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SASAKI METRIC ON THE TANGENT BUNDLE OF A WEYL MANIFOLD

Cornelia-Livia Bejan and İlhan Gül

ABSTRACT. Let $(M,[g])$ be a Weyl manifold of dimension $m > 2$. By using the Sasaki metric G induced by g, we construct a Weyl structure on TM. Then we prove that it is never Einstein–Weyl unless (M,g) is flat. The main theorem here extends to the Weyl context a result of Musso and Tricerri.

1. Introduction

The framework of the tangent bundle TM of a manifold M provides a context in which several geometric objects can be studied. There is a huge literature devoted to the tangent bundle both in mathematics and theoretical physics. Sasaki introduced in [**14**] his well-known Riemannian metric on TM to study some geometric properties of TM endowed with the Sasaki metric. Some extensions of the Sasaki metric were constructed on TM by Abbassi and Sarih [**1**, **2**], Janyska [**8**], Kowalski and Sekizawa [**10**], Oproiu and Papaghiuc [**13**], Munteanu [**11**], Bejan and Druta-Romaniuc [**4**].

For some physical reasons, H. Weyl introduced in 1918 a unified field theory, in order to study a generalized metrical structure. On a manifold M, a Weyl structure is described as a conformal class of metrics $[g]$ preserved by a torsion-free connection D (called a Weyl connection). In the next year will be celebrated a century of a rich literature published on Weyl's geometry, important from various aspects, one of them being the topic of non-Riemannian connections. Roughly speaking, a Weyl manifold is a conformal manifold equipped with a Weyl connection which is a torsion-free connection preserving the conformal structure. The Weyl manifold is said to be Einstein–Weyl if the symmetric part of the Ricci tensor is proportional to the conformal metric. In particular, Einstein–Weyl manifolds appear as the natural background for static Yang–Mills–Higgs theory.

In Proposition 4.1, we obtain the behavior of the Sasaki metric on TM under the gauge transformations of the metrics in the conformal class $[g]$. The method of lifting several geometric objects from the base manifold to the total space of the tangent bundle is a very well known procedure in differential geometry. Starting with a Weyl structure on the base manifold we construct, in Proposition 4.2, a Weyl

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structure on the total space of the tangent bundle whose conformal class of metrics contains the Sasaki metric on TM. By using the curvature tensor field computed previously in Lemmas 4.1 and 4.2, we may state our main result in Theorem 4.1, which characterize (in terms of Sasaki metric) both Weyl structures on M and on TM to be simultaneously Einstein–Weyl.

The present paper is based on the article published by the first author in [**3**]. The study of the present paper, made on the tangent bundle of a Weyl manifold, will be continued by a forthcoming paper [**5**] on the cotangent bundle of a Weyl manifold.

2. Weyl Manifolds

In this section, M denotes a Weyl manifold, that is an m-dimensional manifold endowed with a Weyl structure W, which consists of a Riemannian metric g and a 1-form ϕ on M [**7**]. There exists (as in the Riemannian case) a unique torsion-free affine connection D on M, called the Weyl connection of W, such that $Dg = -2\phi \otimes g$. The Weyl connection is required to be invariant under the gauge transformation $g \mapsto e^{2\lambda} g$, which means that the 1-form ϕ must change as follows: $\phi \mapsto \phi - d\lambda$. Then the conformal class $C(W)$ determined by g forms a primary underlying structure.

The Levi-Civita connection ∇ of any metric $g \in C(W)$ is related to the Weyl connection D by

$$D_X Y = \nabla_X Y + \phi(Y)X + \phi(X)Y - g(X,Y)\xi, \quad \forall X, Y \in \chi(M), \tag{2.1}$$

where ξ is the dual vector field of ϕ with respect to g.

Note that the squared length $\|\xi\|^2$ of ξ with respect to g is given by $\|\xi\|^2 = g(\xi,\xi) = \phi(\xi)$.

