PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 104(118) (2018), 115–120 DOI https://doi.org/10.2298/PIM1818115N

SOME REMARKS ON BAZILEVIČ AND STRONGLY STARLIKE FUNCTIONS

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Abstract. We consider several sufficient conditions for a function $f(z)$, analytic in $|z| < 1$, to be strongly starlike or to be a Bazilevič function of order α.

1. Introduction

Let $\mathcal{A}$ be the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of all univalent functions in $\mathbb{D}$. We say that $f(z)$ is subordinate to $F(z)$ in Δ, written as $f(z) \prec F(z)$, if and only if $f(z) = F(\omega(z))$ for some holomorphic function $\omega(z)$ such that $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in \mathbb{D}$. A lot of applications of theory differential subordinations can be found in [**5**].

If $f \in \mathcal{A}$ satisfies $\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0$, $z \in \mathbb{D}$, then $f(z)$ is said to be starlike with respect to the origin in $\mathbb{D}$ and it is denoted by $f(z) \in \mathcal{S}^*$. It is known that $\mathcal{S}^* \subset \mathcal{S}$. We say that a function $f \in \mathcal{A}$ is strongly starlike of order β if and only if

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\pi}{2}\beta \quad (z \in \mathbb{D}), \tag{1.1}$$

for some β $(0 < \beta \leqslant 1)$. Let $\mathcal{SS}^*(\beta)$ denote the class of strongly starlike functions of order β. The class $\mathcal{SS}^*(\beta)$ was introduced independently by Stankiewicz [**7, 8**] and by Brannan and Kirvan [**1**], see also [**3**]. If there exists a starlike function $g(z)$ for which the function $f(z)$ satisfies the condition

$$\left|\arg\left(\frac{zf'(z)}{g(z)}\right)\right| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{D}),$$

then we say $f(z)$ is strongly close-to-convex of order α, $0 < \alpha \leqslant 1$. Moreover, if $f(z)$ satisfies the condition

$$\left|\arg\left(\frac{zf'(z)}{f^{1-\beta}(z)g^{\beta}(z)}\right)\right| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{D}),$$

2010 *Mathematics Subject Classification:* Primary 30C45; Secondary 30C80.

Key words and phrases: Bazilevič function; close-to-convex functions; convex functions; starlike functions; strongly starlike functions.

Communicated by Stevan Pilipović.

then we say $f(z)$ is strongly Bazilevič of type β, $0<\beta$ of order α, $0<\alpha\leqslant 1$.

LEMMA 1.1. [**6**] *Let $p(z)$ be of the form*

$$p(z)=1+\sum_{n=m\geqslant 1}^{\infty}a_nz^n,\qquad a_m\neq 0,\quad (|z|<1),$$

with $p(z)\neq 0$ in $|z|<1$. If there exists a point z_0, $|z_0|<1$, such that

$$|\arg\{p(z)\}|<\pi\alpha/2\quad \textit{in}\quad |z|<|z_0|$$

and $|\arg\{p(z_0)\}|=\pi\alpha/2$ for some $\alpha>0$, then we have

$$\frac{z_0p'(z_0)}{p(z_0)}=ik\alpha,$$

where

$$k\geqslant m(a^2+1)/(2a)\quad \textit{when}\quad \arg\{p(z_0)\}=\pi\alpha/2, \tag{1.2}$$

$$k\leqslant -m(a^2+1)/(2a)\quad \textit{when}\quad \arg\{p(z_0)\}=-\pi\alpha/2, \tag{1.3}$$

where $\{p(z_0)\}^{1/\alpha}=\pm ia$, $a>0$.

2. Main results

THEOREM 2.1. *Let $\{p(z)\}^{1/\beta}=1+c_1z+\cdots$, $\{p(z)\}^{1/\beta}\neq -1$, $p(z)\neq 0$, be analytic in $\mathbb{D}$ and suppose that there exists a point $z_0\in\mathbb{D}$, $z_0\neq 0$ for which*

$$\Big|\frac{zp'(z)}{p(z)}\Big|<\frac{2\beta}{|z_0|+|z_0|^{-1}}\quad (z\in\mathbb{D}), \tag{2.1}$$

for some positive real β. Then we have

$$p(z)\prec\Big(\frac{1+zz_0}{1+zz_0}\Big)^{\beta}\quad (z\in\mathbb{D}), \tag{2.2}$$

