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ON GEOMETRIC FORMALITY OF RATIONALLY ELLIPTIC MANIFOLDS IN DIMENSIONS 6 AND 7

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Abstract. We discuss the question of geometric formality for rationally elliptic manifolds of dimension 6 and 7. We prove that a geometrically formal six-dimensional biquotient with $b_2 = 3$ has the real cohomology of a symmetric space. We also show that a rationally hyperbolic six-dimensional manifold with $b_2 \leqslant 2$ and $b_3 = 0$ can not be geometrically formal. As it follows from their real homotopy classification, the seven-dimensional geometrically formal rationally elliptic manifolds have the real cohomology of symmetric spaces as well.

1. Introduction

The notion of geometric formality of a closed compact manifold M is defined by an existence of a metric g on M such that the exterior product of harmonic forms are again harmonic forms. It is proved in [**15**] that a geometrically formal manifold of dimension $\leqslant 4$ has the real cohomology of a symmetric space. Afterwards this notion has been further studied and there were provided many examples of non-geometrically formal homogeneous spaces [**17, 18, 11**], but also the examples of geometrically formal homogeneous spaces which are not homotopy symmetric spaces [**18**]. The notion of geometric formality has also been studied from the point of view of its relation to the different positive curvatures [**2, 1**].

In this note we investigate the question of geometric formality of rationally elliptic manifolds in small dimensions. The reason for considering rationally elliptic manifolds is that a rationally hyperbolic manifold has many relations in its real cohomology algebra comparing to the number of generators, which very often may appear as an obstruction to geometric formality. In addition, the same estimation on the Betti numbers that holds for the rationally elliptic manifolds [**8**] holds for the geometrically formal manifolds as well [**15**].

In Subsubection 2.2.1 and Section 4 we show that, from the classification of the rationally elliptic manifolds in dimensions five and seven it directly follows that in these dimensions any geometrically formal manifold has the real cohomology of

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a symmetric space. In Section 3 we consider the biquotients of dimension six for which $b_2 = 3$ and prove that any such geometrically formal biquotient has the real cohomology algebra of a symmetric space. We also show that a rationally hyperbolic six-dimensional manifold with $b_2 \leqslant 2$ and $b_3 = 0$ can not be geometrically formal.

2. Rationally elliptic manifolds and geometric formality

2.1. Notion of geometric formality. Let (M, g) be a closed oriented Riemannian manifold and $\Omega^*(M)$ its de Rham algebra of differential forms. A differential form $\omega \in \Omega^k(M)$ is said to be harmonic if $\Delta\omega = d\delta\omega + \delta d\omega = (d+\delta)^2\omega = 0$, where d is the exterior derivative, δ is coderivative and Δ is the Laplace-de Rham operator. To recall this in more detail, let $[,] : \Omega_x^k(M) \to \mathbb{R}$ be the scalar product in the space of differential forms at T_xM defined by

$$[\alpha_x, \beta_x] = \frac{1}{k!} \sum_{i_1,\dots,i_k,j_1,\dots,j_k} g^{i_1j_1} \cdots g^{i_kj_k} a_{i_1\dots i_k} b_{j_1\dots j_k},$$

where $\alpha = \frac{1}{k!} \sum\limits_{i_1,\dots,i_k} a_{i_1\dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$ and $\beta = \frac{1}{k!} \sum\limits_{j_1,\dots,j_k} b_{j_1\dots j_k} dx^{j_1} \wedge \dots \wedge dx^{j_k}$.

The scalar product on the space $\Omega^k(M)$ is defined by $\langle \alpha, \beta \rangle = \int_M [\alpha_x, \beta_x]\, d\,\mathrm{vol}_g$.

The Hodge star operator $* : \Omega^k(M) \to \Omega^{n-k}(M)$, $n = \dim M$, is defined by

$$\alpha_x \wedge (*\beta)_x = [\alpha_x, \beta_x]\, d\,\mathrm{vol}_{g_x}.$$

Then for $\alpha \in \Omega^{k-1}$ and $\beta \in \Omega^k$ it holds $\langle d\alpha, \beta \rangle = (-1)^k \langle \alpha, (*^{-1} d*)\beta \rangle$. It implies that the operator $\delta = (-1)^k *^{-1} d*$ is adjoint to d in the space of k-forms.

Denote by $\Upsilon(M, g) \subseteq \Omega^*(M)$ the graded linear subspace of harmonic forms. It is well known that any harmonic form is closed and no harmonic form is exact. In addition, the Hodge theorem states that any cohomology class $[\omega] \in H^*(M, \mathbb{R})$ contains unique harmonic representative. Thus, there exists an isomorphism between the graded vector spaces $\Upsilon(M, g)$ and $H^*(M, \mathbb{R})$.

