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REMARKS ON THE SPARK OF A MATRIX AND THE NULL SPACE PROPERTY

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Abstract. We discuss the behavior of the spark of a matrix under multiplication by invertible matrices. We also give a necessary and sufficient condition for an NSP constant of a matrix to be finite.

1. Introduction and preliminaries

The notions of spark of a matrix (introduced by Donoho and Elad) and null space property (introduced by Cohen, Dahmen and DeVore) come from the mathematical theory of Compressed Sensing. The spark is a dual notion for the rank of a matrix. The null space property plays an essential role in the ℓ^1 norm minimization methods of Compressed Sensing.

Two natural questions arose in a seminar discussion: "What are the relationships between the sparks of two equivalent matrices?" and "When is an NSP constant of a matrix infinite?". We were not able to find explicit answers to these questions in the literature of the subject.

The present note provides a partial answer to the first question and the complete answer to the second.

We will use only elementary linear algebra, matrix theory, and functional analysis.

Throughout the text, m and n are positive integers, $N = \{1, \dots, n\}$, and $\mathbb{F}$ is a field. The elements of the $\mathbb{F}$-vector space $\mathbb{F}^n$ are understood to be columns. We denote the zero vector in $\mathbb{F}^n$ by $\mathbf{0}$. For a positive integer ℓ and vectors $v_1, \dots, v_\ell \in \mathbb{F}^n$, we define $\mathrm{Span}_{\mathbb{F}}(v_1, \dots, v_\ell)$ to be the linear subspace of the space $\mathbb{F}^n$ spanned by $v_1, \dots, v_\ell$. We denote by $\mathcal{M}_{m\times n}(\mathbb{F})$ the set of all the $m \times n$ matrices over $\mathbb{F}$. The general linear group of degree n over the field $\mathbb{F}$ is denoted by $\mathcal{GL}_n(\mathbb{F})$ (i.e., $\mathcal{GL}_n(\mathbb{F}) = \{V \in \mathcal{M}_{n\times n}(\mathbb{F}) : \det(V) \neq 0\}$). Moreover, $\#S$ denotes the cardinality of a set S.

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2. Some properties of the spark of a matrix

We start with a definition, which was originally formulated in [**2**].

Definition 2.1. Let $C_1, \dots, C_n \in \mathbb{F}^m$ be the columns of a matrix $A \in \mathcal{M}_{m\times n}(\mathbb{F})$. The spark of A is defined to be the infimum of the set of all positive integers ℓ such that

$$\exists j_1, \dots, j_\ell \in N : \begin{cases} j_1 < \cdots < j_\ell, \\ C_{j_1}, \dots, C_{j_\ell} \text{ are linearly dependent (over } \mathbb{F}). \end{cases}$$

Let us also recall the definition of the Hamming weight.

Definition 2.2. The Hamming weight of a vector $x = [x_1, \dots, x_n]^{\mathrm{T}} \in \mathbb{F}^n$ is defined by $\|x\|_0 = \#\{j \in N : x_j \neq 0\}$.

The following properties of the spark of a matrix are obvious or well known [**2**].

Proposition 2.1. *Let* $A \in \mathcal{M}_{m\times n}(\mathbb{F})$. *Then*

(i) $\operatorname{spark}(A) \in N \cup \{+\infty\}$,
(ii) $\operatorname{spark}(A) = +\infty$ *if and only if* $\operatorname{rank}(A) = n$,
(iii) $\operatorname{spark}(A) = 1$ *if and only if A has a zero column,*
(iv) $\operatorname{spark}(A) \leqslant \operatorname{rank}(A) + 1$ *whenever* $\operatorname{spark}(A) \neq +\infty$.

Moreover, $\operatorname{spark}(A)$ *coincides with the infimum of the set* $\{\|x\|_0 : x \in \mathbb{F}^n \setminus \{\mathbf{0}\}, Ax = \mathbf{0}\}$.

We will denote by $\operatorname{Ker}(A)$ the kernel of a matrix $A \in \mathcal{M}_{m\times n}(\mathbb{F})$ (i.e., $\operatorname{Ker}(A) = \{x \in \mathbb{F}^n : Ax = \mathbf{0}\}$). For a vector $x = [x_1, \dots, x_n]^{\mathrm{T}} \in \mathbb{F}^n$ and a set $S \subseteq N$, we define $x_S = [y_1, \dots, y_n]^{\mathrm{T}} \in \mathbb{F}^n$ by

$$y_j = \begin{cases} x_j, & \text{if } j \in S, \\ 0, & \text{otherwise.} \end{cases}$$

In the sequel, we make use of a simple lemma.