$$|\arg\{p(z)\}|\leqslant\frac{\pi\beta}{2}\quad (z\in\mathbb{D}). \tag{2.3}$$

PROOF. Let us put

$$w(z)=z_0^{-1}\frac{(p(z))^{1/\beta}-1}{(p(z))^{1/\beta}+1},\quad w(0)=0$$

or

$$p(z)=\Big(\frac{1+z_0w(z)}{1-z_0w(z)}\Big)^{\beta},$$

then it follows that $w(z)$ is analytic in $\mathbb{D}$, $w(0)=0$, $z_0w\neq\pm 1$ and

$$\frac{zp'(z)}{p(z)}=\frac{2\beta z_0zw'(z)}{1-(z_0w(z))^2}.$$

To prove (2.2) we need $|w(z)|<1$ in $\mathbb{D}$. If there exists a point $z_1\in\mathbb{D}$ such that

$$|w(z)|<1\quad\text{for}\quad |z|<|z_1|$$

$$|w(z_1)|=1,\quad w(z_1)=e^{i\alpha},$$

for some real α, $0 \leqslant \alpha < 2\pi$, then from Jack [**4**] or from Fukui and Sakaguchi's lemma [**2**] or from Lema 1.1 with $\alpha = 1$, we have that

$$\frac{z_1 w'(z_1)}{w(z_1)} = k \geqslant 1.$$

Then it follows that

$$\frac{z_1 p'(z_1)}{p(z_1)} = \frac{2\beta z_0 z_1 w'(z_1)}{1-(z_0 w(z_1))^2} = \frac{2\beta k z_0 e^{i\alpha}}{1-(z_0 e^{i\alpha})^2} = \frac{2\beta k r e^{i\theta}}{1-(re^{i\theta})^2}$$

where $0 < r = |z_0| < 1$ and $\theta = \alpha + \arg\{z_0\}$. On the other hand, we have

$$\begin{aligned}\Big|\frac{2\beta k r e^{i\theta}}{1-(re^{i\theta})^2}\Big| &= 2\beta k \Big|\frac{1}{(re^{i\theta})^{-1} - re^{i\theta}}\Big| \\ &= 2\beta k \bigg\{\frac{1}{\left(\frac{1}{r}-r\right)^2\cos^2\theta + \left(\frac{1}{r}+r\right)^2\sin^2\theta}\bigg\}^{1/2} \\ &= 2\beta k \bigg\{\frac{1}{\frac{1}{r^2}+r^2+2\sin^2\theta}\bigg\}^{1/2} \\ &\geqslant 2\beta \bigg\{\frac{1}{\frac{1}{r^2}+r^2+2}\bigg\}^{1/2} = \frac{2\beta}{r+\frac{1}{r}} = \frac{2\beta}{|z_0| + \frac{1}{|z_0|}}\end{aligned}$$

This contradicts (2.1) and so, we have (2.2) and (2.3). □

In the same way as the above theorem, we can prove the following result.

THEOREM 2.2. *Let* $\{p(z)\}^{1/\beta} = 1 + c_1 z + \cdots$, $\{p(z)\}^{1/\beta} \neq -1$, $p(z) \neq 0$, *be analytic in* $\mathbb{D}$ *and suppose that*

$$\Big|\frac{zp'(z)}{p(z)}\Big| < \beta \quad (z \in \mathbb{D}), \tag{2.4}$$

for some positive real β. *Then we have*

$$p(z) \prec \Big(\frac{1+z}{1+z}\Big)^{\beta} \quad (z \in \mathbb{D}), \tag{2.5}$$

$$|\arg\{p(z)\}| \leqslant \frac{\pi\beta}{2} \quad (z \in \mathbb{D}).$$

COROLLARY 2.1. *Assume that* $f(z) \in \mathcal{A}$ *and* $\{zf'(z)/f(z)\}^{1/\beta} = 1 + c_1 z + \cdots$ *is analytic in* $\mathbb{D}$, $\{zf'(z)/f(z)\}^{1/\beta} \neq -1$, $zf'(z)/f(z) \neq 0$, *and suppose that*

$$\Big|1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\Big| < \beta \quad (z \in \mathbb{D}), \tag{2.6}$$

for some positive real β, $0 < \beta \leqslant 1$. *Then we have*

$$\frac{zf'(z)}{f(z)} \prec \Big(\frac{1+z}{1+z}\Big)^{\beta} \quad (z \in \mathbb{D}) \tag{2.7}$$

or $f(z)$ *is a strongly starlike function of order* β.

