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ON LORENTZIAN SPACES OF CONSTANT SECTIONAL CURVATURE

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Abstract. We investigate Osserman-like conditions for Lorentzian curvature tensors that imply constant sectional curvature. It is known that Osserman (moreover zwei-stein) Lorentzian manifolds have constant sectional curvature. We prove that some generalizations of the Rakić duality principle (Lorentzian totally Jacobi-dual or four-dimensional Lorentzian Jacobi-dual) imply constant sectional curvature. Moreover, any four-dimensional Jacobi-dual algebraic curvature tensor such that the Jacobi operator for some nonnull vector is diagonalizable, is Osserman. Additionally, any Lorentzian algebraic curvature tensor such that the reduced Jacobi operator for all nonnull vectors has a single eigenvalue has a constant sectional curvature.

1. Introduction

We investigate some sufficient conditions that a manifold is of constant sectional curvature, especially in the Lorentzian setting. According to Schur's lemma for a pseudo-Riemannian manifold (M, g) of dimension $n \geqslant 3$, if the sectional curvature $\kappa(\sigma)$ does not depend on the plane $\sigma \subseteq T_pM$, but only on the point $p \in M$, then κ is constant. It allows us to put the problem into a pure algebraic concept with an algebraic curvature tensor.

Let $(\mathcal{V}, g)$ be a (possibly indefinite) scalar product space of dimension n. The squared norm of a vector $X \in \mathcal{V}$ is the real number $\varepsilon_X = g(X, X)$. The sign of the squared norm separates all vectors $X \in \mathcal{V}$ into three different types. A vector $X \in \mathcal{V}$ is spacelike if $\varepsilon_X > 0$; timelike if $\varepsilon_X < 0$; null if $\varepsilon_X = 0$. Especially, a vector $X \in \mathcal{V}$ is unit if $\varepsilon_X \in \{-1, 1\}$ and it is nonnull if $\varepsilon_X \neq 0$.

An algebraic curvature tensor on $(\mathcal{V}, g)$ is a quadri-linear map $R\colon \mathcal{V}^4 \to \mathbb{R}$ that satisfies usual $\mathbb{Z}_2$-symmetries and the first Bianchi identity. In the presence of an orthonormal basis $(E_1, \dots, E_n)$ in $\mathcal{V}$ we can use shortcuts $R_{ijkl} = R(E_i, E_j, E_k, E_l)$ and $\varepsilon_i = \varepsilon_{E_i}$ for $1 \leqslant i, j, k, l \leqslant n$.

The formula $\mathcal{R}(X, Y)Z = \sum_{i=1}^n \varepsilon_i R(X, Y, Z, E_i)E_i$ defines the associated algebraic curvature operator. We can define the polarized Jacobi operator using

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$\mathcal{J}(X,Y)Z = \frac{1}{2}(\mathcal{R}(Z,X)Y + \mathcal{R}(Z,Y)X)$, while the Jacobi operator is given by $\mathcal{J}_X(Y) = \mathcal{J}(X,X)Y = \mathcal{R}(Y,X)X$. In the case of nonnull X, the orthogonal $X^\perp$ is a nondegenerate hyperspace in $\mathcal{V}$, while $\mathcal{J}_X(X) = 0$. Thus, the Jacobi operator for nonnull $X \in \mathcal{V}$ is completely determined by its restriction $\widetilde{\mathcal{J}}_X\colon X^\perp \to X^\perp$, called the reduced Jacobi operator. We say that R is Jacobi-diagonalizable if $\mathcal{J}_X$ is diagonalizable for any nonnull X.

2. Motivation

Given an algebraic curvature tensor R on $(\mathcal{V}, g)$, its sectional curvature is defined on a non-degenerate two-dimensional subspace σ of $\mathcal{V}$ as follows

$$\kappa(\sigma) = \kappa(X,Y) = \frac{R(X,Y,Y,X)}{\varepsilon_X\varepsilon_Y - (g(X,Y))^2},$$

where $\{X, Y\}$ is any basis of σ. Recall that $\sigma = \mathrm{Span}\{X,Y\}$ is non-degenerate if and only if $\varepsilon_X\varepsilon_Y \neq (g(X,Y))^2$.

Let $A, B, T \in \mathcal{V}$ be orthonormal with $\varepsilon_A = \varepsilon_B = 1 = -\varepsilon_T$. Since $\varepsilon_{T+\theta B}\varepsilon_A\kappa(T+\theta B, A) = R(T+\theta B, A, A, T+\theta B)$ we have

$$\kappa(T+\theta B, A) = \frac{\theta^2\kappa(B,A) - \kappa(T,A) + 2\theta R(T,A,A,B)}{\theta^2 - 1}.$$

If $\kappa \geqslant m$ is bounded from below, then $\theta^2\kappa(B,A) - \kappa(T,A) + 2\theta R(T,A,A,B)$ is greater than $m(\theta^2-1)$ for $|\theta| > 1$ and less than $m(\theta^2-1)$ for $|\theta| < 1$, so by continuity it is zero for $\theta = 1$ and $\theta = -1$, which implies $\kappa(B,A) = \kappa(T,A)$ and $R(T,A,A,B) = 0$. Using this argument, Kulkarni [**12**] proved the following theorem.

