interest and helpful suggestions. Supported by Scientific Research Project in Amasya University FMB-BAP 15-107.

References

1. K. Al-Takhman, C. Lomp, R. Wisbauer, *τ-complemented and τ-supplemented modules*, Algebra Discr. Math. **3** (2006), 1–16.
2. E. Büyükaşık, *Modules whose maximal submodules have τ-supplements*, Algebra Discr. Math. **10**(2) (2010), 1–9.
3. E. Büyükaşık, C. Lomp, *On a recent generalization of semiperfect rings*, Bull. Aust. Math. Soc. **78**(2) (2008), 317–325.
4. E. Büyükaşık, E. Mermut, S. Özdemir, *Rad-supplemented modules*, Rend. Semin. Mat. Univ. Padova **124** (2010), 157–177.
5. E. Büyükaşık, E. Türkmen, *Strongly radical supplemented modules*, Ukr. Math. J. **106** (2011), 25–30.
6. J. Clark, C. Lomp, N. Vanaja, R. Wisbauer, *Lifting modules. Supplements and projectivity in module theory*, Frontiers in Mathematics, Birkhäuser, Basel, 2006, 406.

7. E. Türkmen, A. Pancar, *Some properties of Rad-supplemented modules*, Internat. J. Phys. Sci. **6**(35) (2011), 7904–7909.
8. A. Ç. Özcan, A. Harmacı, P. F. Smith, *Duo modules*, Glasgow Math. J. **48** (2006), 533–545.
9. D. W. Sharpe, P. Vamos, *Injective Modules*, Cambridge University Press, Cambridge, 1972.
10. R. Wisbauer, *Foundations of Modules and Rings*, Gordon and Breach, 1991.
11. H. Zöschinger, *Moduln, die in jeder Erweiterung ein Komplement haben*, Math. Scand. **35** (1974), 267–287.

Vocational School of Technical Sciences
Amasya University
Amasya
Turkey
engin.kaynar@amasya.edu.tr

(Received 18 06 2015)
(Revised 14 06 2017, and 21 11 2017)

Department of Mathematics
Amasya University
Amasya
Turkey
ergulturkmen@hotmail.com

Department of Mathematics
Ondokuz Mayıs University
Samsun
Turkey
yildizaydin60@hotmail.com