PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 104(118) (2018), 127–138 DOI: https://doi.org/10.2298/PIM1818127L

ON HERMITE–HADAMARD TYPE INEQUALITIES FOR THE PRODUCT OF TWO CONVEX MAPPINGS DEFINED ON TOPOLOGICAL GROUPS

Muhammad Amer Latif, Sever Silvestru Dragomir, and Ebrahim Momoniat

Abstract. We study Hermite–Hadamard type inequalities for the product of two midconvex and quasi-midconvex functions and give some applications of our results.

1. Introduction

Let $f\colon I \to \mathbb{R}$ be a cnovex mapping defined on the interval I of real numbers and a, $b \in I$ with $a < b$. The following double inequality:

$$f\Big(\frac{a+b}{2}\Big) \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx \leqslant \frac{f(a)+f(b)}{2} \tag{1.1}$$

is known as Hermite–Hadamard inequality. The history of this inequality goes back to the papers of Hermite [**9**] and Hadamard [**8**] in 1883 and 1893, respectively. This inequality produces some classical inequalities of means for particular choice of the mapping f. Inequality (1.1) has attracted a number of mathematicians and it has been generalized, extended, and refined it in a number of ways (see e.g. [**3, 4, 7**]). Also some mappings naturally connected with (1.1) are defined and the properties of these mappings are discussed by many mathematicians (see e.g. [**5, 6**]). We only discuss recent studies in this paper.

A generalization of the left-hand side of (1.1) for convex functions defined on a convex subset of $\mathbb{R}^n$ is the following inequality from [**13**]

$$f(0) \leqslant \frac{1}{\mu(X)}\int_X f(x)\,dx, \tag{1.2}$$

where $X \subset \mathbb{R}^n$ is a convex bounded symmetric set that is, if $x \in X$ then $-x \in X$, f is a lower semicontinuous convex function $f\colon X \to \mathbb{R}$ and $\mu(X)$ is the volume of the set X.

2010 *Mathematics Subject Classification:* Primary 26A51, 22A10; Secondary 26D07, 26D15.
Key words and phrases: convex functions, Hermite–Hadamard inequality, midconvex function, quasi-midconvex function, topological groups.
Communicated by Gradimir Milovanović.

In [**12**], Morassaei established the Hermite–Hadamard type inequality for midconvex and quasi-midconvex functions in topological groups and discussed some of the properties of the mapping naturally connected with the Hermite–Hadamard inequality for globally midconvex functions defined in a topological group. Some of the main results of [**12**] are stated in the following theorems.

THEOREM 1.1. [**12**] *Let G be a locally compact group and $\Omega \subset G$ an open symmetric set relative to $a \in G$ with $0 < \mu(\Omega) < \infty$. Let $f\colon \Omega \to \mathbb{R}$ be measurable and locally midconvex in a and $f \in L_1(\Omega)$. If $\omega\colon \Omega \to \mathbb{R}$ is non-negative, symmetric with respect to a and $\omega \in L_1(\Omega)$ such that $f\omega \in L_1(\Omega)$; then*

$$f(a)\int_\Omega \omega(az)\,d\mu(z) \leqslant \int_\Omega f(az)\omega(az)\,d\mu(z),$$

where μ is the Haar measure.

THEOREM 1.2. [**12**] *Suppose that G is a locally compact group and $\Omega \subset G$ an open symmetric set relative to $a \in G$ with $0 < \mu(\Omega) < \infty$ and $e \in \Omega$. Let f be measurable and quasi-midconvex real-valued function on Ω such that $f \in L_2(\Omega)$. If $\omega\colon \Omega \to \mathbb{R}$ is non-negative and symmetric with respect to a and $\omega \in L_2(\Omega)$, then*

$$f(a)\int_\Omega \omega(az)\,d\mu(z) \leqslant \int_\Omega f(az)\omega(az)\,d\mu(z) + I(a),$$

where

$$I(a) = \frac{1}{2}\int_\Omega |f(az) - f(az^{-1})|\omega(az)\,d\mu(z).$$

Furthermore, $I(a)$ satisfies the following inequality

$$\begin{aligned}0 &\leqslant I(a)\\ &\leqslant \min\left\{\begin{matrix}\frac{1}{\sqrt{2}}\big(\int_\Omega f^2(az)\,d\mu(z) - \int_\Omega f^2(az)f(az^{-1})\,d\mu(z)\big)^{\frac{1}{2}}\big(\int_\Omega \omega^2(az)\,d\mu(z)\big)^{\frac{1}{2}},\\ \int_\Omega |f(az)|\omega(az)\,d\mu(z).\end{matrix}\right\}\end{aligned}$$

THEOREM 1.3. *Let G be a locally compact group and $\Omega \subset G$ an open symmetric set relative to $a \in G$ with $0 < \mu(\Omega) < \infty$. Let f be measurable real valued P-function on Ω such that $f \in L_1(\Omega)$. If $\omega\colon \Omega \to \mathbb{R}$ is non-negative symmetric with respect to a, $\omega \in L_1(\Omega)$ and $f\omega \in L_1(\Omega)$, then*

$$f(a)\int_\Omega \omega(az)\,d\mu(z) \leqslant 2\int_\Omega f(az)\omega(az)\,d\mu(z).$$

We give a result similar to (1.2) for the product of two convex functions defined on a convex bounded symmetric subset X of $\mathbb{R}^n$. We will also give our results for the product of two midconvex and qusi-midconvex mappings defined on topological groups in Section 3. Applications of the obtained results are given in Section 3 as well.