Theorem 2.1 (Kulkarni, 1979 [**12**]). *Let κ be the sectional curvature function of an algebraic curvature tensor R on an indefinite $(\mathcal{V}, g)$. If κ is either bounded from above or bounded from below, then κ is constant.*

Theorem 2.1 inspired Dajczer and Nomizu [**7**] to improve it with the following.

Theorem 2.2 (Dajczer and Nomizu, 1980 [**7**]). *Let κ be the sectional curvature function of an algebraic curvature tensor R on an indefinite $(\mathcal{V}, g)$. If κ of indefinite planes are bounded both above and below, then κ is constant.*

We say that R is timelike (or spacelike) Osserman if the characteristic polynomial of $\mathcal{J}_X$ is independent of unit timelike (or spacelike) X. Let R be Lorentzian timelike Osserman and let $T \in \mathcal{V}$ be unit timelike. Since $T^\perp \leqslant \mathcal{V}$ is positive definite it has an orthonormal basis $V_1, \dots, V_{n-1}$ consisting of eigenvectors of $\mathcal{J}_T$. However, $\mathcal{J}_T(V_i) = \lambda_i V_i$ holds for $1 \leqslant i \leqslant n-1$ where $\lambda_1, \dots, \lambda_{n-1}$ are independent of T since they are roots of the characteristic polynomial. Thus, for arbitrary $X = \sum_{i=1}^{n-1}\alpha_i V_i \in T^\perp$ we have $\kappa(T,X) = g(\mathcal{J}_T(X), X)/(\varepsilon_T\varepsilon_X) = -\sum_i \alpha_i^2\lambda_i / \sum_i \alpha_i^2$, and therefore $-\max_{1\leqslant i\leqslant n-1}\lambda_i \leqslant \kappa(T,X) \leqslant -\min_{1\leqslant i\leqslant n-1}\lambda_i$. García-Río, Kupeli, and Vázquez-Abal [**8**] used this argument, so Theorem 2.2 implies that κ is constant.

THEOREM 2.3 (García-Río, Kupeli, and Vázquez-Abal, 1997 [**8**]). *Lorentzian timelike Osserman algebraic curvature tensor has constant sectional curvature.*

The concepts of timelike Osserman and spacelike Osserman are equivalent (see [**9**]), so we can say that any Lorentzian Osserman R has constant sectional curvature. However, the Osserman property is further equivalent with a generalization of the Einstein property named k-stein. We say that R is k-stein if there exist constants $C_1, C_2, \ldots, C_k$ such that $\mathrm{Tr}((\mathcal{J}_X)^j) = C_j$ holds for each $1 \leqslant j \leqslant k$ and $X \in \mathcal{V}$. An algebraic curvature tensor R is Osserman if and only if it is k-stein for any positive integer k (see Gilkey [**11**, Lemma 1.7.3]). Therefore, one can say that R is Osserman if $\det(\varepsilon_X \lambda \,\mathrm{Id} - \mathcal{J}_X) = 0$ is the same equation for all nonnull X.

For Theorem 2.3 to hold, it is enough to assume that R is Lorentzian zwei-stein (2-stein). Let R be zwei-stein, then $\mathrm{Tr}((\mathcal{J}_{X+tY})^2) = (\varepsilon_{X+tY})^2 C_2$, for $X \perp Y$ implies $\mathrm{Tr}((\mathcal{J}_X + 2t\mathcal{J}(X,Y) + t^2\mathcal{J}_Y)^2) = (\varepsilon_X + t^2\varepsilon_Y)^2 C_2$ for all $t \in \mathbb{R}$. Thus $\mathrm{Tr}({\mathcal{J}_X}^2) = \varepsilon_X^2 C_2$ and $\mathrm{Tr}(\mathcal{J}_X\mathcal{J}_Y) + 4\,\mathrm{Tr}((\mathcal{J}(X,Y))^2) + \mathrm{Tr}(\mathcal{J}_Y\mathcal{J}_X) = 2\varepsilon_X\varepsilon_Y C_2$. For an arbitrary orthonormal basis $(E_1, \ldots, E_n)$ in $\mathcal{V}$ and for $1 \leqslant x \neq y \leqslant n$ we can conclude

$$\sum_{1 \leqslant i,j \leqslant n} \varepsilon_i\varepsilon_j \big((R_{ixyj} + R_{iyxj})^2 - \varepsilon_x\varepsilon_y(\varepsilon_x R_{ixxj} - \varepsilon_y R_{iyyj})^2\big) = 0,$$

For $\varepsilon_x\varepsilon_y < 0$ it becomes $\sum_{i,j} \varepsilon_i\varepsilon_j \big((R_{ixyj} + R_{iyxj})^2 + (R_{ixxj} + R_{iyyj})^2\big) = 0$. For Lorentzian R we have $\varepsilon_i = 1$ for all $i \notin \{x, y\}$, some parts in our sum disappear, so

$$\sum_{i,j \notin \{x,y\}} \big((R_{ixyj} + R_{iyxj})^2 + (R_{ixxj} + R_{iyyj})^2\big) = 0.$$

Therefore $R_{ixxj} + R_{iyyj} = 0$ holds for all $i, j \notin \{x, y\}$. Especially for $i = j$ we have

$$\kappa(E_i, E_x) = \varepsilon_i\varepsilon_x R_{ixxi} = \varepsilon_i\varepsilon_y R_{iyyi} = \kappa(E_i, E_y),$$

whenever i, x, y are different with $\varepsilon_x\varepsilon_y = -1$, for example $\varepsilon_x = -1$.