It naturally arises the question about the existence of the metric g on M such that $\Upsilon(M, g)$ has an algebra structure under the exterior product $\wedge$. For such a metric the algebras $(\Upsilon(M, g), \wedge)$ and $(H^*(M, \mathbb{R}), \wedge)$ are isomorphic. This is defined in [**15**]:

Definition 2.1. A Riemannian metric g on M is said to be formal if the exterior product of its harmonic forms are harmonic forms.

Definition 2.2. A closed Riemannian manifold M is said to be geometrically formal if it admits a formal Riemannian metric.

The following examples of geometrically formal manifolds are well known: the real cohomology spheres are geometrically formal since they have, up to constant, just one harmonic form; the symmetric spaces G/H are geometrically formal for an invariant metric g. The second one follows from the observations [**6**] that any G-invariant form on a symmetric space G/H is closed and none is exact. In addition, invariant forms $\Omega^G(G/H)$ form an algebra under the exterior product. Since harmonic forms for an invariant metric g are G-invariant, it follows that $\Omega^G(G/H)$ coincides with $\Upsilon(G/H, g)$ and, thus, $(\Upsilon(G/H, g), \wedge)$ is an algebra.

We found useful to note the following:

LEMMA 2.1. *Assume that the manifold M is not geometrically formal. Then the product metric $g = g_M \times g_N$ on $M \times N$ can not be formal for any closed manifold N and any Riemannian metrics g_M on M and g_N on N.*

PROOF. Assume that product metric g on $M \times N$ is a formal metric for some closed manifold N and some Riemannian metrics g_M on M and g_N on N. We claim that the metric g_M is also formal. To see that let α be a harmonic form on M relative to the metric g_M and let $*_M$ be the corresponding star operator. Then α is a harmonic form on $M \times N$ relative to the metric g. Namely, since T_xM and T_xN are orthogonal for the metric g we have that $[\beta_{(x,y)}, \alpha_{(x,y)}] = [\beta^M_{(x,y)}, \alpha_x]$, where $\alpha_{(x,y)} = \alpha_x$ and $\beta^M_{(x,y)}$ is the restriction of the from $\beta_{(x,y)}$ on $T_xM \subset T_{(x,y)}(M \times N)$. More precisely, if

$$\beta_{(x,y)} = \frac{1}{k!} \sum_{j_1,\dots,j_k} \sum_{s=0}^{k} b_{j_1\dots j_k}(x,y) dx^{j_1} \wedge \cdots dx^{j_s} \wedge dy^{j_{s+1}} \wedge \cdots dy^{j_k},$$

then

$$\beta^M_{(x,y)} = \frac{1}{k!} \sum_{j_1,\dots,j_k} b_{j_1\dots j_k}(x,y) dx^{j_1} \wedge \cdots \wedge dx^{j_k}.$$

Since

$$\begin{aligned} \beta_{(x,y)} \wedge (*_M\alpha)_x \wedge (\mathrm{vol}_N)_y &= \beta^M_{(x,y)} \wedge (*_M\alpha)_x \wedge (\mathrm{vol}_N)_y \\ = [\beta^M_{(x,y)}, \alpha_x](\mathrm{vol}_M)_x(\mathrm{vol}_N)_y &= [\beta_{(x,y)}, \alpha_{(x,y)}](\mathrm{vol}_{M\times N})_{(x,y)}, \end{aligned}$$

we obtain that on $M \times N$ it holds $*\alpha = *_M\alpha \wedge \mathrm{vol}_N$. It further implies that $d(*\alpha) = d(*_M\alpha) \wedge \mathrm{vol}_N \pm *_M \alpha \wedge d(\mathrm{vol}_N) = 0$, since obviously $d(\mathrm{vol}_N) = 0$ and $d(*_M\alpha) = 0$. Therefore if α and β are harmonic forms on M then $\alpha \wedge \beta$ is a harmonic form on $M \times N$. The restriction of $\alpha \wedge \beta$ on M is the same form, so it follows that $\alpha \wedge \beta$ is a harmonic form on M and the metric g_M is formal, which is a contradiction. □

REMARK 2.1. Let us point out one useful observation. Assume that a manifold M is geometrically formal and consider its cohomology ring $H^*(M,\mathbb{R})$ with its generators and relations. Choose harmonic form in each generator for $H^*(M,\mathbb{R})$. Then these harmonic forms satisfy the same relations as the corresponding generators in $H^*(M,\mathbb{R})$. In many cases the existence of such forms leads to the contradiction meaning that the cohomology structure is often an obstruction to geometric formality.