2. A Secondary Result

THEOREM 2.1. *Let f, g be two convex functions defined on a convex bounded symmetrical subset X of $\mathbb{R}^n$. Then*

$$f(0)g(0) \leqslant \frac{1}{2\mu(X)} \int_X [f(x)g(x) + f(x)g(-x)]dx = \frac{1}{2\mu(X)} \int_X [f(x)g(x) + f(-x)g(x)]dx. \tag{2.1}$$

PROOF. Consider the transformation $h\colon \mathbb{R}^n \to \mathbb{R}^n$, $h = (h_1, \dots, h_n)$, given by $h_i(x_1, \dots, x_n) = -x_i \quad i = 1, 2, \dots, n$. Then $h(X) = X$ and

$$\frac{D(h_1, \dots, h_n)}{D(x_1, \dots, x_n)} = \begin{vmatrix} -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{vmatrix} = (-1)^n.$$

Thus we have, by the change of variables that

$$\begin{aligned} &\int_X f(x_1, \dots, x_n)g(x_1, \dots, x_n)\, dx_1 \dots dx_n \\ &= \int_X f(h_1(x_1, \dots, x_n), \dots, h_n(x_1, \dots, x_n))g(h_1(x_1, \dots, x_n), \dots, h_n(x_1, \dots, x_n)) \\ &\times \Big|\frac{D(h_1, \dots, h_n)}{D(x_1, \dots, x_n)}\Big| dx_1 \dots dx_n = \int_X f(-x_1, \dots, -x_n)g(-x_1, \dots, -x_n)\, dx_1 \dots dx_n, \end{aligned}$$

$$\begin{aligned} &\int_X f(x_1, \dots, x_n)g(-x_1, \dots, -x_n)\, dx_1 \dots dx_n \\ &= \int_X f(h_1(x_1, \dots, x_n), \dots, h_n(x_1, \dots, x_n))g(-h_1(x_1, \dots, x_n), \dots, -h_n(x_1, \dots, x_n)) \\ &\times \Big|\frac{D(h_1, \dots, h_n)}{D(x_1, \dots, x_n)}\Big| dx_1 \dots dx_n = \int_X f(-x_1, \dots, -x_n)g(x_1, \dots, x_n)\, dx_1 \dots dx_n. \end{aligned}$$

Now by the convexity of f and g on X, we get

$$\begin{aligned} &f(0, \dots, 0)g(0, \dots, 0) = f\big(\tfrac{1}{2}(x_1 - x_1), \dots, \tfrac{1}{2}(x_n - x_n)\big)g\big(\tfrac{1}{2}(x_1 - x_1), \dots, \tfrac{1}{2}(x_n - x_n)\big) \\ &= f\big(\tfrac{1}{2}((x_1, \dots, x_n) + (-x_1, \dots, -x_n))\big)g\big(\tfrac{1}{2}((x_1, \dots, x_n) + (-x_1, \dots, -x_n))\big) \\ &\leqslant \tfrac{1}{4}[f(x_1, \dots, x_n) + f(-x_1, \dots, -x_n)][g(x_1, \dots, x_n) + g(-x_1, \dots, -x_n)] \\ &= \tfrac{1}{4}[f(x_1, \dots, x_n)g(x_1, \dots, x_n) + f(-x_1, \dots, -x_n)g(-x_1, \dots, -x_n) \\ &\qquad + f(-x_1, \dots, -x_n)g(x_1, \dots, x_n) + f(x_1, \dots, x_n)g(-x_1, \dots, -x_n)] \end{aligned}$$

which gives by integration on X that

$$\int_X f(0, \dots, 0)g(0, \dots, 0)\, dx_1 \dots dx_n \tag{2.2}$$

$$\leqslant \frac{1}{4}\Bigg[2\int_X f(x_1,\dots,x_n)g(x_1,\dots,x_n)\,dx_1\dots dx_n + \int_X f(x_1,\dots,x_n)g(-x_1,\dots,-x_n)\,dx_1\dots dx_n + \int_X f(-x_1,\dots,-x_n)g(x_1,\dots,x_n)\,dx_1\dots dx_n\Bigg].$$

Hence (2.1) follows from (2.2). □

3. Main Results

Now we prove Hermite–Hadamard type inequalities for products of midconvex and quasi-convex functions defined in a topological groups. Before we proceed to prove our results we give some definitions from [**1, 11, 12**].

We recall that for a group $(G,\cdot,e)$, a topology on G is compatible with the group structure when the maps $G\times G\to G\colon (x,y)\mapsto xy$ (multiplication) and $G\to G\colon x\mapsto x^{-1}$ (inverse) are continuous. A group together with a topology compatible with its group structure is a topological group. A compact group is a topological group that is a compact topological space.