In order to complete the proof we shall use the following argument, missed by other authors (see [**5**]), which significantly speeds up things. Since, zwei-stein is Einstein by definition, we have

$$C_1 = \sum_{1 \leqslant y \leqslant n} \varepsilon_i\varepsilon_y R_{yiiy} = \sum_{y \neq i} \varepsilon_i\varepsilon_x R_{ixxi} = (n-1)\kappa(E_i, E_x),$$

and therefore $\kappa(E_i, E_x) = \kappa(E_i, E_y) = C_1/(n-1)$ for any orthonormal basis, so R has constant sectional curvature.

THEOREM 2.4 (Blažić, Bokan, and Gilkey, 1997 [**5**]). *Lorentzian zwei-stein algebraic curvature tensor has constant sectional curvature.*

3. Duality principle

One natural property of a Riemannian Osserman curvature tensor is the Rakić duality principle, where $\mathcal{J}_X(Y) = \lambda Y \implies \mathcal{J}_Y(X) = \lambda X$ holds for unit $X, Y \in \mathcal{V}$

(see Rakić [**14**]). This implication is inaccurate in a pseudo-Riemannian setting so we corrected it with the following implication (see [**3**]):

$$\mathcal{J}_X(Y) = \varepsilon_X \lambda Y \;\Rightarrow\; \mathcal{J}_Y(X) = \varepsilon_Y \lambda X. \tag{3.1}$$

If (3.1) holds for all $X, Y \in \mathcal{V}$ with $\varepsilon_X \neq 0$, we say that R is Jacobi-dual (see [**4**]) or that R satisfies the duality principle. The concept with no restrictions on X and Y can be reformulated with the following definition. We say that an algebraic curvature tensor R is totally Jacobi-dual if the equivalence

$$Y \text{ belongs to an eigenspace of } \mathcal{J}_X \;\Leftrightarrow\; X \text{ belongs to an eigenspace of } \mathcal{J}_Y$$

holds for all $X, Y \in \mathcal{V}$.

Let us emphasize some known results in this topic. Some sufficient conditions for an algebraic curvature tensor to be Jacobi-dual are to be Riemannian Osserman (Rakić, 1999 [**14**]), four-dimensional Osserman (Andrejić, 2010 [**2**]), or Jordan-Osserman (the special case of Osserman which keeps the Jordan normal form) (Nikolayevsky and Rakić, 2016 [**13**]). In this paper we deal with the converse question. Whether being Jacobi-dual necessarily implies being Osserman? We have an affirmative answer in the following cases: the three-dimensional case (Andrejić, 2009 [**2**]), the Riemannian four-dimensional case (Brozos-Vázquez and Merino, 2012 [**6**]), and the Jacobi-diagonalizable case (Nikolayevsky and Rakić, 2016 [**13**]). According to Theorem 2.4 it is natural to have the following theorem.

Theorem 3.1 (Andrejić and Rakić [**4**]). *A Lorentzian totally Jacobi-dual curvature tensor has constant sectional curvature.*

Proof. Let T be a unit timelike vector in a Lorentzian space $\mathcal{V}$, then $T^\perp$ is positive definite, so $\widetilde{\mathcal{J}}_T$ is diagonalizable. Let $S_1, \dots, S_{n-1}$ be orthonormal ($\varepsilon_{S_i} = -\varepsilon_T = 1$) eigenvectors of $\widetilde{\mathcal{J}}_T$. Then $\mathcal{J}_T(S_i) = \varepsilon_T \lambda_i S_i$ and Jacobi-duality gives $\mathcal{J}_{S_i}(T) = \varepsilon_{S_i} \lambda_i T$ for all $1 \leqslant i \leqslant n-1$.