Lemma 2.1. *Let* $A \in \mathcal{M}_{m\times n}(\mathbb{F})$. *Assume that* $r := \operatorname{rank}(A) < n$. *Then the following conditions are equivalent for an integer* $s \in \{1, \dots, r+1\}$*:*

(1) $\exists\, V \in \mathcal{GL}_n(\mathbb{F}) : \operatorname{spark}(AV) = s$,
(2) *there exists a linear subspace L of the space $\mathbb{F}^n$ such that* $\dim_{\mathbb{F}} L = n - r$ *and* $s = \min_{x \in L \setminus \{\mathbf{0}\}} \|x\|_0$.

Proof. Suppose that condition (2) is satisfied. Since $n - r = \dim_{\mathbb{F}} \operatorname{Ker}(A)$, there exists a matrix $V \in \mathcal{GL}_n(\mathbb{F})$ such that $L = \{V^{-1}x : x \in \operatorname{Ker}(A)\}$. Notice that $\operatorname{Ker}(AV) = \{V^{-1}x : x \in \operatorname{Ker}(A)\}$. Therefore, by Proposition 2.1, we get

$$s = \min\{\|x\|_0 : x \in \operatorname{Ker}(AV) \setminus \{\mathbf{0}\}\} = \operatorname{spark}(AV).$$

Condition (1) follows. If condition (1) is satisfied, define $L = \operatorname{Ker}(AV)$. □

We are in a position to state and prove the main result of the section ("What are the relationships between the sparks of two equivalent matrices?").

Theorem 2.1. *The following properties hold true for a matrix* $A \in \mathcal{M}_{m\times n}(\mathbb{F})$*:*

(i) $\forall U \in \mathcal{GL}_m(\mathbb{F}) : \operatorname{spark}(UA) = \operatorname{spark}(A)$,
(ii) $\operatorname{spark}(AP) = \operatorname{spark}(A)$ *for every permutation matrix* $P \in \mathcal{M}_{n\times n}(\mathbb{F})$,
(iii) *if* $\operatorname{rank}(A) = n-1$, *then* $\forall s \in N \exists V \in \mathcal{GL}_n(\mathbb{F}) : \operatorname{spark}(AV) = s$,
(iv) *if A has no zero column, then* $\exists V \in \mathcal{GL}_n(\mathbb{F}) : \operatorname{spark}(AV) = \operatorname{spark}(A) - 1$.

Moreover, if the field $\mathbb{F}$ has at least $n+1$ elements and $\operatorname{rank}(A) = n-2$, *then*

$$\forall s \in \{1, \dots, n-1\} \, \exists V \in \mathcal{GL}_n(\mathbb{F}) : \operatorname{spark}(AV) = s.$$

Proof. Observe first that if $U \in \mathcal{GL}_m(\mathbb{F})$, then $\operatorname{Ker}(UA) = \operatorname{Ker}(A)$. Hence property (i) follows from Proposition 2.1.

If $P \in \mathcal{M}_{n\times n}(\mathbb{F})$ is a permutation matrix, then A and AP differ only in the order of columns. Property (ii) follows.

Suppose that $\operatorname{rank}(A) = n-1$. Let $s \in N$, and let $v \in \mathbb{F}^n$ be such that $\|v\|_0 = s$. Then $L = \{\lambda v : \lambda \in \mathbb{F}\}$ is a 1-dimensional linear subspace of the space $\mathbb{F}^n$. Observe that $\|x\|_0 = s$ for every vector $x \in L \setminus \{\mathbf{0}\}$. Therefore, property (iii) follows from Lemma 2.1.