A Haar measure on compact group G is a measure $\mu\colon \Sigma\to[0,\infty)$, with a σ-algebra Σ containing all Borel subsets of G, such that $\mu(G)=1$ and $\mu(\gamma S)=\mu(S)$ for all $\gamma\in G$, $S\in\Sigma$, where $\gamma S=\{\gamma\alpha:\alpha\in S\}$.

Definition 3.1. [**1**] Let G be a topological group, Ω a non-empty open subset of G and f a real-valued function on Ω. We say that f is globally (right) midconvex if $2f(a)\leqslant f(az)+f(az^{-1})$ for all a, $z\in G$ such that $a, az, az^{-1}\in\Omega$. We say that f is locally (right) midconvex for every $a\in\Omega$ if there exists an open symmetric neighborhood $V=V^{-1}$ of e such that $2f(a)\leqslant f(az)+f(az^{-1})$ for all $z\in V$ such that $az, az^{-1}\in\Omega$.

Definition 3.2. [**1**] Let G be a topological group, Ω a non-empty open subset of G and f a real-valued function on Ω. The mapping f is called quasi-(right) midconvex, if $f(az)\leqslant\max\{f(a),f(az^2)\}$ for every $a,z\in G$ so that $a,az,az^2\in\Omega$. Note that a is the midpoint of az^{-1} and az, and az is the midpoint of a and az^2.

Definition 3.3. [**12**, Definition 1, page 4] Let Ω be an open subset of a topological group G, and $a\in G$. Ω is said to be symmetric relative to a, if $a^{-1}\Omega$ is symmetric and $e\in a^{-1}\Omega$.

Definition 3.4. [**12**, Definition 2, Page 4] Let G be a topological group and $\Omega\subset G$ an open set. A function $\omega\colon\Omega\to R$ is called symmetric relative to $a\in G$, if for every $z\in G$, $az, az^{-1}\in\Omega$ and $\omega(az)=\omega(az^{-1})$.

We now give our main result.

Theorem 3.1. *Suppose that G is a locally compact group and $\Omega\subset G$ an open symmetric set relative to $a\in G$ with $0<\mu(\Omega)<\infty$. Let f, $g\colon\Omega\to\mathbb{R}_+$ be measurable and locally midconvex in a and $f,g\in L_1(\Omega)$. If $\omega\colon\Omega\to\mathbb{R}$ is non-negative symmetric with respect to a and $\omega\in L_1(\Omega)$ such that $fg\omega\in L_1(\Omega)$, then* *we have*

$$f(a)g(a)\int_\Omega \omega(az)\,d\mu(z) \leqslant \frac{1}{2}\Big[\int_\Omega f(az)g(az)\omega(az)\,d\mu(z)+\int_\Omega f(az)g(az^{-1})\omega(az)\,d\mu(z)\Big] = \frac{1}{2}\Big[\int_\Omega f(az)g(az)\omega(az)\,d\mu(z)+\int_\Omega f(az^{-1})g(az)\omega(az)\,d\mu(z)\Big], \tag{3.1}$$

where μ is the Haar measure.

Proof. Since f and g are midconvex in a, we have $2f(a)\leqslant f(az)+f(az^{-1})$ and $2g(a)\leqslant g(az)+g(az^{-1})$ for any $z\in\Omega$. From these inequalities we get

$$4f(a)g(a)\leqslant f(az)g(az)+f(az^{-1})g(az)+f(az)g(az^{-1})+f(az^{-1})(g(az^{-1}).$$

Since ω is non-negative and symmetric relative to a, we have

$$4f(a)g(a)\omega(az)\leqslant f(az)g(az)\omega(az)+f(az^{-1})g(az)\omega(az^{-1}) + f(az)g(az^{-1})\omega(az)+f(az^{-1})g(az^{-1})\omega(az^{-1}).$$

Integrating this inequality over Ω, we get

$$\begin{aligned}
&4f(a)g(a)\int_\Omega \omega(az)\,d\mu(z)\\
&\leqslant \int_\Omega f(az)g(az)\omega(az)\,d\mu(z)+\int_\Omega f(az^{-1})g(az)\omega(az^{-1})\,d\mu(z)\\
&\quad+\int_\Omega f(az)g(az^{-1})\omega(az)\,d\mu(z)+\int_\Omega f(az^{-1})g(az^{-1})\omega(az^{-1})\,d\mu(z)\\
&=\int_{a^{-1}\Omega} f(z)g(z)\omega(z)\,d\mu(z)+\int_{a^{-1}\Omega} f(z^{-1})g(z)\omega(z^{-1})\,d\mu(z)\\
&\quad+\int_{a^{-1}\Omega} f(z)g(z^{-1})\omega(z)\,d\mu(z)+\int_{a^{-1}\Omega} f(z^{-1})g(z^{-1})\omega(z^{-1})\,d\mu(z)\\
&=\int_G f(z)g(z)\omega(z)\chi_{a^{-1}\Omega}(z)\,d\mu(z)+\int_G f(z^{-1})g(z)\omega(z^{-1})\chi_{a^{-1}\Omega}(z)\,d\mu(z)\\
&\quad+\int_G f(z)g(z^{-1})\omega(z)\chi_{a^{-1}\Omega}(z)\,d\mu(z)+\int_G f(z^{-1})g(z^{-1})\omega(z^{-1})\chi_{a^{-1}\Omega}(z)\,d\mu(z)\\
&=\int_G f(z)g(z)\omega(z)\chi_{a^{-1}\Omega}(z)\,d\mu(z)+\int_G f(z^{-1})g(z)\omega(z^{-1})\chi_{a^{-1}\Omega}(z^{-1})\,d\mu(z)\\
&\quad+\int_G f(z)g(z^{-1})\omega(z)\chi_{a^{-1}\Omega}(z)\,d\mu(z)+\int_G f(z^{-1})g(z^{-1})\omega(z^{-1})\chi_{a^{-1}\Omega}(z^{-1})\,d\mu(z)
\end{aligned}$$