Since $\mathcal{J}_{T\pm S_i}(T) = \mathcal{R}(T, T\pm S_i)(T\pm S_i) = \mp\mathcal{J}_T(S_i) + \mathcal{J}_{S_i}(T) = \lambda_i(T\pm S_i)$, and similarly $\mathcal{J}_{T\pm S_i}(S_i) = \mp\lambda_i(T\pm S_i)$, we have $\mathcal{J}_{T\pm S_i}(\mathcal{U}_i) \subseteq \mathcal{U}_i = \operatorname{Span}\{T, S_i\}$. Since $\mathcal{J}_{T\pm S_i}$ is self-adjoint, $g(\mathcal{J}_{T\pm S_i}(\mathcal{U}_i^\perp), \mathcal{U}_i) = g(\mathcal{U}_i^\perp, \mathcal{J}_{T\pm S_i}(\mathcal{U}_i)) \subseteq g(\mathcal{U}_i^\perp, \mathcal{U}_i) = \{0\}$, and therefore $\mathcal{J}_{T\pm S_i}(\mathcal{U}_i^\perp) \subseteq \mathcal{U}_i^\perp = \operatorname{Span}\bigcup_{j\neq i}\{S_j\}$ holds.

The restriction $\mathcal{J}_{T\pm S_i}\restriction_{\mathcal{U}_i^\perp}$ is diagonalizable as a self-adjoint operator on a definite space. If M is a nonnull eigenvector of $\mathcal{J}_{T\pm S_i}\restriction_{\mathcal{U}_i^\perp}$, then $\mathcal{J}_{T\pm S_i}(M) = \mu M$ together with the totally Jacobi-dual condition gives $\mathcal{J}_M(T\pm S_i) = \nu(T\pm S_i)$. However, $\mu\varepsilon_M = g(\mathcal{J}_{T\pm S_i}(M), M) = g(\mathcal{J}_M(T\pm S_i), T\pm S_i) = \nu\varepsilon_{T\pm S_i} = 0$, so $\mu = 0$, and therefore $\mathcal{J}_{T\pm S_i}(M) = 0$. Consequently, $\mathcal{J}_{T\pm S_i}(\mathcal{U}_i^\perp) = \{0\}$ and therefore $\mathcal{J}_{T\pm S_i}(S_j) = 0$ holds, for all $1 \leqslant i \neq j \leqslant n-1$.

We can express $\mathcal{J}_{T\pm S_i} = \mathcal{J}_T + \mathcal{J}_{S_i} \pm 2\mathcal{J}(T, S_i)$ and get $\mathcal{J}_{T+S_i} + \mathcal{J}_{T-S_i} = 2(\mathcal{J}_T + \mathcal{J}_{S_i})$. Thus $\mathcal{J}_{T\pm S_i}(S_j) = 0$ implies $\mathcal{J}_{S_i}(S_j) = -\mathcal{J}_T(S_j) = \lambda_j S_j$. Comparing this equation after (i,j)-symmetry $\mathcal{J}_{S_j}(S_i) = \lambda_i S_i$ and after the Jacobi-dual property $\mathcal{J}_{S_j}(S_i) = \lambda_j S_i$, we easily conclude that $\lambda_i = \lambda_j$ for $1 \leqslant i \neq j \leqslant n-1$, which proves that R has constant sectional curvature. □

4. Four-dimensional Jacobi-dual

Let R be a four-dimensional Jacobi-dual algebraic curvature tensor on $(\mathcal{V}, g)$. We have been working under the assumption that $\mathcal{J}_X$ is diagonalizable for some nonnull X. Then there exists an orthonormal basis (E_1, E_2, E_3, E_4) in $\mathcal{V}$ such that $\mathcal{J}_{E_1}(E_i) = \varepsilon_1\lambda_{1i}E_i$ holds for $i = 2, 3, 4$.

Without loss of generality we can assume $\varepsilon_3 = \varepsilon_4$. Since Jacobi-duality gives $\mathcal{J}_{E_2}(E_1) = \varepsilon_2\lambda_{12}E_1$ then $\mathcal{J}_{E_2}(\mathrm{Span}\{E_3, E_4\})$ is a subset of definite $\mathrm{Span}\{E_3, E_4\}$. Therefore, $\mathcal{J}_{E_2}$ is diagonalizable with two mutually orthogonal eigenvectors $\alpha_3E_3 + \alpha_4E_4$ and $\varepsilon_4\alpha_4E_3 - \varepsilon_3\alpha_3E_4$ for some $\alpha_3, \alpha_4 \in \mathbb{R}$ such that $\alpha_3^2\varepsilon_3 + \alpha_4^2\varepsilon_4 \neq 0$. If we set the relation $\sim$ for collinear vectors, then

$$\alpha_3^2\mathcal{J}_{E_3}(E_2) + \alpha_4^2\mathcal{J}_{E_4}(E_2) + 2\alpha_3\alpha_4\mathcal{J}(E_3, E_4)(E_2) \sim E_2,$$
$$\alpha_4^2\mathcal{J}_{E_3}(E_2) + \alpha_3^2\mathcal{J}_{E_4}(E_2) - 2\varepsilon_3\varepsilon_4\alpha_3\alpha_4\mathcal{J}(E_3, E_4)(E_2) \sim E_2,$$

and therefore

$$(\varepsilon_3\varepsilon_4\alpha_3^2 + \alpha_4^2)\mathcal{J}_{E_3}(E_2) + (\varepsilon_3\varepsilon_4\alpha_4^2 + \alpha_3^2)\mathcal{J}_{E_4}(E_2) \sim E_2.$$