Let $R_{[g]}$ and R_g denote respectively the curvature of the Weyl connection D and the curvature of the Levi-Civita connection ∇, defined by

$$R_{[g]}(X,Y) = D_X D_Y - D_Y D_X - D_{[X,Y]} \ \ \forall X, Y \in \chi(M),$$

and similarly for R_g. Then the relation between these curvature tensor fields is given by

$$\begin{aligned} R_{[g]}(X,Y)Z = {} & R_g(X,Y)Z + d\phi(X,Y)Z - ((\nabla_Y \phi)(Z))X + ((\nabla_X \phi)(Z))Y \\ & + \phi(Y)\phi(Z)X - g(Y,Z)\nabla_X \xi - g(Y,Z)\phi(\xi)X \\ & + g(Y,Z)\phi(X)\xi - \phi(X)\phi(Z)Y + g(X,Z)\nabla_Y \xi \\ & + g(X,Z)\phi(\xi)Y - g(X,Z)\phi(Y)\xi, \ \ \forall X, Y, Z \in \chi(M). \end{aligned} \tag{2.2}$$

From (2.2), it follows that the Ricci tensor field $\mathrm{Ric}_{[g]}$ of the Weyl connection D and the Ricci tensor field Ric_g of the Levi-Civita connection ∇ are related by

$$\begin{aligned} \mathrm{Ric}_{[g]}(X,Y) = {} & \mathrm{Ric}_g(X,Y) + d\phi(X,Y) + \big(\delta\phi - (m-2)\|\xi\|^2\big) g(X,Y) \\ & - (m-2)(\nabla_X \phi)Y + (m-2)\phi(X)\phi(Y), \ \ \forall X, Y, \in \chi(M), \end{aligned}$$

where the co-differential $\delta\phi$ of ϕ is defined by

$$\delta\phi = -\operatorname{trace}_g\{(U,V) \to (\nabla_U \phi)V\}.$$

Hence the symmetric part $\mathrm{Ric}^{\mathrm{sym}}_{[g]}$ of $\mathrm{Ric}_{[g]}$ is given by

$$\begin{aligned}\mathrm{Ric}^{\mathrm{sym}}_{[g]}(X,Y) &= \mathrm{Ric}_g(X,Y) + (\delta\phi - (m-2)\|\xi\|^2)g(X,Y) \\ &\quad - \frac{1}{2}(m-2)[(\nabla_X\phi)Y + (\nabla_Y\phi)X] \\ &\quad + (m-2)\phi(X)\phi(Y), \quad \forall X, Y, \in \chi(M).\end{aligned} \tag{2.3}$$

We recall the following:

DEFINITION 2.1. A manifold M endowed with a Weyl structure (g, ϕ) and the Weyl connection D is called an Einstein–Weyl manifold if the symmetrized Ricci tensor field $\mathrm{Ric}^{\mathrm{sym}}_{[g]}$ is proportional to a metric g representing the class $[g]$, that is there exists a smooth function α, such that

$$\mathrm{Ric}^{\mathrm{sym}}_{[g]}(X,Y) = \alpha g(X,Y), \quad \forall X, Y, \in \chi(M).$$

Since not every Weyl connection is Levi-Civita, it follows that Einstein–Weyl manifolds provide a natural generalization of Einstein geometry, see [**6**].

3. Geometric objects on M lifted to TM

Let M be a connected smooth m-dimensional manifold ($m > 2$), whose tangent bundle TM has the natural projection $\pi : TM \mapsto M$ defined by $\pi(x,u) = x$ for any $x \in M$ and $(x,u) \in TM$. To a local coordinate system $(U; x^1, \ldots, x^m)$ on M around $x \in M$ will correspond a local coordinate system $(\pi^{-1}(U); x^1, \ldots, x^m, u^1, \ldots, u^m)$ on TM around $(x,u) \in TM$, where for any $i = \overline{1,m}$, we identify the function $x^i \circ \pi$ on $\pi^{-1}(U)$ with x^i on U and we denote $u = \sum_{i=1}^m u^i(\frac{\partial}{\partial x^i})$ at any point $(x,u) \in \pi^{-1}(U)$. Then

$$\Big\{\Big(\frac{\partial}{\partial x^1}\Big)_{(x,u)}, \ldots, \Big(\frac{\partial}{\partial x^m}\Big)_{(x,u)}, \Big(\frac{\partial}{\partial u^1}\Big)_{(x,u)}, \ldots, \Big(\frac{\partial}{\partial u^m}\Big)_{(x,u)}\Big\}$$

is a basis for the tangent space $T_{(x,u)}(TM)$.