that is

$$f(a)g(a)\int_\Omega \omega(az)\,d\mu(z) \leqslant 2\int_G f(z)g(z)\omega(z)\chi_{a^{-1}\Omega}(z)\,d\mu(z)+2\int_G f(z)g(z^{-1})\omega(z)\chi_{a^{-1}\Omega}(z)\,d\mu(z) \tag{3.2}$$

$$= 2\int_{a^{-1}\Omega} f(z)g(z)\omega(z)\,d\mu(z) + 2\int_{a^{-1}\Omega} f(z)g(z^{-1})\omega(z)\,d\mu(z)$$
$$= \frac{1}{2}\bigg[\int_{\Omega} f(az)g(az)\omega(az)\,d\mu(z) + \int_{\Omega} f(az)g(az^{-1})\omega(az)\,d\mu(z)\bigg].$$

Since

$$\int_{\Omega} f(az)g(az^{-1})\omega(az)\,d\mu(z) = \int_{\Omega} f(az)g(az^{-1})\omega(az^{-1})\,d\mu(z)$$
$$= \int_{a^{-1}\Omega} f(z)g(z^{-1})\omega(z^{-1})\,d\mu(z) = \int_{G} f(z)g(z^{-1})\omega(z^{-1})\chi_{a^{-1}\Omega}(z)\,d\mu(z)$$
$$= \int_{G} f(z)g(z^{-1})\omega(z^{-1})\chi_{a^{-1}\Omega}(z^{-1})\,d\mu(z) = \int_{G} f(z^{-1})g(z)\omega(z)\chi_{a^{-1}\Omega}(z)\,d\mu(z)$$
$$= \int_{a^{-1}\Omega} f(z^{-1})g(z)\omega(z)\,d\mu(z) = \int_{\Omega} f(az^{-1})g(az)\omega(az)\,d\mu(z),$$

we also have

$$(3.3)\quad f(a)g(a)\int_{\Omega}\omega(az)\,d\mu(z) \leqslant \frac{1}{2}\bigg[\int_{\Omega} f(az)g(az)\omega(az)\,d\mu(z) + \int_{\Omega} f(az^{-1})g(az)\omega(az)\,d\mu(z)\bigg].$$

Consequently inequality (3.1) follows from (3.2) and (3.3). □

REMARK 3.1. If we take $a = e$ and $\omega \equiv 1$ on Ω in Theorem 3.1, then we have

$$f(e)g(e) \leqslant \frac{1}{2\mu(\Omega)}\bigg[\int_{\Omega} f(z)g(z)\omega(z)\,d\mu(z) + \int_{\Omega} f(z)g(z^{-1})\omega(z)\,d\mu(z)\bigg]$$
$$= \frac{1}{2\mu(\Omega)}\bigg[\int_{\Omega} f(z)g(z)\omega(z)\,d\mu(z) + \int_{\Omega} f(z^{-1})g(z)\omega(z)\,d\mu(z)\bigg]$$

which is similar to (2.1).

REMARK 3.2. If $g \equiv 1$ on Ω in Theorem 3.1, then we have

$$f(a)\int_{\Omega}\omega(az)\,d\mu(z) \leqslant \int_{\Omega} f(az)g(az)\omega(az)\,d\mu(z)$$

which is a similar result to that in [**12**, Theorem 1].

REMARK 3.3. If we take $a = e$, $\omega \equiv 1$ and $g \equiv 1$ on Ω in Theorem 3.1, then we have $f(e) \leqslant \frac{1}{\mu(\Omega)}\int_{\Omega} f(z)\,d\mu(z)$ which is the same result to that in [**12**, Remark 1].

THEOREM 3.2. *Suppose that G is a locally compact group and $\Omega \subset G$ an open symmetric set relative to $a \in G$ with $0 < \mu(\Omega) < \infty$ and $e \in \Omega$. Let f and g be measurable and quasi-midconvex non-negative real-valued functions on Ω such that $fg \in L_2(\Omega)$. If $\omega\colon \Omega \to \mathbb{R}$ is non-negative and symmetric with respect to a and $\omega \in L_2(\Omega)$, we have*

$$(3.4)\qquad f(a)g(a)\int_{\Omega}\omega(az)\,d\mu(z)$$

$$\leqslant \frac{1}{2}\int_\Omega f(az)g(az)\omega(az)\,d\mu(z)+\frac{1}{2}\int_\Omega f(az)g(az^{-1})\omega(az)\,d\mu(z)+I(a)$$
$$=\frac{1}{2}\int_\Omega f(az)g(az)\omega(az)\,d\mu(z)+\frac{1}{2}\int_\Omega f(az^{-1})g(az)\omega(az)\,d\mu(z)+I(a),$$

where

$$\begin{aligned}I(a)=&\frac{1}{2}\int_\Omega f(az)|g(az)-g(az^{-1})|\omega(az)\,d\mu(z)\\&+\frac{1}{2}\int_\Omega |f(az)-f(az^{-1})|g(az)\omega(az)\,d\mu(z)\\&+\frac{1}{4}\int_\Omega |f(az)-f(az^{-1})||g(az)-g(az^{-1})|\omega(az)\,d\mu(z).\end{aligned}$$