Hence $\mathcal{J}_{E_3}(E_2) \perp E_4$ which together with $\mathcal{J}_{E_3}(\mathrm{Span}\{E_2, E_4\}) \subseteq \mathrm{Span}\{E_2, E_4\}$ (from $\mathcal{J}_{E_3}(E_1) = \varepsilon_3\lambda_{13}E_1$) implies $\mathcal{J}_{E_3}(E_2) \sim E_2$ and consequently $\mathcal{J}_{E_3}(E_4) \sim E_4$. Thus $\mathcal{J}_{E_2}(E_3) \sim E_3$ and $\mathcal{J}_{E_2}(E_4) \sim E_4$, which means that our basis diagonalizes all of operators $\mathcal{J}_{E_1}, \mathcal{J}_{E_2}, \mathcal{J}_{E_3}, \mathcal{J}_{E_4}$, so we have $\mathcal{J}_{E_i}(E_j) = \varepsilon_i\lambda_{ij}E_j$ where $\lambda_{ij} = \lambda_{ji}$ for all $1 \leqslant i, j \leqslant 4$.

This allows simple calculations,

$$\mathcal{J}(E_1, E_2)(E_3) = \sum_i \varepsilon_i \frac{1}{2}(R_{312i} + R_{321i})E_i = \frac{1}{2}\varepsilon_4(R_{3124} + R_{3214})E_4,$$

which holds in all permutations of indices $\{1, 2, 3, 4\}$. Since,

$$\mathcal{J}_{\alpha_1E_1+\alpha_2E_2}(E_3) = (\alpha_1^2\varepsilon_1\lambda_{13} + \alpha_2^2\varepsilon_2\lambda_{23})E_3 + \alpha_1\alpha_2\varepsilon_4(R_{3124} + R_{3214})E_4,$$
$$\mathcal{J}_{\alpha_1E_1+\alpha_2E_2}(E_4) = (\alpha_1^2\varepsilon_1\lambda_{14} + \alpha_2^2\varepsilon_2\lambda_{24})E_4 + \alpha_1\alpha_2\varepsilon_3(R_{4123} + R_{4213})E_3,$$

after the substitution $Q_{12} = R_{3124} + R_{3214} = R_{4123} + R_{4213}$, we have

$$\begin{aligned}\mathcal{J}_{\alpha_1E_1+\alpha_2E_2}(\alpha_3E_3 + \alpha_4E_4) = {} & (\alpha_3(\alpha_1^2\varepsilon_1\lambda_{13} + \alpha_2^2\varepsilon_2\lambda_{23}) + \alpha_4\alpha_1\alpha_2\varepsilon_3Q_{12})E_3 \\ & + (\alpha_4(\alpha_1^2\varepsilon_1\lambda_{14} + \alpha_2^2\varepsilon_2\lambda_{24}) + \alpha_3\alpha_1\alpha_2\varepsilon_4Q_{12})E_4.\end{aligned}$$

We should discuss two cases $Q_{12} \neq 0$ and $Q_{12} = 0$. Let us suppose first that $Q_{12} \neq 0$. Since $\mathcal{J}_{E_1}(E_2) = \varepsilon_1\lambda_{12}E_2$ and $\mathcal{J}_{E_2}(E_1) = \varepsilon_2\lambda_{12}E_1$, a straightforward calculations give $\mathcal{J}_{\alpha_1E_1+\alpha_2E_2}(\varepsilon_2\alpha_2E_1 - \varepsilon_1\alpha_1E_2) = \varepsilon_{\alpha_1E_1+\alpha_2E_2}\lambda_{12}(\varepsilon_2\alpha_2E_1 - \varepsilon_1\alpha_1E_2)$ (see Andrejić [**1**, Lemma 1]). Therefore for any $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $\alpha_1^2\varepsilon_1 + \alpha_2^2\varepsilon_2 \neq 0$ there exist $\alpha_3, \alpha_4 \in \mathbb{R}$ such that $\alpha_3^2\varepsilon_3 + \alpha_4^2\varepsilon_4 \neq 0$ and $\mathcal{J}_{\alpha_1E_1+\alpha_2E_2}(\alpha_3E_3 + \alpha_4E_4) \sim \alpha_3E_3 + \alpha_4E_4$. Thus

$$\begin{aligned}&\alpha_4(\alpha_3(\alpha_1^2\varepsilon_1\lambda_{13} + \alpha_2^2\varepsilon_2\lambda_{23}) + \alpha_4\alpha_1\alpha_2\varepsilon_3Q_{12}) \\ &= \alpha_3(\alpha_4(\alpha_1^2\varepsilon_1\lambda_{14} + \alpha_2^2\varepsilon_2\lambda_{24}) + \alpha_3\alpha_1\alpha_2\varepsilon_4Q_{12}),\end{aligned}$$

which implies

$$\alpha_3\alpha_4(\alpha_1^2\varepsilon_1(\lambda_{13} - \lambda_{14}) + \alpha_2^2\varepsilon_2(\lambda_{23} - \lambda_{24})) = \alpha_1\alpha_2Q_{12}(\alpha_3^2\varepsilon_4 - \alpha_4^2\varepsilon_3).$$