Let us turn to (iv). Suppose that A has no zero column. This means that $\operatorname{spark}(A) \geqslant 2$. If $\operatorname{rank}(A) = n$, then $\operatorname{spark}(A) = +\infty$ and

$$\forall V \in \mathcal{GL}_n(\mathbb{F}) : \operatorname{rank}(AV) = n,$$

and hence $\operatorname{spark}(AV) = +\infty = \operatorname{spark}(A) - 1$ for every matrix $V \in \mathcal{GL}_n(\mathbb{F})$. If $\operatorname{rank}(A) = n-1$, then the assertion follows from property (iii). Assume, therefore, that $\operatorname{rank}(A) \leqslant n-2$. Then $n \geqslant 3$, $t := \operatorname{spark}(A) \neq +\infty$, and $d := n - \operatorname{rank}(A) = \dim_{\mathbb{F}} \operatorname{Ker}(A) \geqslant 2$. Let $w \in \operatorname{Ker}(A) \setminus \{\mathbf{0}\}$ be such that

$$\|w\|_0 = \min\{\|x\|_0 : x \in \operatorname{Ker}(A) \setminus \{\mathbf{0}\}\} = t.$$

Moreover, let $j_0 \in N$ be such that the j_0th coordinate of the vector w is different from 0. Define $\tilde{w} = w_{N \setminus \{j_0\}}$. Choose vectors $u_1, \dots, u_{d-1}$ so that the sequence $(w, u_1, \dots, u_{d-1})$ is a basis for $\operatorname{Ker}(A)$, and define $L = \operatorname{Span}_{\mathbb{F}}(\tilde{w}, u_1, \dots, u_{d-1})$. Since $\|\tilde{w}\|_0 = t-1$, we have $\tilde{w} \notin \operatorname{Ker}(A)$. Thus, $\tilde{w}, u_1, \dots, u_{d-1}$ are linearly independent, which yields that $\dim_{\mathbb{F}} L = d$. Let $x = \lambda_0 \tilde{w} + \sum_{i=1}^{d-1} \lambda_i u_i \in L \setminus \{\mathbf{0}\}$. Then $y := \lambda_0 w + \sum_{i=1}^{d-1} \lambda_i u_i \in \operatorname{Ker}(A) \setminus \{\mathbf{0}\}$, and hence $\|y\|_0 \geqslant t$. It is obvious that $|\|x\|_0 - \|y\|_0| \leqslant 1$. We have therefore $\|x\|_0 \geqslant t-1 = \|\tilde{w}\|_0$. Consequently, $t-1 = \min_{x \in L \setminus \{\mathbf{0}\}} \|x\|_0$. Property (iv) follows now from Lemma 2.1.

Finally, suppose that the field $\mathbb{F}$ has at least $n+1$ elements and $\operatorname{rank}(A) = n-2$. Let $s \in \{1, \dots, n-1\}$. Choose some pairwise distinct scalars $\mu_1, \dots, \mu_{s+1} \in \mathbb{F} \setminus \{0\}$. Define $p, q \in \mathbb{F}^n$ by

$$p = [\mu_1, \dots, \mu_{s+1}, 0, \dots, 0]^{\mathrm{T}} \quad \text{and} \quad q = [\underbrace{1, \dots, 1}_{s+1}, 0, \dots, 0]^{\mathrm{T}}.$$

Observe that the vectors p and q are linearly independent. Let $L = \operatorname{Span}_{\mathbb{F}}(p, q)$. Since $\|p - \mu_1 q\|_0 = s$, and $\alpha\mu_1 + \beta, \dots, \alpha\mu_{s+1} + \beta$ are pairwise distinct for all $\alpha \in \mathbb{F} \setminus \{0\}$ and all $\beta \in \mathbb{F}$, we have $\min_{x \in L \setminus \{\mathbf{0}\}} \|x\|_0 = s$. Therefore, Lemma 2.1 yields the existence of a matrix $V \in \mathcal{GL}_n(\mathbb{F})$ such that $\operatorname{spark}(AV) = s$. □

Let us illustrate the above theorem with an example.

Example 2.1. The rank of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \in \mathcal{M}_{2\times 3}(\mathbb{F})$$

is equal to 2. Hence, Theorem 2.1(iii) guarantees that

$$\forall s \in \{1,2,3\}\, \exists V_s \in \mathcal{GL}_3(\mathbb{F}) : \operatorname{spark}(AV_s) = s.$$

It is easy to see that we can take

$$V_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad V_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

and $V_2 = I_3$, the unit matrix of size 3.

Note that the "moreover" part of Theorem 2.1 does not work over an arbitrary field.