Furthermore,

$$0\leqslant I(a)\leqslant \min\{A_1,A_2,A_3,A_4\}, \tag{3.5}$$

where

$$\begin{aligned}A_1=&\frac{1}{2}\Big(\int_\Omega (f(az^{-1})(g(az)-g(az^{-1}))\Big)^2 d\mu(z))^{\frac{1}{2}}\Big(\int_\Omega \omega^2(az)\,d\mu(z)\Big)^{\frac{1}{2}}\\&+\frac{1}{2}\Big(\int_\Omega (g(az^{-1})(f(az)-f(az^{-1})))^2 d\mu(z)\Big)^{\frac{1}{2}}\Big(\int_\Omega \omega^2(az)\,d\mu(z)\Big)^{\frac{1}{2}}\\&+\frac{1}{4}\Big(\int_\Omega ((f(az^{-1})-f(az))(g(az)-g(az^{-1})))^2 d\mu(z)\Big)^{\frac{1}{2}}\Big(\int_\Omega \omega^2(az)\,d\mu(z)\Big)^{\frac{1}{2}},\\A_2=&\frac{1}{\sqrt{2}}\Big(\int_\Omega f^2(az)\,d\mu(z)\Big)^{\frac{1}{2}}\Big(\int_\Omega g^2(az)\omega^2(az)\,d\mu(z)\\&-\int_\Omega g(az)g(az^{-1})\omega^2(az)\,d\mu(z)\Big)^{\frac{1}{2}}+\frac{1}{\sqrt{2}}\Big(\int_\Omega g^2(az)\,d\mu(z)\Big)^{\frac{1}{2}}\\&\times\Big(\int_\Omega f^2(az)\omega^2(az)\,d\mu(z)-\int_\Omega f(az)f(az^{-1})\omega^2(az)\,d\mu(z)\Big)^{\frac{1}{2}}\\&+\frac{1}{2}\Big(\int_\Omega f^2(az)\,d\mu(z)-\int_\Omega f(az)f^{-1}(az)\,d\mu(z)\Big)^{\frac{1}{2}}\\&\times\Big(\int_\Omega g^2(az)\omega^2(az)\,d\mu(z)-\int_\Omega g(az)g(az^{-1})\omega^2(az)\,d\mu(z)\Big)^{\frac{1}{2}},\\A_3=&\frac{1}{\sqrt{2}}\Big(\int_\Omega g^2(az)\,d\mu(z)-\int_\Omega g^2(az)g(az^{-1})\,d\mu(z)\Big)^{\frac{1}{2}}\Big(\int_\Omega f^2(az)\omega^2(az)\,d\mu(z)\Big)^{\frac{1}{2}}\\&+\frac{1}{\sqrt{2}}\Big(\int_\Omega f^2(az)\,d\mu(z)-\int_\Omega f^2(az)f(az^{-1})\,d\mu(z)\Big)^{\frac{1}{2}}\Big(\int_\Omega g^2(az)\omega^2(az)\,d\mu(z)\Big)^{\frac{1}{2}}\\&+\frac{1}{2}\Big(\int_\Omega g^2(az)\,d\mu(z)-\int_\Omega g^2(az)g(az^{-1})\,d\mu(z)\Big)^{\frac{1}{2}}\end{aligned}$$

$$\times\left(\int_{\Omega} f^2(az)\omega^2(az)\,d\mu(z)-\int_{\Omega} f^2(az)f(az^{-1})\omega^2(az)\,d\mu(z)\right)^{\frac{1}{2}},$$

$$A_4+\frac{3}{2}\int_{\Omega} f(az)g(az)\omega(az)\,d\mu(z)+\frac{3}{2}\int_{\Omega} f(az)g(az^{-1})\omega(az)\,d\mu(z).$$

PROOF. Since Ω is a symmetric set relative to a, for any $z\in G$ and by the quasi-midconvexity of f and g, we have

$$f(a)=\max\{f(az),f(az^{-1})\}=\frac{f(az)+f(az^{-1})+|f(az)-f(az^{-1})|}{2},$$

$$g(a)=\max\{g(az),g(az^{-1})\}=\frac{g(az)+g(az^{-1})+|g(az)-g(az^{-1})|}{2}.$$

Now, by the non-negativity of f and g, we get

$$\begin{aligned}(3.6)\quad f(a)g(a)\leqslant \tfrac{1}{4}[&f(az)g(az)+f(az)g(az^{-1})+f(az)|g(az)-g(az^{-1})|\\ &+f(az^{-1})g(az)+f(az^{-1})g(az^{-1})+f(az^{-1})|g(az)-g(az^{-1})|\\ &+|f(az)-f(az^{-1})|g(az)+|f(az)-f(az^{-1})|g(az^{-1})\\ &+|f(az)-f(az^{-1})||g(az)-g(az^{-1})|].\end{aligned}$$

