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THE MODULE MULTIPLIER EMBEDDING PROBLEM

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Abstract. We precise properties, embedding and isomorphism theorems of Banach module multipliers between a Banach algebra and modules on it. This task for Banach dual valued multipliers is a problem of interest because of its connection with the theory of induced Banach representations of Banach algebras and Frobenius reciprocity theorems.

1. Introduction

The concept of *multipliers* appeared in harmonic analysis in connection with the theory of Fourier series [**6**]. Since their formal introduction in 1956, operators of Banach algebras which commute with left or right multiplications are a matter of huge research [**7**]. Let $\mathcal{A}$ be an abelian semisimple Banach algebra, considered as an algebra of continuous functions over its regular maximal ideal space $X_{\mathcal{A}}$. By a *multiplier* of $\mathcal{A}$ is meant a function f on $X_{\mathcal{A}}$ such that $fG(\mathcal{A}) \subset G(\mathcal{A})$, where G denotes the Gélfand transform of $\mathcal{A}$. In this context it is readily seen that f determines a unique function m of $\mathcal{A}$ into $\mathcal{A}$ so that $fG(a) = G(m(a))$ if $a \in \mathcal{A}$. Moreover, m becomes a bounded linear operator on $\mathcal{A}$ and

$$m(a)b = am(b) \quad \text{if } a, b \in \mathcal{A}. \tag{1.1}$$

This motivates to say that a linear operator of any algebra that satisfies the equation (1.1) is a *multiplier*. For a first systematic study of multipliers on some commutative Banach algebras the reader can see [**19**].

Let ${}_{\mathcal{A}}H(\mathcal{A}, M)$ be the Banach space, endowed with the uniform norm, of *left M-valued module multipliers on a Banach algebra $\mathcal{A}$*, or else bounded linear operators T between $\mathcal{A}$ and a left Banach $\mathcal{A}$-module M so that the identity $T(am) = aT(m)$ holds if $a \in \mathcal{A}$, $m \in M$. The structure of ${}_{\mathcal{A}}H(\mathcal{A}, M)$ reveals relevant information concerning $\mathcal{A}$, M and their conjugate spaces. Perhaps, on a nondiscrete locally compact group G, Wendel's isomorphism theorem ${}_{L^1(G)}H(L^1(G), L^1(G)) \approx M(G)$

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is one of the major insights in this area [**21**]. This theorem was restated in a more general setting succeeding a characterization of ${}_{L^1(G)}H(L^1(G), M)$ for M within a wide class of Banach $L^1(G)$-modules [**8**, **9**, **11**]. For consideration of more general bounded $\mathcal{A}$-module multipliers between Banach $\mathcal{A}$-modules and the important notion of $\mathcal{A}$-tensor products of Banach $\mathcal{A}$-modules the reader can see [**14**]. Given a left Banach $\mathcal{A}$-module M and a right Banach $\mathcal{A}$-module N the formula $(M \otimes_{\mathcal{A}} N)^* \approx_{\mathcal{A}} H(M, N^*)$ assimilates both constructions (cf. see [**14**, Corollary 2.13] or [**15**, (1.1)]).

Let G be a topological group and let π and ϱ be representations of G on Banach spaces X_π and X_ϱ respectively. Let $\mathcal{B}_G(\pi, \varrho)$ be the space of bounded operators $A \in \mathcal{B}(X_\pi, X_\varrho)$ so that $A\pi(g) = \varrho(g)A$ if $g \in G$. Let G be a locally compact separable group and let K be a closed subgroup of G. Let us assume that G/K (right cosets) has a measure invariant under the operation of G by right translations on G/K. Let π and ϱ be representations of K and G respectively. In these conditions Moore proved the existence of an isometric isomorphism of $\mathcal{B}_K(\pi, \varrho|_K)$ onto $\mathcal{B}_G(\pi^{\text{ind}}, \varrho)$, where $\varrho|_K$ and π^{ind} denote the restriction of the representation ϱ to K and the induced representation by π of G [**12**, **13**]. In particular, Moore remarked that his development works if X_ϱ is reflexive or, more generally, a dual Banach space.