2.1.1. *Relation between rational formality and geometric formality.*

DEFINITION 2.3. A manifold M is formal in the sense of rational homotopy theory if $\Omega^*(M)$ is weakly equivalent to $H^*(M,\mathbb{R})$:

$$(\Omega^*(M), d) \leftarrow (C, d) \rightarrow (H^*(M), d = 0), \tag{2.1}$$

where both homomorphisms induce isomorphisms in cohomology.

The first well known examples of formal spaces are the manifolds having free cohomology algebras, then Kaehler manifolds, compact symmetric spaces, etc. Note that the first proof of formality of compact symmetric spaces is based on the fact we already recalled that an invariant metric on a compact symmetric space is formal. Thus, in this case to prove formality one can take $(C, d) = (\Upsilon(G/H), 0)$ in (2.1), where $\Upsilon(G/H)$ is an algebra of harmonic forms for an invariant metric.

In addition it is known: all homogeneous spaces G/H with $\operatorname{rk} H = \operatorname{rk} G$ are formal [**20**], all closed simply connected manifolds of dimension $\leqslant 6$ are formal [**19**], all closed simply connected 7-dimensional manifolds M with $b_2(M) \leqslant 1$ are formal [**10**].

REMARK 2.2. A geometrically formal manifold M is formal:

$$(\Omega(M), d) \leftarrow (\Upsilon(M), d) \rightarrow (H^*(M), d = 0).$$

The converse is not true. For example, it is proved in [**17**] that the complete flag manifolds $SU(n+1)/T^n$ are not geometrically formal, although they are formal since $\operatorname{rk} SU(n+1) = \operatorname{rk} T^n = n$. Moreover, none of the complete flag manifolds of a simple compact Lie group is geometrically formal, although they are all formal. This is proved in [**17**] for the classical Lie groups and G_2 and in [**11**] for the exceptional Lie groups. For all these spaces their cohomology ring structure is an obstruction for geometric formality. On the other hand, in [**18**] are provided the series of Stiefel manifolds for which it is proved to be geometrically formal and not homotopy equivalent to a symmetric space.

2.2. Rationally elliptic manifolds and geometric formality. Let X be a simply connected topological space of finite type, that is $\dim H_k(X) < \infty$ for any k.

DEFINITION 2.4. X is said to be rationally elliptic if $\operatorname{rk} \pi_x(X) = \dim_{\mathbb{Q}} \pi_*(X) \otimes \mathbb{Q}$ is finite and it is said to be rationally hyperbolic if $\operatorname{rk} \pi_k(X) = \dim_{\mathbb{Q}} \pi_*(X) \otimes \mathbb{Q}$ is infinite.

EXAMPLE 2.1. The compact homogeneous spaces and the biquotients of compact Lie groups are rationally elliptic spaces, see [**8**].

The ranks of the homotopy groups of a rationally elliptic space X, $\dim X = n$ satisfy [**8**]:

$$\sum_k 2k \cdot \operatorname{rk} \pi_{2k}(X) \leqslant n, \quad \sum_k (2k+1) \cdot \operatorname{rk} \pi_{2k+1}(X) \leqslant 2n - 1. \tag{2.2}$$

We want to consider the question of geometric formality, or more precisely the weaker question of the real cohomology structure of geometrically formal manifolds, for rationally elliptic spaces. Why to consider rationally elliptic spaces?

The first reason comes from the fact that the Betti numbers of a geometrically formal manifold M satisfy [**15**]: $b_i(M) \leqslant b_i(T^{\dim M})$, $1 \leqslant i \leqslant \dim M$. It implies that

$$\sum_{i=1}^{\dim M} b_i(M) \leqslant 2^{\dim M}. \tag{2.3}$$

On the other hand, it is known [**8**] that the Betti numbers of a rationally elliptic space X satisfy inequality (2.3) as well.

The second reason is that a rationally hyperbolic space has many relations in its real cohomology algebra comparing to the number of generators. Namely, let us recall [**8**] that a free algebra $(\wedge V, d)$ is said to be a minimal model for a commutative differential graded algebra $(\mathcal{A}, d_{\mathcal{A}})$ if $d(V) \subset \wedge^{\geqslant 2} V$ and there exists a morphism $f : (\wedge V, d) \to (\mathcal{A}, d_{\mathcal{A}})$, which induces an isomorphism in cohomology. The minimal model $\mu(X)$ of a simply connected topological space X of a finite type is defined to be the minimal model of $A_{PL}(X)$. It is well known that $\mu(X)$ is unique up to isomorphism and it classifies the rational homotopy type of X. Moreover, the ranks of the homotopy groups for X are given by the numbers of the generators of the corresponding degree in the minimal model $\mu(X)$.