Since ω is non-negative and symmetric relative to a, we have from (3.6)

$$\begin{aligned}f(a)g(a)&\int_{\Omega}\omega(az)\,d\mu(z)\\ &\leqslant \frac{1}{2}\int_{\Omega} f(az)g(az)\omega(az)\,d\mu(z)+\frac{1}{2}\int_{\Omega} f(az)g(az^{-1})\omega(az)\,d\mu(z)+I(a)\\ &=\frac{1}{2}\int_{\Omega} f(az)g(az)\omega(az)\,d\mu(z)+\frac{1}{2}\int_{\Omega} f(az^{-1})g(az)\omega(az)\,d\mu(z)+I(a).\end{aligned}$$

Hence, (3.4) is proved, where

$$\begin{aligned}(3.7)\quad I(a)=\frac{1}{2}&\int_{\Omega} f(az)|g(az)-g(az^{-1})|\omega(az)\,d\mu(z)\\ &+\frac{1}{2}\int_{\Omega}|f(az)-f(az^{-1})|g(az)\omega(az)\,d\mu(z)\\ &\quad+\frac{1}{4}\int_{\Omega}|f(az)-f(az^{-1})||g(az)-g(az^{-1})|\omega(az)\,d\mu(z).\end{aligned}$$

Now by th Cauchy–Schwarz inequality, we observe from (3.7) that

$$\begin{aligned}(3.8)\quad 0\leqslant I(a)\leqslant \frac{1}{2}&\left(\int_{\Omega}(f(az)(g(az)-g(az^{-1})))^2d\mu(z)\right)^{\frac{1}{2}}\left(\int_{\Omega}\omega^2(az)\,d\mu(z)\right)^{\frac{1}{2}}\\ &+\frac{1}{2}\left(\int_{\Omega}(g(az)(f(az)-f(az^{-1})))^2d\mu(z)\right)^{\frac{1}{2}}\left(\int_{\Omega}\omega^2(az)\,d\mu(z)\right)^{\frac{1}{2}}\\ &+\frac{1}{4}\left(\int_{\Omega}((f(az^{-1})-f(az))(g(az)-g(az^{-1})))^2d\mu(z)\right)^{\frac{1}{2}}\left(\int_{\Omega}\omega^2(az)\,d\mu(z)\right)^{\frac{1}{2}}.\end{aligned}$$

Again by the Cauchy–Schwarz inequality, we have from (3.7) the following inequality

$$\begin{aligned}0 \leqslant I(a) \leqslant& \frac{1}{2}\Big(\int_\Omega f^2(az)\,d\mu(z)\Big)^{\frac{1}{2}}\Big(\int_\Omega \Big(g(az)-g(az^{-1})\Big)^2\omega^2(az)\,d\mu(z)\Big)^{\frac{1}{2}}\\ &+\frac{1}{2}\Big(\int_\Omega g^2(az)\,d\mu(z)\Big)^{\frac{1}{2}}\Big(\int_\Omega (f(az)-f(az^{-1}))^2\omega^2(az)\,d\mu(z)\Big)^{\frac{1}{2}}\\ &+\frac{1}{4}\Big(\int_\Omega (f(az)-f(az^{-1}))^2 d\mu(z)\Big)^{\frac{1}{2}}\Big(\int_\Omega (g(az)-g(az^{-1}))^2\omega^2(az)\,d\mu(z)\Big)^{\frac{1}{2}}\end{aligned}$$

which is equivalent to

$$\begin{aligned}0 \leqslant I(a) \leqslant& \frac{1}{\sqrt{2}}\Big(\int_\Omega f^2(az)\,d\mu(z)\Big)^{\frac{1}{2}}\Big(\int_\Omega g^2(az)\omega^2(az)\,d\mu(z)\\ &-\int_\Omega g(az)g(az^{-1})\omega^2(az)\,d\mu(z)\Big)^{\frac{1}{2}}+\frac{1}{\sqrt{2}}\Big(\int_\Omega g^2(az)\,d\mu(z)\Big)^{\frac{1}{2}}\\ &\times\Big(\int_\Omega f^2(az)\omega^2(az)\,d\mu(z)-\int_\Omega f(az)f(az^{-1})\omega^2(az)\,d\mu(z)\Big)^{\frac{1}{2}}\\ &+\frac{1}{2}\Big(\int_\Omega f^2(az)\,d\mu(z)-\int_\Omega f(az)f^{-1}(az)\,d\mu(z)\Big)^{\frac{1}{2}}\\ &\times\quad(\int_\Omega g^2(az)\omega^2(az)\,d\mu(z)-\int_\Omega g(az)g(az^{-1})\omega^2(az)\,d\mu(z)\Big)^{\frac{1}{2}}.\end{aligned}\tag{3.9}$$