Throughout this article let M be a left Banach $\mathcal{A}$-module. As usualy M is said to be an *essential* $\mathcal{A}$-module if $\mathcal{A} \cdot M$, the linear manifold spanned by elements of the form am with $a \in \mathcal{A}$ and $m \in M$, is dense in M. We shall write $M_e = (\mathcal{A} \cdot M)^-$ to the *essential submodule* of M. Further, the set $\{m \in M : \mathcal{A}m = \{0_M\}\}$ is a submodule of M, which we call the *order submodule* M_0 of M. We say that M is *order-free* if $M_0 = \{0_M\}$.

If $\mathcal{A}$ is an approximately bounded Banach algebra and M is a left $\mathcal{A}$-module, then there is an isometric module isomorphism $[{}_{\mathcal{A}}H(\mathcal{A}, M)]_e \approx M_e$ (see [**14**, Theorem 4.5]). Moreover, if M is an essential left $\mathcal{A}$-module, then there is an isometric right $\mathcal{A}$-module isomorphism ${}_{\mathcal{A}}H(\mathcal{A}, (M^*)_e) \approx M^*$, while if besides M is reflexive, then ${}_{\mathcal{A}}H(\mathcal{A}, M) \approx M$ (Ibid., Theorem 8.9 and Corollary 8.10 respectively). The question (cf. [**14**, 8.17]) concerning whether this last isomorphism even holds if M is an essential dual Banach space was raised in connection with Moore's work. The following two conditions, each sufficient for an affirmative answer, are known: Let χ_M be the natural immersion of M into its second dual space M^{**}. If $\mathcal{A}\chi_M(M) \subseteq \chi_M(M)$ or the left $\mathcal{A}$-module M^{***} is essential, then $H_{\mathcal{A}}(\mathcal{A}, M^*) \approx M^*$ [**3**].

In our main result (Theorem 2.1), we describe several properties of ${}_{\mathcal{A}}H(\mathcal{A}, M)$. The connection with the above problem will be established in Theorem 2.2. Then we will determine ${}_{\mathcal{A}}H(\mathcal{A}, M)$ if M is a dual Banach space, under suitable conditions on the underlying algebra, depending on whether M is essential or not. We shall analyze various related examples in the context of group and measure algebras, W^* algebras, operator algebras, reflexive spaces, etc..

2. On the class ${}_{\mathcal{A}}H(\mathcal{A}, M)$

THEOREM 2.1. *The following assertions hold:*

(1) ${}_{\mathcal{A}}H(\mathcal{A}, M)$ *has a left Banach* $\mathcal{A}$*-module structure.*

(2) *Let* $\theta : M \to_{\mathcal{A}} H(\mathcal{A}, M)$ *be given as* $\theta(m)(a) = am$ *if* $a \in \mathcal{A}$, $m \in M$. *Then* $\theta \in {}_{\mathcal{A}}H(M, {}_{\mathcal{A}}H(\mathcal{A}, M))$.

(3) M *is order-free if and only if* θ *is injective.*

Further, let us assume that $\mathcal{A}$ *has a right bounded approximate identity. Then:*

(4) *If* $T \in {}_{\mathcal{A}}H(\mathcal{A}, M)$ *there is* $\bar{m}^{**} \in M^{**}$ *so that* $\chi_M(T(a)) = a\bar{m}^{**}$ *if* $a \in \mathcal{A}$.

(5) *The set* $\mathcal{I}_{\mathcal{A}}(M) = \{m^{**} \in M^{**} : \mathcal{A}m^{**} \subseteq \chi_M(M)\}$ *is a Banach subspace of* M^{**} *and there is an isomorphism of Banach left* $\mathcal{A}$*-modules*

$$ {}_{\mathcal{A}}H(\mathcal{A}, M) \approx \mathcal{I}_{\mathcal{A}}(M)/(M^{**})_0. $$

(6) *If*

$$ \mathcal{A} \cdot M^{**} \subseteq \chi_M(M) \tag{2.1} $$

then ${}_{\mathcal{A}}H(\mathcal{A}, M) \approx (M^* \cdot \mathcal{A})^*$.