However, Jacobi-duality follows that $\alpha_1 E_1 + \alpha_2 E_2$ is an eigenvector of $\mathcal{J}_{\alpha_3 E_3 + \alpha_4 E_4}$, and since $Q_{34} = R_{2341} + R_{2431} = R_{1342} + R_{1432} = Q_{12}$, we can use the symmetric formula to get

$$\alpha_1\alpha_2(\alpha_3^2\varepsilon_3(\lambda_{31} - \lambda_{32}) + \alpha_4^2\varepsilon_4(\lambda_{41} - \lambda_{42})) = \alpha_3\alpha_4 Q_{12}(\alpha_1^2\varepsilon_2 - \alpha_2^2\varepsilon_1).$$

Let us rescale this with $\alpha_1 = x, \alpha_2 = 1$ and $\alpha_3 = y, \alpha_4 = 1$, which means that for any x (with $x^2 \neq -\varepsilon_1\varepsilon_2$) there exists y (with $y^2 \neq -\varepsilon_3\varepsilon_4$) such that

$$(\varepsilon_4 Q_{12}x)y^2 - (\varepsilon_1 x^2(\lambda_{13} - \lambda_{14}) + \varepsilon_2(\lambda_{23} - \lambda_{24}))y - \varepsilon_3 Q_{12}x = 0,$$
$$(\varepsilon_3(\lambda_{13} - \lambda_{23})x)y^2 - (Q_{12}(\varepsilon_2 x^2 - \varepsilon_1))y + \varepsilon_4(\lambda_{14} - \lambda_{24})x = 0.$$

Since the solutions goes in pairs (mutually orthogonal eigenvectors), two equations above are the same, which implies

$$\begin{aligned} \varepsilon_4 Q_{12}x &= K\varepsilon_3(\lambda_{13} - \lambda_{23})x, \\ \varepsilon_1 x^2(\lambda_{13} - \lambda_{14}) + \varepsilon_2(\lambda_{23} - \lambda_{24}) &= KQ_{12}(\varepsilon_2 x^2 - \varepsilon_1), \\ -\varepsilon_3 Q_{12}x &= K\varepsilon_4(\lambda_{14} - \lambda_{24})x, \end{aligned}$$

for infinitely many x and some $K = K(x)$. Immediately $\lambda_{13} - \lambda_{23} = -(\lambda_{14} - \lambda_{24})$, that is

$$\lambda_{13} + \lambda_{14} = \lambda_{23} + \lambda_{24}.$$

Additionally,

$$\varepsilon_3(\lambda_{13} - \lambda_{23})(\varepsilon_1 x^2(\lambda_{13} - \lambda_{14}) + \varepsilon_2(\lambda_{23} - \lambda_{24})) = \varepsilon_4 Q_{12}^2(\varepsilon_2 x^2 - \varepsilon_1),$$

and the polynomial

$$(\varepsilon_1\varepsilon_3(\lambda_{13} - \lambda_{23})(\lambda_{13} - \lambda_{14}) - \varepsilon_2\varepsilon_4 Q_{12}^2)x^2 + \varepsilon_2\varepsilon_3(\lambda_{13} - \lambda_{23})(\lambda_{23} - \lambda_{24}) + \varepsilon_1\varepsilon_4 Q_{12}^2 = 0$$

has infinitely many solutions, so

$$\begin{aligned} (\lambda_{13} - \lambda_{23})(\lambda_{13} - \lambda_{14}) &= \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 Q_{12}^2, \\ (\lambda_{13} - \lambda_{23})(\lambda_{23} - \lambda_{24}) &= -\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 Q_{12}^2. \end{aligned}$$

Therefore $\lambda_{13} - \lambda_{14} = \lambda_{24} - \lambda_{23}$, which together with $\lambda_{13} + \lambda_{14} = \lambda_{24} + \lambda_{23}$ implies

$$\lambda_{13} = \lambda_{24} \quad \text{and} \quad \lambda_{14} = \lambda_{23}. \tag{4.1}$$

Moreover, we have

$$(\lambda_{13} - \lambda_{14})^2 = \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 Q_{12}^2,$$

that immediately implies $\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 = 1$, which means that $Q_{12} \neq 0$ is impossible in the Lorentzian setting.

We turn now to the case $Q_{12} = 0$, so $\mathcal{J}_{\alpha_1 E_1 + \alpha_2 E_2}(E_3) = (\alpha_1^2\varepsilon_1\lambda_{13} + \alpha_2^2\varepsilon_2\lambda_{23})E_3$ holds, and duality gives

$$\begin{aligned} \alpha_1\varepsilon_3\lambda_{13}E_1 + \alpha_2\varepsilon_3\lambda_{23}E_2 &= \mathcal{J}_{E_3}(\alpha_1 E_1 + \alpha_2 E_2) \\ &= \varepsilon_3 \frac{\alpha_1^2\varepsilon_1\lambda_{13} + \alpha_2^2\varepsilon_2\lambda_{23}}{\alpha_1^2\varepsilon_1 + \alpha_2^2\varepsilon_2}(\alpha_1 E_1 + \alpha_2 E_2), \end{aligned}$$

which implies $\lambda_{13} = \lambda_{23}$. Similarly, $\mathcal{J}_{\alpha_1 E_1 + \alpha_2 E_2}(E_4) = (\alpha_1^2\varepsilon_1\lambda_{14} + \alpha_2^2\varepsilon_2\lambda_{24})E_4$ implies $\lambda_{14} = \lambda_{24}$.