Using the Cauchy-Schwarz inequality again, from (3.7) we infer

(3.10)

$$\begin{aligned}&0 \leqslant I(a)\\ &\leqslant \frac{1}{\sqrt{2}}\Big(\int_\Omega g^2(az)\,d\mu(z)-\int_\Omega g^2(az)g(az^{-1})\,d\mu(z)\Big)^{\frac{1}{2}}\Big(\int_\Omega f^2(az)\omega^2(az)\,d\mu(z)\Big)^{\frac{1}{2}}\\ &+\frac{1}{\sqrt{2}}\Big(\int_\Omega f^2(az)\,d\mu(z)-\int_\Omega f^2(az)f(az^{-1})\,d\mu(z)\Big)^{\frac{1}{2}}\Big(\int_\Omega g^2(az)\omega^2(az)\,d\mu(z)\Big)^{\frac{1}{2}}g\\ &+\frac{1}{2}\Big(\int_\Omega g^2(az)\,d\mu(z)-\int_\Omega g^2(az)g(az^{-1})\,d\mu(z)\Big)^{\frac{1}{2}}\\ &\qquad\times\Big(\int_\Omega f^2(az)\omega^2(az)\,d\mu(z)-\int_\Omega f^2(az)f(az^{-1})\omega^2(az)\,d\mu(z)\Big)^{\frac{1}{2}}.\end{aligned}$$

Finally, by using the properties of absolute value, we have

$$0 \leqslant I(a) \leqslant \frac{3}{2}\int_\Omega f(az)g(az)\omega(az)\,d\mu(z)+\frac{3}{2}\int_\Omega f(az)g(az^{-1})\omega(az)\,d\mu(z).\tag{3.11}$$

Inequality (3.5) follows from (3.8)–(3.11). □

COROLLARY 3.1. *Suppose the assumptions of Theorem 3.2 are satisfied. If $g \equiv 1$ on Ω in Theorem 3.2, we have*

$$f(a)\int_\Omega \omega(az)\,d\mu(z) \leqslant \int_\Omega f(az)\omega(az)\,d\mu(z) + I(a), \tag{3.12}$$

where $I(a) = \frac{1}{2}\int_\Omega |f(az) - f(az^{-1})|\omega(az)\,d\mu(z)$. *Furthermore,*

$$0 \leqslant I(a) \leqslant \min\{B_1, B_2, B_3\},$$

where

$$B_1 = \frac{1}{\sqrt{2}}\sqrt{\mu(\Omega)}\bigg(\int_\Omega f^2(az)\omega^2(az)\,d\mu(z) - \int_\Omega f(az)f(az^{-1})\omega^2(az)\,d\mu(z)\bigg)^{\frac{1}{2}},$$

$$B_2 = \frac{1}{\sqrt{2}}\bigg(\int_\Omega f^2(az)\,d\mu(z) - \int_\Omega f^2(az)f(az^{-1})\,d\mu(z)\bigg)^{\frac{1}{2}}\bigg(\int_\Omega \omega^2(az)\,d\mu(z)\bigg)^{\frac{1}{2}},$$

$$B_3 = \int_\Omega f(az)\omega(az)\,d\mu(z).$$

DEFINITION 3.5. [**12**] The function $f\colon\ \Omega \to \mathbb{R}$ is said to be a P-function in Ω, if $f(a) \leqslant f(az) + f(az^{-1})$ for all $a \in \Omega$ and $z \in G$ such that $az, az^{-1} \in \Omega$.

THEOREM 3.3. *Let G be a locally compact group and $\Omega \subset G$ an open symmetric set relative to $a \in G$ with $0 < \mu(\Omega) < \infty$. Let f, g be measurable non-negative real valued P-functions on Ω such that $fg \in L_1(\Omega)$. If $\omega\colon \Omega \to \mathbb{R}$ is non-negative symmetric to a, $\omega \in L_1(\Omega)$ and $fg\omega \in L_1(\Omega)$, we have*

$$\begin{aligned} f(a)g(a)\int_\Omega \omega(az)\,d\mu(z) &\leqslant 2\int_\Omega f(az)g(az)\omega(az)d\mu(z) + 2\int_\Omega f(az)g(az^{-1})\omega(az)\,d\mu(z) \\ &= 2\int_\Omega f(az)g(az)\omega(az)\,d\mu(z) + 2\int_\Omega f(az^{-1})g(az)\omega(az)d\mu(z). \end{aligned}$$

PROOF. Since f and g are P-functions and ω is non-negative and symmetric to a, we have

$$\begin{aligned} f(a)g(a)\omega(az) &\leqslant (f(az) + f(az^{-1}))(g(az) + g(az^{-1}))\omega(az) \\ &= f(az)g(az)\omega(az) + f(az^{-1})g(az^{-1})\omega(az) \\ &\quad + f(az)g(az^{-1})\omega(az) + f(az^{-1})g(az)\omega(az). \end{aligned}$$

Integrating this inequality on Ω, we get

$$\begin{aligned} &f(a)g(a)\int_\Omega \omega(az)\,d\mu(z) \\ &\leqslant \int_\Omega f(az)g(az)\omega(az)d\mu(z) + \int_\Omega f(az^{-1})g(az^{-1})\omega(az)\,d\mu(z) \\ &\quad + \int_\Omega f(az)g(az^{-1})\omega(az)\,d\mu(z) + \int_\Omega f(az^{-1})g(az)\omega(az)\,d\mu(z) \end{aligned}$$

$$= 2\int_\Omega f(az)g(az)\omega(az)\,d\mu(z) + 2\int_\Omega f(az)g(az^{-1})\omega(az)\,d\mu(z)$$
$$= 2\int_\Omega f(az)g(az)\omega(az)d\mu(z) + 2\int_\Omega f(az^{-1})g(az)\omega(az)\,d\mu(z).$$