(7) *If* M^* *is an essential right* $\mathcal{A}$*-Banach module then* ${}_{\mathcal{A}}H(\mathcal{A}, M) \approx \mathcal{I}_{\mathcal{A}}(M)$.

(8) *Let* $F\colon {}_{\mathcal{A}}H(\mathcal{A}, M) \to (M^* \cdot \mathcal{A})^*$, $F(T) = (\imath_{M^* \cdot \mathcal{A}})^*(\bar{m}^{**})$, *where the element* $\bar{m}^{**} \in M^{**}$ *is given according to* 2.1(4) *and* $\imath_{M^* \cdot \mathcal{A}}$ *denotes the inclusion map* $M^* \cdot \mathcal{A} \hookrightarrow M^*$. *Then* $F \in {}_{\mathcal{A}}H[{}_{\mathcal{A}}H(\mathcal{A}, M), (M^* \cdot \mathcal{A})^*]$ *is a monomorphism. Moreover, it is an isomorphism of Banach spaces if and only if* $\mathcal{I}_{\mathcal{A}}(M) = M^{**}$.

Proof. (1) Given $a \in \mathcal{A}$ and $T \in {}_{\mathcal{A}}H(\mathcal{A}, M)$ let $(a * T)(b) = bT(a)$ if $a, b \in \mathcal{A}$. It is easy to see that $a * T$ is $\mathbb{C}$-linear bounded operator and that

$$ \|a * T\| \leqslant \|T(a)\| \leqslant \|T\|\|a\|. \tag{2.2} $$

Given $b, c \in \mathcal{A}$ we get $(a * T)(bc) = (bc)T(a) = b(cT(a)) = b(a * T)(c)$, i.e., $a * T \in {}_{\mathcal{A}}H(\mathcal{A}, M)$. Further,

$$ (a * (b * T))(c) = c(b * T)(a) = c(aT(b)) = cT(ab) = ((ab) * T)(c) $$

and we obtain that $(ab) * T = a(b * T)$. By (2.2) ${}_{\mathcal{A}}H(\mathcal{A}, M)$ becomes a left Banach $\mathcal{A}$- module.

(2) Clearly $\theta(m)$ is a $\mathbb{C}$-linear operator if $m \in M$ and $\|\theta(m)\| \leqslant \|m\|$. Besides, given $a, b \in \mathcal{A}$ it is seen that $\theta(m)(ab) = (ab)m = a(bm) = a\theta(m)(b)$. Consequently, $\theta(m) \in {}_{\mathcal{A}}H(\mathcal{A}, M)$ and $\theta \in B(M, {}_{\mathcal{A}}H(\mathcal{A}, M))$. Moreover, as

$$ (a * \theta(m))(b) = b\theta(m)(a) = b(am) = \theta(am)(b) $$

our claim holds.

(3) It is obvious.

(4) Let $\{e_i\}_{i \in I}$ be a right bounded approximate identity of $\mathcal{A}$. As it is bounded by Alaoglu's theorem there is a subnet $\{e_{i_j}\}_{j \in J}$ and an element $\bar{m}^{**} \in \mathcal{A}^{**}$ so that $\bar{m}^{**} = w^* - \lim_{j \in J} \chi_M\{T(e_{i_j})\}$. Given $T \in {}_{\mathcal{A}}H(\mathcal{A}, M)$, $a \in \mathcal{A}$ and $m^* \in M^*$ we now obtain

$$ \begin{aligned} \langle m^*, \chi_M(T(a))\rangle &= \langle T(a), m^*\rangle = \lim_{j \in J} \langle T(ae_{i_j}), m^*\rangle = \lim_{j \in J} \langle aT(e_{i_j}), m^*\rangle \\ &= \lim_{j \in J} \langle (Te_{i_j)}, m^* a\rangle = \langle m^* a, \bar{m}^{**}\rangle = \langle m^*, a\bar{m}^{**}\rangle \end{aligned} $$

and the fourth assertion follows.