Example 2.2. Let us consider the binary field $\mathbb{Z}_2$ and the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \in \mathcal{M}_{2\times 4}(\mathbb{Z}_2).$$

Observe that $\operatorname{spark}(A) = 2 = \operatorname{rank}(A)$. If $v, w \in \mathbb{Z}_2^4$ are such that

$$\min\{\|v\|_0, \|w\|_0\} \geqslant 3,$$

then $\|v + w\|_0 \leqslant 2$. Consequently, $\min_{x\in L\setminus\{\mathbf{0}\}} \|x\|_0 \leqslant 2$ for every 2-dimensional linear subspace L of the space $\mathbb{Z}_2^4$. Hence, by Lemma 2.1, there is no matrix $V \in \mathcal{GL}_4(\mathbb{Z}_2)$ such that $\operatorname{spark}(AV) = 3$.

3. The spark of a matrix and the null space property

We will denote by $\mathbb{R}_+$ the set of all the positive real numbers. Moreover, let $\|\cdot\|_1$ be the ℓ^1 norm on the space $\mathbb{C}^n$ (i.e., $\|[x_1,\dots,x_n]^{\mathrm{T}}\|_1 = \sum_{j=1}^n |x_j|$).

Definition 3.1. For a matrix $A \in \mathcal{M}_{m\times n}(\mathbb{C})$ and a non-negative integer k, we define $\gamma_k(A)$, the NSP constant of order k, to be the infimum of the set of all $\gamma \in \mathbb{R}_+$ such that

$$\forall x \in \mathbb{C}^n \forall S \subseteq N : (Ax = \mathbf{0}, \#S \leqslant k) \Rightarrow \|x_S\|_1 \leqslant \gamma \|x_{N\setminus S}\|_1.$$

The null space property first appeared in [**1**]. The above definition is based on [**3**].

Notice that $\gamma_k(A)$ is either a non-negative real number, or $+\infty$. If $A = O_{m\times n}$, the $m \times n$ zero matrix, then $\gamma_k(A) = +\infty$ for every positive order k.

Proposition 3.1. *The following properties hold true for an arbitrary matrix* $A \in \mathcal{M}_{m\times n}(\mathbb{C})$*:*

(i) $\gamma_0(A) = 0$,
(ii) *if* $\operatorname{rank}(A) = n$, *then* $\forall k \in \mathbb{N} \cup \{0\} : \gamma_k(A) = 0$,
(iii) *if there exists an integer* $k \geqslant n$ *such that* $\gamma_k(A) \neq +\infty$, *then* $\operatorname{rank}(A) = n$,
(iv) $\forall k \in \mathbb{N} \cup \{0\} : \gamma_k(A) \neq +\infty \implies \{x \in \mathbb{C}^n : Ax = \mathbf{0}, \|x\|_0 \leqslant k\} = \{\mathbf{0}\}$.

Proof. If $S = \emptyset$, then $x_S = \mathbf{0}$ for all $x \in \mathbb{C}^n$. Assertion (i) follows.

To prove (ii), it suffices to recall that $\operatorname{Ker}(A) = \{\mathbf{0}\}$ whenever $\operatorname{rank}(A) = n$.

If $\gamma_k(A) \neq +\infty$ for an integer $k \geqslant n$, then

$$\exists \gamma \in \mathbb{R}_+ \forall x \in \operatorname{Ker}(A) : \|x_N\|_1 \leqslant \gamma \|x_{N \setminus N}\|_1,$$

hence $\operatorname{Ker}(A) = \{\mathbf{0}\}$, which means that $\operatorname{rank}(A) = n$.

Finally, suppose that $\gamma_k(A) \neq +\infty$ for some $k \in \mathbb{N} \cup \{0\}$. Let $x = [x_1, \ldots, x_n]^{\mathrm{T}} \in \operatorname{Ker}(A)$ be such that $\|x\|_0 \leqslant k$. Define $S = \{j \in N : x_j \neq 0\}$. Since $\#S \leqslant k$, we have

$$\|x\|_1 = \|x_S\|_1 \leqslant \gamma_k(A) \|x_{N \setminus S}\|_1 = \gamma_k(A) \|\mathbf{0}\|_1 = 0.$$

Assertion (iv) follows. □

Example 3.1. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \in \mathcal{M}_{2\times 3}(\mathbb{C}).$$

Observe that $\operatorname{spark}(A) = 2$. Since $\operatorname{rank}(A) = 2$, assertion (iii) of Proposition 3.1 yields that $\gamma_k(A) = +\infty$ for every integer $k \geqslant 3$. Since

$$\operatorname{Ker}(A) = \{[\lambda, -\lambda, 0]^{\mathrm{T}} : \lambda \in \mathbb{C}\},$$

we have $\gamma_1(A) = 1$ and $\gamma_2(A) = +\infty$.