From [**15**], some geometric objects on the manifold M can be lifted to TM as follows.

If f is a function on M, then the vertical lift f^v of f is given by $f^v = f \circ \pi$. The horizontal lift f^h of f is $f^h = 0$.

Let X be a vector field on M which is locally represented by

$$X = \sum_{i=1}^m X^i \frac{\partial}{\partial x^i}.$$

If g is a Riemannian metric on M, whose Levi-Civita connection is ∇ with $\{\Gamma^i_{jk}\}$ its Christoffel symbols, then the vertical and horizontal lifts X^v and X^h of X are given respectively by

$$X^v_{(x,u)} = \sum_{i=1}^m X^i \frac{\partial}{\partial u^i}, \quad X^h_{(x,u)} = \sum_{i=1}^m X^i \frac{\partial}{\partial x^i} - \sum_{i,j,k=1}^m \Gamma^i_{jk} X^j u^k \frac{\partial}{\partial u^i}.$$

Let ω be a 1-form on M. Then the horizontal lift ω^h of ω is defined by

$$\omega^h(X^h) = 0, \ \omega^h(X^v) = (\omega(X))^v.$$

The vertical lift ω^v of ω is defined by

$$\omega^v(X^v) = 0, \ \ \omega^v(X^h) = (\omega(X))^v.$$

The Lie bracket operation of vector fields on the tangent bundle is given by

$$[X^v, Y^v]_{(x,u)} = 0, \ [X^h, Y^v]_{(x,u)} = (\nabla_X Y)^v_{(x,u)},$$
$$[X^h, Y^h]_{(x,u)} = [X, Y]^h_{(x,u)} - (R(X, Y)u)^v.$$

4. Sasaki metric on the tangent bundle

Any Riemannian metric g on a manifold M defines the Sasaki metric G on TM at any point $(x, u) \in TM$ by

$$G_{(x,u)}(X^h, Y^h) = g_x(X, Y) = G_{(x,u)}(X^v, Y^v),$$
$$G_{(x,u)}(X^h, Y^v) = 0, \ \forall X, Y \in \chi(M).$$

PROPOSITION 4.1. *The Sasaki metrics on TM, corresponding to any representative of the conformal class $[g]$ on M, form a class which is invariant under the vertical conformal change. That is, if g is a metric on the manifold M and G is its corresponding Sasaki metric on TM, then to any conformal change $g \mapsto e^\lambda g$ on M, will correspond the change of the Sasaki metric $G \mapsto (e^\lambda)^v G$ on TM.*

PROPOSITION 4.2. *Let (g, ϕ) be a Weyl structure on a manifold M and let G be the Sasaki metric on TM induced by g. Then (G, ϕ^v) is a Weyl structure on TM, whose Weyl connection $\bar{D}$ is given by*

$$\bar{D}_{X^h} Y^h = (D_X Y)^h - \frac{1}{2}(R_g(X, Y)u)^v,$$
$$\bar{D}_{X^h} Y^v = (D_X Y - \phi(Y)X + g(X, Y)\xi)^v + \frac{1}{2}(R_g(u, Y)X)^h,$$
$$\bar{D}_{X^v} Y^h = \frac{1}{2}(R_g(u, X)Y)^h + \phi(Y)X^v,$$
$$\bar{D}_{X^v} Y^v = -g(X, Y)\xi^h, \quad \forall X, Y \in \chi(M),$$

where D, R_g, ξ are respectively the Weyl connection on M, the curvature tensor field of g and the dual vector field of ϕ with respect to g.

PROOF. Since (M, g) is a Riemannian manifold whose tangent bundle TM is endowed with the Sasaki metric G, then the Levi-Civita connection $\bar{\nabla}$ of G is defined at any point $(x, u) \in TM$ by

$$\begin{aligned} &\bar{\nabla}_{X^h} Y^h = (\nabla_X Y)^h - \frac{1}{2}(R_g(X, Y)u)^v, \\ &\bar{\nabla}_{X^h} Y^v = (\nabla_X Y)^v + \frac{1}{2}(R_g(u, Y)X)^h, \ \ \bar{\nabla}_{X^v} Y^h = \frac{1}{2}(R_g(u, X)Y)^h, \\ &\bar{\nabla}_{X^v} Y^v = 0, \quad \forall X, Y \in \chi(M). \end{aligned} \tag{4.1}$$