Hence, the proof of the theorem is completed. □

Corollary 3.2. *Suppose that the assumptions of Theorem* 3.3 *are satisfied. If* $g \equiv 1$ *on* Ω*, then we have*

$$f(a)\int_\Omega \omega(az)\,d\mu(z) \leqslant 4\int_\Omega f(az)\omega(az)\,d\mu(z).$$

Corollary 3.3. *If we take* $a = e$ *and* $\omega \equiv 1$ *on* Ω *in Corollary* 3.2*, then we have*

$$f(e) \leqslant \frac{4}{\mu(\Omega)}\int_\Omega f(z)\,d\mu(z).$$

Some of the applications of our results are given in the following remarks.

Remark 3.4. Set $G = \mathbb{R}$. Since $\mathbb{R}$ is an abelian additive group, for all $a, z \in \mathbb{R}$, $a - z$ and $a + z$ are points for which a is the midpoint. Now, if $a - z = y$ and $a + z = x$, then $a = \frac{x+y}{2}$. If we take $\Omega = [-b, b]$, we get $a = 0$ and $y = -x$. Hence, from Theorem 3.1 we have

$$f(0)g(0)\int_{-b}^{b}\omega(x)\,dx \leqslant \frac{1}{2}\left[\int_{-b}^{b} f(x)g(x)\omega(x)\,dx + \int_{-b}^{b} f(x)g(-x)\omega(x)\,dx\right] \tag{3.13}$$
$$= \frac{1}{2}\left[\int_{-b}^{b} f(x)g(x)\omega(x)\,dx + \int_{-b}^{b} f(-x)g(x)\omega(x)\,dx\right].$$

If $\omega(x) \equiv 1$ for all $x \in [-b, b]$ in (3.13), we obtain

$$f(0)g(0) \leqslant \frac{1}{4b}\left[\int_{-b}^{b} f(x)g(x)\,dx + \int_{-b}^{b} f(x)g(-x)\,dx\right]$$
$$= \frac{1}{4b}\left[\int_{-b}^{b} f(x)g(x)\,dx + \int_{-b}^{b} f(-x)g(x)\,dx\right].$$

Remark 3.5. If in the Theorem 3.1, $G = \mathbb{R}^n$ with the operation of additive and $\Omega = X$ is an open bounded symmetric and convex subset of $\mathbb{R}^n$, then the result of Theorem 2.1 holds.

References

1. A. Chademan, F. Mirzapour, *Midconvex functions in locally compact groups*, Proc. Am. Math. Soc. **127**(10) (1999), 2961–2968.
2. K. Chandrasekharan, *A course on topological groups*, Volume 9 of Texts Read. Math. **9** (1996), 117 pp.
3. S. S. Dragomir, C. E. M. Pearce, *Quasi-convex functions and Hadamard's inequalities*, Bull. Aust. Math. Soc. **57** (1998), 377–385.
4. ______, *Select topics on Hermite–Hadamard inequalities and applications*, Melbourne and Adelaide, 2000.
5. S. S. Dragomir, *A mapping in connection to Hadamard's inequality*, An Ostro. Akad. Wiss. Math.-Natur (Wien) **128** (1991), 17-20.

6. ______, *Two mappings in connection to Hadamard's inequality*, J. Math. Anal. Appl. **167** (1992), 49–56.
7. ______, *On Hadamard's inequality for convex functions*, Math. Balk., New Ser. **6** (1992), 215–222.
8. J. Hadamard, *Étude sur les propriétés des fonctions entières et en particulier d'une fonction considerée par Riemann*, J. Math. (4) **9** (1893), 171–215.
9. Ch. Hermite, *Sur deux limites d'une intégrale définie*, Mathesis **3** (1883), 82.
10. P. J. Higgins, *An Introduction to Topological Groups*, Lond. Math. Soc. Lect. Note Ser. **15**, Cambridge University Press, London–New York, 1974, 109 pp.
11. F. Mirzapour, A. Morassaei, *Quasi-convex functions in topological groups*, Int. J. Appl. Math. **16**(4) (2004), 281–290.
12. A. Morassaei, *On Hadamard's inequality for convex mappings defined in topological groups and connected results*, J. Math. Inequal. **4**(3) (2010), 445–452.
13. A. M. Rubinov, *Abstract convexity and global optimization*, Kluwer, Dordrecht, 2000.

Department of Basic Sciences
Deanship of Preparatory Year Program
University of Hail
Hail, Saudi Arabia
m_amer_latif@hotmail.com

(Received 27 06 2015)

School of Engineering and Science
Victoria University
Melbourne City
Australia

School of Computational and Applied Mathematics
University of the Witwatersrand
Johannesburg
South Africa
sever.dragomir@vu.edu.au

School of Computational and Applied Mathematics
University of the Witwatersrand
Johannesburg
South Africa
Ebrahim.Momoniat@wits.ac.za