Using symmetries we can conclude the following. If $Q_{12} = R_{3124} + R_{3214} \neq 0$ then $\lambda_{13} = \lambda_{24}$ and $\lambda_{14} = \lambda_{23}$, otherwise $\lambda_{13} = \lambda_{23}$ and $\lambda_{14} = \lambda_{24}$. If $Q_{13} = R_{2134} + R_{2314} \neq 0$ then $\lambda_{12} = \lambda_{34}$ and $\lambda_{14} = \lambda_{23}$, otherwise $\lambda_{12} = \lambda_{23}$ and $\lambda_{14} = \lambda_{34}$. If $Q_{14} = R_{3142} + R_{3412} \neq 0$ then $\lambda_{13} = \lambda_{24}$ and $\lambda_{12} = \lambda_{34}$, otherwise $\lambda_{13} = \lambda_{34}$ and $\lambda_{12} = \lambda_{24}$. This is enough to conclude that R is Einstein since for $X = \sum_i \alpha_i E_i$ we have

$$\begin{aligned}\mathrm{Tr}(\mathcal{J}_X) &= \sum_i \varepsilon_i R(E_i, X, X, E_i) = \sum_{i,p,q} \varepsilon_i \alpha_p \alpha_q R_{ipqi} \\ &= \sum_{i,p} \varepsilon_i \alpha_p^2 R_{ippi} = \sum_{i,p} \alpha_p^2 \varepsilon_p \lambda_{ip} = \varepsilon_X C\end{aligned}$$

The simplest case is $Q_{12} = Q_{13} = Q_{14} = 0$ which implies $\lambda_{12} = \lambda_{13} = \lambda_{14} = \lambda_{23} = \lambda_{24} = \lambda_{34} = C/3$, so R has constant sectional curvature. Notice that this is the only possible case in the Lorentzian setting.

THEOREM 4.1. *Any four-dimensional Lorentzian Jacobi-dual algebraic curvature tensor has constant sectional curvature.*

Since the first Bianchi identity gives $Q_{12} + Q_{13} + Q_{14} = 0$, it is impossible to have only one Q different than zero. However, $Q_{12} = 0$ and $Q_{13}Q_{14} \neq 0$ implies $\lambda_{13} = \lambda_{23} = \lambda_{14} = \lambda_{24} \neq \lambda_{12} = \lambda_{34}$, so (4.1) holds anyway. This means that we left with the most general case where $\lambda_{12} = \lambda_{34}$, $\lambda_{13} = \lambda_{24}$, and $\lambda_{14} = \lambda_{23}$ with additional $\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 = 1$, $(\lambda_{14} - \lambda_{13})^2 = Q_{12}^2$, $(\lambda_{12} - \lambda_{14})^2 = Q_{13}^2$, $(\lambda_{13} - \lambda_{12})^2 = Q_{14}^2$.

Changing the sign of one vector in our basis, if necessary, we can set

$$\lambda_{14} - \lambda_{13} = Q_{12}, \quad \lambda_{12} - \lambda_{14} = Q_{13}, \quad \lambda_{13} - \lambda_{12} = Q_{14}. \tag{4.2}$$

For example, if $Q_{12} \neq 0$ then $\lambda_{14} - \lambda_{13} = -Q_{12}$, after the change E_4 with $-E_4$, gives $\lambda_{14} - \lambda_{13} = Q_{12}$. Now, if $\lambda_{12} - \lambda_{14} \neq Q_{13}$ it is $\lambda_{12} - \lambda_{14} = -Q_{13}$, and $\pm(\lambda_{13} - \lambda_{12}) = Q_{14} = -Q_{12} - Q_{13} = \lambda_{12} - 2\lambda_{14} + \lambda_{13}$. Since $\lambda_{14} \neq \lambda_{13}$ it follows $\lambda_{14} = \lambda_{12}$, so $Q_{13} = 0 = \lambda_{12} - \lambda_{14}$.

From (4.2), it follows

$$\begin{aligned}\lambda_{14} - \lambda_{13} &= Q_{12} = -R_{1324} + R_{1432} = R_{1234} - 2R_{1324}, \\ \lambda_{12} - \lambda_{14} &= Q_{13} = -R_{1234} + R_{1423} = -2R_{1234} + R_{1324},\end{aligned}$$

and therefore

$$R_{1234} = -\frac{2}{3}\lambda_{12} + \frac{1}{3}\lambda_{13} + \frac{1}{3}\lambda_{14}, \quad R_{1324} = -\frac{1}{3}\lambda_{12} + \frac{2}{3}\lambda_{13} - \frac{1}{3}\lambda_{14}.$$

We calculate the final two of the curvature tensor components which completely determines R. Is such R Osserman? We know that there exists an Osserman algebraic curvature tensor with our conditions (eigenvalues). Any four-dimensional Osserman is Jacobi-dual (see Andrejić [**2**]), so since it is unique, our R is Osserman.