Let us state and prove the main result of the section ("When is an NSP constant of a matrix infinite?").

Theorem 3.1. *For a matrix $A \in \mathcal{M}_{m\times n}(\mathbb{C})$ and a non-negative integer k, the following conditions are equivalent:* (1) $\gamma_k(A) \neq +\infty$, (2) $k < \operatorname{spark}(A)$.

Proof. Suppose first that $\gamma_k(A) \neq +\infty$. Then, by assertion (iv) of Proposition 3.1, $\|x\|_0 > k$ whenever $x \in \operatorname{Ker}(A) \setminus \{\mathbf{0}\}$. Consequently, $\operatorname{spark}(A) = \inf\{\|x\|_0 : x \in \operatorname{Ker}(A) \setminus \{\mathbf{0}\}\} > k$.

Suppose now that $k < \operatorname{spark}(A)$. If $\operatorname{spark}(A) = +\infty$, then $\operatorname{rank}(A) = n$, and hence assertion (ii) of Proposition 3.1 yields $\gamma_k(A) = 0$. Therefore, assume additionally that $\operatorname{spark}(A) \neq +\infty$. If $k = 0$, then $\gamma_k(A) = 0$. So, assume moreover that $k \geqslant 1$. Then $A \neq O_{m\times n}$ (because $\operatorname{spark}(O_{m\times n}) = 1$). Since $k < \operatorname{spark}(A)$ and $\operatorname{spark}(A) \neq +\infty$, we have $1 \leqslant k < n$. Let $C_1, \ldots, C_n$ be the columns of A. Consider a k-element set $E = \{j_1, \ldots, j_k\} \subseteq N$, where $j_1 < \cdots < j_k$. It follows from the inequality $k < \operatorname{spark}(A)$ that the columns $C_{j_1}, \ldots, C_{j_k}$ are linearly independent (over $\mathbb{C}$). The linear map

$$f_E \colon \mathbb{C}^k \ni [y_1, \ldots, y_k]^{\mathrm{T}} \longmapsto \sum_{\nu=1}^{k} y_\nu C_{j_\nu} \in \mathbb{C}^m$$

is therefore injective. In virtue of the continuity of the inverse map, we can choose a positive real number δ_E such that $\forall y \in \mathbb{C}^k : \delta_E \|y\|_1 \leqslant \|f_E(y)\|_1$. Let $\delta = \min\{\delta_E : E \subseteq N, \#E = k\}$. Moreover, let $\tilde{\gamma}$ be the operator norm of the matrix A induced by the ℓ^1 norms on $\mathbb{C}^n$ and $\mathbb{C}^m$. Define $\gamma = \tilde{\gamma}/\delta$. Since $A \neq O_{m\times n}$, we have $\gamma \in \mathbb{R}_+$. Finally, let $x \in \operatorname{Ker}(A)$, and let $S \subseteq N$ be such that $\#S \leqslant k$. To complete the proof, it is enough to show that $\|x_S\|_1 \leqslant \gamma \|x_{N \setminus S}\|_1$. Since $\mathbf{0} = Ax = A(x_S + x_{N \setminus S}) = Ax_S + Ax_{N \setminus S}$, we get

$$\|Ax_S\|_1 = \|Ax_{N \setminus S}\|_1 \leqslant \tilde{\gamma} \|x_{N \setminus S}\|_1.$$

Let E_0 be a k-element subset of N such that $S \subseteq E_0$. It is obvious that

$$\exists z \in \mathbb{C}^k : \begin{cases} \|z\|_1 = \|x_S\|_1, \\ f_{E_0}(z) = Ax_S. \end{cases}$$

Consequently,

$$\tilde{\gamma} \|x_{N \setminus S}\|_1 \geqslant \|Ax_S\|_1 = \|f_{E_0}(z)\|_1 \geqslant \delta_{E_0} \|z\|_1 \geqslant \delta \|x_S\|_1,$$

which yields $\|x_S\|_1 \leqslant \gamma \|x_{N \setminus S}\|_1$. □

We conclude the note with a straightforward but quite interesting consequence of Theorem 3.1.

COROLLARY 3.1. *If* $A \in \mathcal{M}_{m \times n}(\mathbb{C})$, *then* $\operatorname{spark}(A) = 1 + \sup\{k \in \mathbb{N} \cup \{0\} : \gamma_k(A) \neq +\infty\}$.

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