By using twice relation (2.1) on M and similarly on TM, then we complete the proof. □

From (4.1), by straightforward calculations we obtain:

LEMMA 4.1. *Let g be a Riemannian metric on a manifold M and let G be its induced Sasaki metric on TM. Then the relation between the curvature tensor field R_g of the metric g on M and the curvature tensor field $\bar{R}_G$ of the metric G is given at any point $(x,u) \in TM$ by*

$$
\begin{aligned}
\bar{R}_G(X^h,Y^h)Z^h &= (R_g(X,Y)Z)^h + \frac{1}{4}(R_g(u,R_g(X,Z)u)Y \\
&\quad - R_g(u,R_g(Y,Z)u)X + 2R_g(u,R_g(X,Y)u)Z)^h \\
&\quad + \frac{1}{2}((\nabla_Z R_g)(X,Y)u)^v, \\
\bar{R}_G(X^h,Y^h)Z^v &= [R_g(X,Y)Z + \frac{1}{4}R_g(Y,R_g(u,Z)X)u \\
&\quad - R_g(X,R_g(u,Z)Y)u]^v + \frac{1}{2}[(\nabla_X R_g)(u,Z)Y \\
&\quad - (\nabla_Y R_g)(u,Z)X]^h, \\
\bar{R}_G(X^h,Y^v)Z^h &= \frac{1}{2}[(\nabla_X R_g)(u,Y)Z]^h + \frac{1}{2}[R_g(X,Z)Y \\
&\quad - \frac{1}{2}R_g(X,R_g(u,Y)Z)u]^v, \\
\bar{R}_G(X^h,Y^v)Z^v &= -\frac{1}{2}(R_g(Y,Z)X)^h - \frac{1}{4}(R_g(u,Y)R_g(u,Z)X)^h, \\
\bar{R}_G(X^v,Y^v)Z^h &= (R_g(X,Y)Z)^h + \frac{1}{4}[R_g(u,X)R_g(u,Y)Z \\
&\quad - R_g(u,Y)R_g(u,X)Z]^h, \\
\bar{R}_G(X^v,Y^v)Z^v &= 0, \quad \forall X,Y,Z \in \chi(M).
\end{aligned} \tag{4.2}
$$

LEMMA 4.2. *Let M be an m-dimensional manifold ($m > 2$) endowed with the Weyl structure (g,ϕ) and let (G,ϕ^v) be the induced Weyl structure on TM, where G is the Sasaki metric. Then the symmetric part $\overline{\mathrm{Ric}}^{\mathrm{sym}}_{[G]}$ of the Ricci tensor field of the Weyl structure (G,ϕ^v) on TM satisfies:*

$$
\begin{aligned}
\overline{\mathrm{Ric}}^{\mathrm{sym}}_{[G]}(X^h,Y^h) &= \overline{\mathrm{Ric}}_G(X^h,Y^h) - \frac{m}{m-2}\delta\phi\; g(X,Y) \\
&\quad + \frac{2(m-1)}{m-2}(\mathrm{Ric}^{\mathrm{sym}}_{[g]}(X,Y) - \mathrm{Ric}_g(X,Y))
\end{aligned} \tag{4.3}
$$

$$
\overline{\mathrm{Ric}}^{\mathrm{sym}}_{[G]}(X^v,Y^h) = \frac{1}{2}\sum_{i=1}^{m} g((\nabla_{e_i}R)(X,e_i)u,Y) + \frac{1}{2}\phi(R(u,X)Y), \tag{4.4}
$$

$$
\begin{aligned}
\overline{\mathrm{Ric}}^{\mathrm{sym}}_{[G]}(X^v,Y^v) &= \frac{1}{4}g(R(u,X)e_i,R(u,Y)e_i) \\
&\quad + (\delta\phi - 2(m-1)\|\xi\|^2)g(X,Y), \quad \forall X,Y \in \chi(M),
\end{aligned} \tag{4.5}
$$

where $\{e_i\}_{i=\overline{1,m}}$, ∇, R, Ric_g, $\overline{\mathrm{Ric}}_G$ *and* $\mathrm{Ric}^{\mathrm{sym}}_{[g]}$ *are respectively an orthonormal frame with respect to* g*, the Levi-Civita connection of* g*, the curvature of* g*, the Ricci tensor field of* g *and* G*, and the symmetric part of* $\mathrm{Ric}_{[g]}$ *(and all functions on* M *are identified with their vertical lift on* TM*).*