Proof. If we put $p(z)=\frac{zf'(z)}{f(z)}$ in Theorem 2.1, then (2.4) becomes (2.6) and (2.5) becomes (2.7). This gives

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right|\leqslant\frac{\pi\beta}{2}\quad(z\in\mathbb{D}).\qquad\square$$

Corollary 2.2. *Assume that $f(z)\in\mathcal{A}$ and $\{zf'(z)/f^{1-\beta}(z)g^{\beta}(z)\}^{1/\gamma}=1+c_1z+\cdots$ is analytic in $\mathbb{D}$, $\{zf'(z)/f^{1-\beta}(z)g^{\beta}(z)\}^{1/\gamma}\neq-1$, $zf'(z)/f(z)\neq 0$, and suppose that*

$$\left|1+\frac{zf''(z)}{f'(z)}-(1-\beta)\frac{zf'(z)}{f(z)}-\beta\frac{zg'(z)}{g(z)}\right|<\gamma\quad(z\in\mathbb{D}),\tag{2.8}$$

for some positive real γ, $0<\gamma\leqslant 1$. Then we have

$$\frac{zf'(z)}{f^{1-\beta}(z)g^{\beta}(z)}\prec\left(\frac{1+z}{1+z}\right)^{\gamma}\quad(z\in\mathbb{D})\tag{2.9}$$

or $f(z)$ is strongly Bazilevič function of type β, $0<\beta$ of order γ.

Proof. If we put

$$p(z)=\frac{zf'(z)}{f^{1-\beta}(z)g^{\beta}(z)}$$

in Theorem 2.1, then (2.4) becomes (2.8) and (2.5) becomes (2.9). This gives

$$\left|\arg\left\{\frac{zf'(z)}{f^{1-\beta}(z)g^{\beta}(z)}\right\}\right|\leqslant\frac{\pi\beta}{2}\quad(z\in\mathbb{D}).\qquad\square$$

Theorem 2.3. *Let $p(z)$ be analytic in $\mathbb{D}$ and suppose that*

$$|zp'(z)+p(z)|<\sqrt{1+\alpha^2}|p(z)|\quad(z\in\mathbb{D}),$$

where $0<\alpha<1$. Then we have $|p(z)|\leqslant\left|\frac{1+z}{1-z}\right|^{\alpha}\quad(z\in\mathbb{D})$.

Proof. Let us put $p(z)=\left(\frac{1+w(z)}{1-w(z)}\right)^{\alpha}$, where $w(z)$ is analytic in $\mathbb{D}$ and $w(0)=0$. If there exists a point $z_0\in\mathbb{D}$ such that $|w(z)|<1$ for $|z|<|z_0|$, and $|w(z_0)|=1$, $w(z_0)=e^{i\theta}$, $0\leqslant\theta<2\pi$, then from Jack [**4**] and Fukui and Sakaguchi's [**2**] result, we have

$$\frac{z_0w'(z_0)}{w(z_0)}=k\geqslant 1.$$

Then it follows that

$$\begin{aligned}|z_0p'(z_0)+p(z_0)|&=|p(z_0)|\left|\frac{z_0p'(z_0)}{p(z_0)}+1\right|\\&=|p(z_0)|\left|\frac{2\alpha z_0w'(z_0)}{1-w^2(z_0)}+1\right|=|p(z_0)|\left|\frac{2\alpha kw(z_0)}{1-w^2(z_0)}+1\right|.\end{aligned}$$

On the other hand, it follows that

$$\left|\frac{2w(z_0)}{1-w^2(z_0)}\right|=\left|\frac{2e^{i\theta}}{1-e^{i2\theta}}\right|=\left|\frac{2}{e^{i\theta}-e^{i\theta}}\right|=\left|\frac{2i}{2\sin\theta}\right|=\left|\frac{i}{\sin\theta}\right|\geqslant 1.$$

This shows that

$$|z_0p'(z_0)+p(z_0)|\geqslant|p(z_0)||1+\alpha ki|>\sqrt{1+\alpha^2}|p(z_0)|.$$

This contradicts the hypothesis of the theorem and therefore it completes the proof. $\square$