(5) Since χ_M is isometric and

$$|\langle m^*, \bar{m}^{**}\rangle| = \lim_{j\in J} |\langle T(e_{i_j}), m^*\rangle| \leqslant \|m^*\| \lim_{j\in J} \|T(e_{i_j})\|$$

we see that

$$\|T\| \leqslant \|\bar{m}^{**}\| \leqslant \|T\| \sup_{i\in I} \|e_i\|. \tag{2.3}$$

The set $\mathcal{I}(M)$ is clearly a subspace of M^{**}, and it becomes closed because χ_M is isometric and M is complete. Let

$$\kappa\colon {}_{\mathcal{A}}H(\mathcal{A}, M) \to \mathcal{I}(M)/(M^{**})_0,$$
$$\kappa(T) = m^{**} + (M^{**})_0 \quad \text{if} \quad \chi_M(T(a)) = am^{**} \quad \text{for} \quad a \in \mathcal{A}.$$

Clearly κ is a well defined complex linear functional. So, using (2.3), the former notation and the definition of the quotient norm, we get

$$\|T\| \leqslant \|\kappa(T)\| \leqslant \|T\| \sup_{i\in I} \|e_i\|$$

and so κ becomes a bounded monomorphism. The relation

$$a(m^{**} + (M^{**})_0) = am^{**} + (M^{**})_0 \quad \text{if } a \in \mathcal{A},\ m^{**} \in M^{**}, \tag{2.4}$$

provides to $\mathcal{I}(M)/(M^{**})_0$ of a Banach left $\mathcal{A}$-module structure. By considering $T \in {}_{\mathcal{A}}H(\mathcal{A}, M)$, $a, b \in \mathcal{A}$ and $m^{**} \in \kappa(T)$, it follows that

$$\chi_M((a * T)(b)) = \chi_M(bT(a)) = \chi_M(T(ba)) = (ba)m^{**} = b(am^{**}).$$

Hence $\kappa(a * T) = am^{**} + (M^{**})_0$ because b is arbitrary and κ turns into a left homomorphism by (2.4). Now, if $m^{**} \in \mathcal{I}_{\mathcal{A}}(M)$ and $a \in \mathcal{A}$ there is a unique $m_a \in M$ so that $am^{**} = \chi_M(m_a)$. If $T_{m^{**}}(a) = m_a$ we get a function $T_{m^{**}}\colon \mathcal{A} \to M$ that is $\mathbb{C}$-linear. For, if $z \in \mathbb{C}$ and $a, b \in \mathcal{A}$ we have

$$\begin{aligned}\chi_M(T_{m^{**}}(za + b)) &= (za + b)m^{**} = z(am^{**}) + bm^{**} \\ &= z\chi_M(T_{m^{**}}(a)) + \chi_M(T_{m^{**}}(b)) \\ &= \chi_M(zT_{m^{**}}(a) + T_{m^{**}}(b))\end{aligned}$$

and the claim holds because χ is injective. Indeed, as

$$\|T_{m^{**}}(a)\| = \|am^{**}\| \leqslant \|a\| \|m^{**}\| \quad \text{if } a \in \mathcal{A},$$

then $T_{m^{**}}$ is bounded. Moreover, as $\chi_M \in {}_{\mathcal{A}}H(M, M^{**})$ given $a, b \in \mathcal{A}$ we can write

$$\chi_M(T_{m^{**}}(ab)) = (ab)m^{**} = a(bm^{**}) = a\chi_M(T_{m^{**}}(b)) = \chi_M(aT_{m^{**}}(b)),$$

i.e., $T_{m^{**}} \in {}_{\mathcal{A}}H(\mathcal{A}, M)$. It is now plain that $\kappa(T_{m^{**}}) = m^{**} + (M^{**})_0$.