PROOF. We first apply Proposition 4.2. Then we use relation (2.3), in which we replace the Weyl structure (g,ϕ) on the m-dimensional manifold M, with the Weyl structure (G,ϕ^v) on the $2m$-dimensional manifold TM. From the relation (4.1), if we identify all functions on M with their vertical lift on TM, then we obtain:

$$\begin{aligned}
\overline{\mathrm{Ric}}^{\mathrm{sym}}_{[G]}(X^h,Y^h) &= \overline{\mathrm{Ric}}_G(X^h,Y^h) \\
&\quad + (\delta(\phi^v) - 2(m-1)G(\xi^h,\xi^h))G(X^h,Y^h) \\
&\quad - (m-1)[(\bar{\nabla}^h_X\phi^v)Y^h + (\bar{\nabla}^h_Y\phi^v)X^h] \\
&\quad + 2(m-1)\phi^v(X^h)\phi^v(Y^h) \\
&= \overline{\mathrm{Ric}}_G(X^h,Y^h) + [(\delta\phi - 2(m-1)\|\xi\|^2)g(X,Y)]^v \\
&\quad - (m-1)[((\nabla_X\phi)Y)^v + ((\nabla_Y\phi)X)^v] \\
&\quad + 2(m-1)[\phi(X)\phi(Y)]^v \\
&= \overline{\mathrm{Ric}}_G(X^h,Y^h) + (\delta\phi - 2(m-1)\|\xi\|^2)g(X,Y) \\
&\quad - (m-1)[(\nabla_X\phi)Y + (\nabla_Y\phi)X] \\
&\quad + 2(m-1)\phi(X)\phi(Y), \\
\overline{\mathrm{Ric}}^{\mathrm{sym}}_{[G]}(X^v,Y^h) &= \overline{\mathrm{Ric}}_G(X^v,Y^h) + \frac{1}{2}\phi(R(u,X)Y), \\
\overline{\mathrm{Ric}}^{\mathrm{sym}}_{[G]}(X^v,Y^v) &= \frac{1}{4}g(R(u,X)e_i,R(u,Y)e_i) \\
&\quad + (\delta\phi - 2(m-1)\|\xi\|^2)g(X,Y), \forall X,Y \in \chi(M).
\end{aligned} \tag{4.6}$$

By using (2.3) in the first equation of (4.6), it follows (4.3). Similarly, we obtain (4.4) and (4.5) if we use (4.2) in the last two equations of relation (4.6), which complete the proof. □

Now we state the main result:

THEOREM 4.1. *Let* (g,ϕ) *be a Weyl structure on an* m*-dimensional manifold* M $(m>2)$*, and* G *its induced Sasaki metric on* TM*. Then the Weyl structure* (G,ϕ^v) *is Einstein–Weyl on* TM *if and only if* (M,g) *is flat and* ϕ *satisfies the equation* $(\nabla_X\phi)Y + (\nabla_Y\phi)X = 2\phi(X)\phi(Y)$*.*

PROOF. To prove this statement, let $\{f_i\}_{i=\overline{1,m}}$ be an orthonormal frame around the point $x\in M$ and lift it to the orthonormal frame $\{F_1 = f_1^h,...,F_m = f_m^h, F_{m+1} = f_1^v,...,F_{2m} = f_m^v\}$ around $(x,u)\in TM$. Let R_g (resp. $\bar{R}_G$) be the curvature tensor field of the metric g on M (resp. G on TM), from (4.2). Then $\mathrm{Ric}_g(X,Y) = \sum_{i=1}^m g(R_g(X,f_i)f_i,Y)$ is the Ricci tensor field of (M,g) and similarly for the Ricci tensor field $\bar{R}_G$ of (TM,G). At any point $(x,u)\in TM$, we may write $u = u^i f_i$ and $R(f_i, f_j)f_k = R^h_{ijk} f_h$, with respect to the orthonormal basis $\{f_i\}_{i=\overline{1,m}}$, where the Einstein convention over repeated indices was used here.