For example, if the signature is neutral, then we can set, without loss of generality, $\varepsilon_1 = \varepsilon_2 = -1$, $\varepsilon_3 = \varepsilon_4 = 1$, and use paraquaternionic structure $\{J_1, J_2, J_3\}$

given by

$$\begin{aligned} J_1E_1 &= -E_2, & J_1E_2 &= E_1, & J_1E_3 &= E_4, & J_1E_4 &= -E_3,\\ J_2E_1 &= E_3, & J_2E_2 &= E_4, & J_2E_3 &= E_1, & J_2E_4 &= E_2,\\ J_3E_1 &= E_4, & J_3E_2 &= -E_3, & J_3E_3 &= -E_2, & J_3E_4 &= E_1, \end{aligned}$$

to see that our R is induced by Clifford

$$\mathcal{R} = -\frac{\lambda_{12}}{3}\mathcal{R}^{J_1} + \frac{\lambda_{13}}{3}\mathcal{R}^{J_2} + \frac{\lambda_{14}}{3}\mathcal{R}^{J_3}.$$

THEOREM 4.2. *Any four-dimensional Jacobi-dual algebraic curvature tensor such that $\mathcal{J}_X$ is diagonalizable for some nonnull X is Osserman.*

It is worth noting that our theorem is a generalization of the result given in the Riemannian setting by Brozos-Vázquez and Merino [**6**].

5. Single eigenvalue

In this section we are working under the assumption that the reduced Jacobi operator $\widetilde{\mathcal{J}}_X$ has a single eigenvalue for any nonnull $X \in \mathcal{V}$. Let us suppose that this eigenvalue is $\varepsilon_X\mu_X$.

In the Riemannian setting we have $X^\perp = \mathrm{Ker}(\widetilde{\mathcal{J}}_X - \varepsilon_X\mu_X\,\mathrm{Id})$. Therefore, for mutually orthogonal nonnull $X, Y \in \mathcal{V}$ we have $\mathcal{J}_X(Y) = \varepsilon_X\mu_X Y$ and $\mathcal{J}_Y(X) = \varepsilon_Y\mu_Y X$, so $\varepsilon_X\varepsilon_Y\mu_X = g(\mathcal{J}_X(Y), Y) = \varepsilon_X\varepsilon_Y\kappa(X,Y) = g(\mathcal{J}_Y(X), X) = \varepsilon_X\varepsilon_Y\mu_Y$ which implies a constant sectional curvature $\mu_X = \mu_Y = \kappa(X,Y)$. The same proof remains valid for Jacobi-diagonalizable R (see [**10**, p.102]).

Consider now the same problem in the Lorentzian setting. Here we can use the fact that $\mathcal{J}_T$ is diagonalizable for any timelike T. For spacelike X orthogonal to timelike T we can apply $\mathcal{J}_{X+tT} = \mathcal{J}_X + 2t\mathcal{J}(X,T) + t^2\mathcal{J}_T$ on arbitrary $A \in \mathrm{Span}\{X,T\}^\perp$. Then we have $\mathcal{J}_{X+tT}(A) = \varepsilon_{X+tT}\mu_{X+tT}A$ for $\varepsilon_{X+tT} < 0$, while $\mathcal{J}_X(A) = \alpha A + B$, $\mathcal{J}(X,T)A = \beta A + C$, and $\mathcal{J}_T(A) = \varepsilon_T\mu_T A$, where $B, C \in A^\perp$. Thus $B + 2tC = 0$ holds for all $|t| > \sqrt{-\varepsilon_X/\varepsilon_T}$ which implies $B = C = 0$, and hence A is an eigenvector of $\mathcal{J}_X$. Therefore $\mathcal{J}_X$ is diagonalizable for any spacelike X and like before we have a constant sectional curvature.

THEOREM 5.1. *Any Lorentzian algebraic curvature tensor such that $\widetilde{\mathcal{J}}_X$ has a single eigenvalue for any nonnull $X \in \mathcal{V}$ has constant sectional curvature.*

This problem in the higher signatures is more complex. Potential examples of non-Osserman Jacobi-dual algebraic curvature tensor one can try to find with a minimal set of starting conditions. Then, it is logical to investigate algebraic curvature tensors such that $\widetilde{\mathcal{J}}_X$ has a single eigenvalue with the Jordan block of maximal order $(\dim\mathcal{V} - 1)$ for any nonnull $X \in \mathcal{V}$, because there $\widetilde{\mathcal{J}}_X$ has only one eigenvector (which is null). Is it possible to find that such an algebraic curvature tensor is an open problem for future investigations.

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