(6) It is easily seen that

$$(M^{**})_0 = (M^* \cdot \mathcal{A})^o \tag{2.5}$$

and the hypothesis is equivalent to say that $M^{**} = \mathcal{I}_{\mathcal{A}}(M)$. It is straightforward to see that the well known isometric isomorphism $M^{**}/(M^* \cdot \mathcal{A})^o \approx (M^* \cdot \mathcal{A})^*$ becomes an isomorphism of Banach left $\mathcal{A}$-modules (cf. [**16**, §4.8, p. 96]).

(7) Since M^* is essential by (2.5) M^{**} becomes an order-free left Banach $\mathcal{A}$-module and our claim follows by 2.1(5).

(8) By construction it is easily seen that F is a well defined left bounded multiplier between Banach modules. It is easy to see that F is injective. Given $m^{**} \in M^{**}$, if F is surjective let us choose $m^{**} \in {}_{\mathcal{A}}H(\mathcal{A}, M)$ so that $F(T) = m^{**}$. Given $a \in \mathcal{A}$ and $m^* \in M^*$ we have

$$\begin{aligned}\langle m^*, am^{**}\rangle &= \langle m^*a, m^{**}\rangle = \langle m^*a, F(T)\rangle \\ &= \langle m^*a, \bar{m}^{**}\rangle = \langle m^*, \chi_M(T(a))\rangle\end{aligned}$$

i.e., $m^{**} \in \mathcal{I}_{\mathcal{A}}(M)$. On the other hand, given $\lambda \in (M^* \cdot \mathcal{A})^*$ by the Hahn–Banach theorem there is an extension $\Lambda \in M^{**}$. Thus $\Lambda \in \mathcal{I}_{\mathcal{A}}(M)$ and as in 2.1(5) we can choose $L \in {}_{\mathcal{A}}H(\mathcal{A}, M)$ so that $\kappa(L) = \Lambda + (M^{**})_0$. Consequently $\chi_M(L(a)) = a\Lambda$ if $a \in \mathcal{A}$ and so $\lambda = (\imath_{M^*\cdot\mathcal{A}})^*(\Lambda) = F(L)$. □

Example 2.1. Let G be a locally compact group. The Banach convolution algebra $M(G)$ of complex regular Borel measures on G is unital while the group algebra $L^1(G)$ is approximately unital, being unital only in the discrete case (cf. [**4**, Theorem 3.3.23]). If $\mathcal{A} = M = M(G)$ or $\mathcal{A} = M = L^1(G)$ the condition (2.1) holds if and only if G is finite or compact respectively (cf. [**20**, Theorems 5 and 6]).

Example 2.2. Clearly (2.1) is satisfied when M is reflexive. For instance, again on an underlying locally compact group G, this covers the case of $L^p(G)$ considered as a left-$L^1(G)$-module if $1 < p < \infty$.

Example 2.3. Condition (2.1) holds for W^* algebras, i.e., C^* algebras that are dual Banach algebras. For, any C^* algebra is approximately unital (cf. [**17**, Lemma 1.1]) and a B^* algebra is a dual algebra if and only if its immersion in its second conjugate space is an ideal (cf. [**18**, Theorem 5.1]).

Example 2.4. Let $\mathcal{A} = M = A(X)$ be the Banach algebra of approximable operators on a Banach space X whose dual has the bounded approximation property. Then X itself has the bounded approximation property, $\mathcal{A}$ coincides with the Banach algebra of compact operators on X and it has also a bounded approximate identity (cf. [**1**, **5**]). Now, (2.1) holds if and only if X is reflexive (cf. [**20**, Theorem 4]).

Example 2.5. On a locally compact group G we consider $L^1(G)$ as a left Banach $M(G)$-module (cf. [**10**, Theorem 20.12]). Now (2.1) is no longer true even in the discrete case. For instance, $\chi_{l^1(\mathbb{Z})}(l^1(\mathbb{Z})) \subsetneq M(\mathbb{Z})\cdot l^\infty(\mathbb{Z})^*$. For, let us consider the probability measure $\mu \in M(\mathbb{Z})$ so that $\mu(E) = 3^{-1}\sum_{m\in E} 2^{-|m|}$ if $E \in \mathcal{P}(\mathbb{Z})$. By Alaoglu's theorem there is an increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ that determines a well defined functional