THEOREM 2.4. *Let $p(z)$ be analytic in $\mathbb{D}$, $p(0) = 1$ and suppose that*

$$|zp'(z)/p(z) + p(z)| < \beta(\alpha)|p(z)| \quad (z \in \mathbb{D}), \tag{2.10}$$

for some α, $0 \leqslant \alpha < 1$, where

$$\beta(\alpha) = \begin{cases} \sqrt{\frac{1}{4}\alpha^2(\tilde{a}^{1-\alpha} + \tilde{a}^{-1-\alpha})^2 + \alpha(\tilde{a}^{1-\alpha} + \tilde{a}^{-1-\alpha})\cos\frac{\pi(1-\alpha)}{2} + 1}, & 0 < \alpha < 1, \\ 3/2, & \alpha = 1. \end{cases}$$

and $\tilde{a} = \sqrt{(1+\alpha)/(1-\alpha)}$. Then we have

$$p(z) \prec \Big(\frac{1+z}{1-z}\Big)^{\alpha} \quad (z \in \mathbb{D}). \tag{2.11}$$

PROOF. Let us put $p(z) = \big(\frac{1+w(z)}{1-w(z)}\big)^{\alpha}$, where $w(z)$ is analytic in $\mathbb{D}$ and $w(0) = 0$. To prove (2.11) it suffices to show $|\arg\{p(z)\}| < \pi\alpha/2$ $(z \in \mathbb{D})$. If not, then there exists a point z_0, $|z_0| < 1$, such that $|\arg\{p(z)\}| < \pi\alpha/2$ $(|z| < |z_0|)$ and $|\arg\{p(z_0)\}| = \pi\alpha/2$, then from Nunokawa's lemma 1.1, we have

$$\frac{z_0p'(z_0)}{p(z_0)} = ik\alpha,$$

where $p(z_0)^{1/\alpha} = \pm ia$, $a > 0$ and k is a real number satisfying (1.2) or (1.3) . For the case $\arg\{p(z_0)\} = \pi\alpha/2$, it follows that

$$\begin{aligned}\Big|\frac{z_0p'(z_0)}{p(z_0)} + p(z_0)\Big| &= |p(z_0)|\Big|\frac{z_0p'(z_0)}{p(z_0)}\frac{1}{p(z_0)} + 1\Big| \\ &= |p(z_0)|\Big|ik\alpha\frac{1}{(ia)^{\alpha}} + 1\Big| = |p(z_0)|\Big|\frac{k\alpha}{a^{\alpha}}e^{i\pi(1-\alpha)/2} + 1\Big|\end{aligned}$$

and so when $\alpha = 1$, we have by (1.2)

$$\Big|\frac{z_0p'(z_0)}{p(z_0)} + p(z_0)\Big| = |p(z_0)|\Big|\frac{k\alpha}{a^{\alpha}}e^{i\pi(1-\alpha)/2} + 1\Big| \geqslant |p(z_0)|\frac{3}{2}$$

and when $0 < \alpha < 1$, we have by (1.2)

$$\begin{aligned}\Big|\frac{k\alpha}{a^{\alpha}}e^{i\pi(1-\alpha)/2} + 1\Big|^2 &= \Big(\frac{k\alpha}{a^{\alpha}}\Big)^2 + \frac{2k\alpha}{a^{\alpha}}\cos\{\pi(1-\alpha)/2\} + 1 \\ &\geqslant \Big(\frac{\alpha}{2}(a^{1-\alpha}+a^{-1-\alpha})\Big)^2 + \alpha(a^{1-\alpha}+a^{-1-\alpha})\cos\{\pi(1-\alpha)/2\}+1.\end{aligned}$$

Putting $g(a) = a^{1-\alpha} + a^{-1-\alpha}$, $a > 0$, then $g(a)$ takes the minimum value at $\tilde{a} = \sqrt{(1+\alpha)/(1-\alpha)}$. Therefore, for the case $0 < \alpha < 1$, we have

$$\begin{aligned}&\Big|\frac{z_0p'(z_0)}{p(z_0)} + p(z_0)\Big| \\ &\quad\geqslant |p(z_0)|\sqrt{\Big(\frac{\alpha}{2}(\tilde{a}^{1-\alpha} + \tilde{a}^{-1-\alpha})\Big)^2 + \alpha(\tilde{a}^{1-\alpha} + \tilde{a}^{-1-\alpha})\cos\{\pi(1-\alpha)/2\} + 1}.\end{aligned}$$

This contradicts hypothesis (2.10) of the theorem and therefore it completes the proof for the first case. For the case $\arg\{p(z_0)\} = -\pi\alpha/2$ the proof is analogous. □

It is easy to see that Theorem 2.4 provides the sufficient conditions for a function $f(z) \in \mathcal{A}$ to be strongly starlike of order α, see (1.1).

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(Received 26 05 2016)

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