If (G, ϕ^v) is Einstein–Weyl on TM, i.e., $\overline{\mathrm{Ric}}^{\mathrm{sym}}_{[G]} = \overline{\alpha} G$, then putting $X = Y = f_i$ in (4.3) and (4.5), we obtain

$$\begin{aligned}\overline{\alpha} &= \frac{1}{4}\sum_{j=1}^{m} \|R(u, f_j)f_i\|^2 + (\delta\phi - 2(m-1)\|\xi\|^2)\\ &= -\frac{1}{2}\sum_{j=1}^{m} \|R(f_i, f_j)u\|^2 - \frac{m}{m-2}(\mathrm{Ric}_g(f_i, f_i) + \delta\phi) + \frac{2(m-1)}{m-2}\,\mathrm{Ric}^{\mathrm{sym}}_{[g]}(f_i, f_i).\end{aligned}$$

Restricting the last identity to the zero section of TM, it follows

$$\mathrm{Ric}^{\mathrm{sym}}_{[g]}(f_i, f_i) = \frac{m}{2(m-1)}\,\mathrm{Ric}_g(f_i, f_i) + \big[\delta\phi - (m-2)\|\xi\|^2\big], \tag{4.7}$$

and we deduce that $\sum_{j=1}^{m}(\|R(f_i, f_j)u\|^2 + 2\|R(u, f_j)f_i\|^2) = 0$. Replacing u by f_k in the last formula and summing over i and k, we obtain

$$\sum_{i,j,k=1}^{m} \|R(f_i, f_j)f_k\|^2 = 0.$$

In particular, $\|R(f_i, f_j)f_k\|^2 = 0$, for all $i, j, k = 1, \ldots, m$. Hence $R(f_i, f_j)f_k = 0$, for all $i, j, k = 1, \ldots, m$, and then $R = 0$.

Now (4.7) becomes $\mathrm{Ric}^{\mathrm{sym}}_{[g]}(f_i, f_i) = [\delta\phi - (m-2)\|\xi\|^2]$, and using (2.3), we deduce that $(\nabla_X \phi)Y + (\nabla_Y \phi)X = 2\phi(X)\phi(Y)$.

Conversely, suppose that $(\nabla_X \phi)Y + (\nabla_Y \phi)X = 2\phi(X)\phi(Y)$; then (2.3) reduces to

$$\mathrm{Ric}^{\mathrm{sym}}_{[g]}(X, Y) = \mathrm{Ric}_{[g]}(X, Y) + \big[\delta\phi - (m-2)\|\xi\|^2\big] g(X, Y), \tag{4.8}$$

for all $X, Y \in \chi(M)$. Taking into account (4.8) and the fact that $R = 0$, then formulas (4.3)–(4.5) become

$$\begin{aligned}\overline{\mathrm{Ric}}^{\mathrm{sym}}_{[G]}(X^h, Y^h) &= (\delta\phi - (m-2)\|\xi\|^2) g(X, Y),\\ \overline{\mathrm{Ric}}^{\mathrm{sym}}_{[G]}(X^v, Y^h) &= 0,\\ \overline{\mathrm{Ric}}^{\mathrm{sym}}_{[G]}(X^v, Y^v) &= (\delta\phi - (m-2)\|\xi\|^2) g(X, Y).\end{aligned}$$

We deduce that $\overline{\mathrm{Ric}}^{\mathrm{sym}}_{[G]} = \overline{\alpha} G$, where $\overline{\alpha} = (\delta\phi - (m-2)\|\xi\|^2)^v$, and then (G, ϕ^v) is an Einstein–Weyl structure on TM. □

Remark 1. The fact that (M, g) is flat (and consequently (TM, G) is also flat), from Theorem 4.1, is equivalent to the fact that (TM, G) is Einstein (see [**12**]). Hence, a result obtained by Musso and Tricerri is extended in Theorem 4.1 to Weyl geometry.

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Seminar Matematic
Univ. "Al. I. Cuza",
Iasi, Romania
URL: http://math.etc.tuiasi.ro/bejan/
bejanliv@yahoo.com

Department of Mathematics, Istanbul Technical University
Istanbul, Turkey
igul@itu.edu.tr