$$\Theta = w^* - \lim_{k\to\infty} \chi_{l^1(\mathbb{Z})}[1_{\{-n_k,\ldots,n_k\}}]/(2n_k+1)$$

in $l^\infty(\mathbb{Z})^*$. It will suffice to see that $\mu \cdot \Theta \notin \chi_{l^1(\mathbb{Z})}(l^1(\mathbb{Z}))$. Indeed, we can write $\mu \cdot \Theta = \chi_{l^1(\mathbb{Z})}(f)$ for some $f \in l^1(\mathbb{Z})$ and given $m \in \mathbb{Z}$ we have

$$\begin{aligned} f(m) &= \langle f, 1_{\{m\}}\rangle = \langle 1_{\{m\}}, \mu\cdot\Theta\rangle = \langle 1_{\{m\}}\cdot\mu, \Theta\rangle \\ &= \lim_{k\to\infty}\langle 1_{\{-n_k,\dots,n_k\}}, 1_{\{m\}}\cdot\mu\rangle/(2n_k+1) \\ &= \lim_{k\to\infty}\frac{1}{2n_k+1}\langle 1_{\{-n_k,\dots,n_k\}}, \{3^{-1}\cdot 2^{-|p-m|}\}_{p\in\mathbb{Z}}\rangle \\ &= \lim_{k\to\infty}\frac{3^{-1}}{2n_k+1}\sum_{p\in\mathbb{Z}:|p|\leqslant n_k} 2^{-|p-m|} = \lim_{k\to\infty}\frac{3^{-1}}{2n_k+1}\sum_{q=-n_k-m}^{n_k-m} 2^{-|q|} \\ &= \lim_{k\to\infty}\frac{3^{-1}}{2n_k+1}\Big[\frac{1-2^{-n_k-m-1}}{1/2} - 1 + \frac{1-2^{-n_k+m-1}}{1/2}\Big] = 0, \end{aligned}$$

i.e., $f = 0_{l^1(\mathbb{Z})}$. Nevertheless, $\langle 1_{\mathbb{Z}}, \mu\cdot\Theta\rangle = \langle 1_{\mathbb{Z}}\cdot\mu, \Theta\rangle = \langle 1_{\mathbb{Z}}, \Theta\rangle = 1$ and we get a contradiction.

Theorem 2.2. *Let $\mathcal{A}$ be a Banach algebra with a right bounded approximate identity and let N be a nontrivial right Banach $\mathcal{A}$-module.*

(1) *If N is essential and reflexive then ${}_{\mathcal{A}}H(\mathcal{A}, N^*) \approx N^*$.*
(2) *If N is not essential then ${}_{\mathcal{A}}H(\mathcal{A}, N^*) \approx \mathcal{I}_{\mathcal{A}}(N^*)/(N^{***})_0$, where the quotient is not trivial.*
(3) *If N is not essential and reflexive then ${}_{\mathcal{A}}H(\mathcal{A}, N^*) \approx N^*/(N\cdot\mathcal{A})^o$, where the quotient is not trivial.*

Proof. (1) By Cohen's factorization theorem [**2**] is $N = N\mathcal{A}$, i.e., any element $n \in N$ can be factored as $n = n_1 a$ for suitable $n_1 \in N$ and $a \in \mathcal{A}$. Moreover, if N is reflexive $\chi_N(N) = N^{**}\cdot\mathcal{A}$ and $\mathcal{I}_{\mathcal{A}}(N^*) = N^*$ because N^* becomes reflexive. The result now follows from Theorem 2.1(7).

(2) Clearly it will suffice to prove the existence of $n^{***} \in N^{***}$ so that

$$\{0_{N^{***}}\} \neq \mathcal{A}n^{***} \subseteq \chi_{N^*}(N^*). \tag{2.6}$$

Precisely, $(N\cdot\mathcal{A})^o \neq \{0_{N^*}\}$ because N is not essential. If $(N\cdot\mathcal{A})^o = N^*$ then

$$\{0_{N^{**}}\} = (N\cdot\mathcal{A})^{oo} = \chi_N(N\cdot\mathcal{A})^{-w^*} \supseteq \chi_N(N\cdot\mathcal{A})^{-w}.$$

Hence $\chi_N(N\cdot\mathcal{A})^- = \{0_{N^{**}}\}$ (cf. [**16**, Theorem 3.12]), which is impossible because χ_N is injective and N is a nontrivial $\mathcal{A}$-module. Consequently, let $n^* \in N^* - (N\cdot\mathcal{A})^0$. In particular, there exist $\tilde{a} \in \mathcal{A}$ and $n_0 \in N$ so that $\langle n_0\tilde{a}, n^*\rangle \neq 0$. If $n^{***} \triangleq \chi_{N^*}(n^*)$ then

$$\langle \chi_N(n_0), \tilde{a}n^{***}\rangle = \langle \chi_N(n_0), \chi_{N^*}(\tilde{a}n^*)\rangle = \langle n_0, \tilde{a}n^*\rangle = \langle n_0\tilde{a}, n^*\rangle \neq 0,$$

and clearly n^{***} satisfies (2.6).

(3) By the reflexivity of N the Banach left $\mathcal{A}$-module isomorphism follows by Theorem 2.1(8). Moreover, $N\cdot\mathcal{A} \approx N^{**}\cdot\mathcal{A}$ and $N\cdot\mathcal{A} \subsetneq N$ because N is not essential. Therefore, by the Hahn–Banach theorem there exists a nonzero $n^* \in N^*$ that annihilates on $N\cdot\mathcal{A}$ and the result holds. □

Example 2.6. The condition of reflexivity in Theorem 2.2 is necessary. For instance, the abelian Banach algebra $\mathcal{A} = c_0(\mathbb{N})$ is approximately bounded. Since its conjugate space $l^1(\mathbb{N})$ is a $c_0(\mathbb{N})$-essential Banach module by Theorem 2.1(7) is

${}_{c_0(\mathbb{N})}H(c_0(\mathbb{N}), l^1(\mathbb{N})) \approx \mathcal{I}_{c_0(\mathbb{N})}(l^1(\mathbb{N}))$. However, $\imath_{l^1(\mathbb{N})}(l^1(\mathbb{N})) \subsetneqq \mathcal{I}_{c_0(\mathbb{N})}(l^1(\mathbb{N}))$. For, the semidirect product $l^1(\mathbb{N})^{**} = \imath_{l^1(\mathbb{N})}(l^1(\mathbb{N})) \ltimes [\imath_{c_0(\mathbb{N})}(c_0(\mathbb{N}))]^0$ holds (cf. [**4**, Example 2.6.22(iii)]). Let $\Phi \in l^1(\mathbb{N})^{**}$, $\Phi = \imath_{l^1(\mathbb{N})}(\lambda) + \Psi$, with unique $\lambda \in l^1(\mathbb{N})$ and $\Psi \in [\imath_{c_0(\mathbb{N})}(c_0(\mathbb{N}))]^0$. If $x \in c_0(\mathbb{N})$ it is seen that $x\Psi \in [\imath_{c_0(\mathbb{N})}(c_0(\mathbb{N}))]^0$ and $x\imath_{l^1(\mathbb{N})}(\lambda) = \imath_{l^1(\mathbb{N})}(x\lambda)$. Thus $\Phi \in \mathcal{I}_{c_0(\mathbb{N})}(l^1(\mathbb{N}))$ if and only if $x\Psi = 0_{l^1(\mathbb{N})^{**}}$ for all $x \in c_0(\mathbb{N})$. By the Hahn–Banach theorem we can choose a nonzero functional $\Lambda \in [\imath_{c_0(\mathbb{N})}(c_0(\mathbb{N}))]^0$ and it is plain that $\Lambda \in \mathcal{I}_{c_0(\mathbb{N})}(l^1(\mathbb{N})) - \imath_{l^1(\mathbb{N})}(l^1(\mathbb{N}))$.

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