For a rationally formal simply connected space X, the minimal model $\mu(X)$ coincides with the minimal model of $(H^*(X, \mathbb{Q}), d = 0)$. Therefore, the minimal model of a formal simply connected space can be obtained from its cohomology algebra. One just starts, see [**8**], with the cohomology generators of degree two and builds up the minimal model by adding the generators of higher degree to eliminate the cohomology relations, but in the same time keeping the freeness of the minimal model. Thus, since for a rationally hyperbolic formal space X, $\mu(X)$ has infinite number of generators, the number of relations in $H^*(X, \mathbb{Q})$ is quite large comparing to the number of generators in $H^*(X, \mathbb{Q})$.

Note that $\mu(X)$ and $\mu(X) \otimes_{\mathbb{Q}} \mathbb{R}$ have the same number of generators and $\mu(X) \otimes_{\mathbb{Q}} \mathbb{R}$ is the minimal model for $(H^*(X, \mathbb{R}), d = 0)$ for a formal X. It implies that the number of relations in $H^*(X, \mathbb{R})$ for a rationally hyperbolic formal space X is quite large as well. Therefore, taking into account Remark 2.1, the rationally hyperbolic formal manifolds are hardly to expect to admit a formal metric.

From the side of geometry, it is conjectured by Gromov [**12**] that the estimation (2.3) holds for positively curved manifolds, while there is also conjecture by Bott [**13**] that a simply connected manifold which admits a metric of non-negative sectional curvature is rationally elliptic. This brought attention to the study of the connection between positive curvature and geometric formality. In that context the following results are known.

- It is proved in [**2**] that for a simply connected compact oriented Riemannian 4-manifold M which is geometrically formal and has non-negative sectional curvature one of the following holds: M is homeomorphic to S^4, M is diffeomorphic to $\mathbb{C}P^2$ or M is isometric to $S^2 \times S^2$ with product metric where both factors carry metrics with positive curvature.
- A homogeneous geometrically formal metric of positive curvature is either symmetric or a metric on a rationally homology sphere, see [**1**].
- The normal homogeneous metric on Alloff-Wallach spaces is not geometrically formal [**18**], but it is not positively curved as well. It is proved in [**1**] that no other homogeneous metric is geometrically formal as well.

REMARK 2.3. We further discuss the notion of geometric formality for the rationally elliptic manifolds whose dimension is $\geqslant 5$, because of the more general result of [**15**] which states that a closed oriented geometrically formal manifold of dimension $\leqslant 4$ has the real cohomology algebra of a compact globally symmetric space.

2.2.1. *Five-dimensional rationally elliptic manifolds.* The following results are known:

- All five-dimensional simply connected rationally elliptic manifolds have the rational homotopy type of S^5 or $S^2 \times S^3$ [**21, 24**];
- There are four diffeomorphism types five-dimensional biquotients [**3**]:

$$S^5, \ \ S^2 \times S^3, \ \ X_{-1} = SU(3)/SO(3), \ \ X_\infty.$$

The manifolds X_{-1} and X_∞ are obtained by gluing two copies of non-trivial three dimensional disc bundles over S^2 along the common boundary $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. The Wu manifold X_{-1} is real cohomology sphere S^5, while $H^*(X_\infty, \mathbb{R}) = H^*(S^2 \times S^3, \mathbb{R})$.

Thus, all geometrically formal five-dimensional simply connected rationally elliptic manifolds have the real cohomology of a symmetric space. Among biquotients, $S^5, S^2 \times S^3$ and X_{-1} are geometrically formal, while for X_∞ it is for us an open question.

3. Six-dimensional rationally elliptic manifolds

The second Betti number of a six-dimensional rationally elliptic manifold is by (2.2) less than or equal to 3. The following results are known:

- All six-dimensional rationally elliptic manifolds with $b_2 \leqslant 1$ have the real cohomology of S^6, $S^3 \times S^3$, $S^2 \times S^4$ and $\mathbb{C}P^3$ [**14, 23**].
- All six-dimensional rationally elliptic manifolds with $b_2 = 2$ have the real homotopy type of $\mathbb{C}P^2 \times S^2$, $SU(3)/T^2$ or $\mathbb{C}P^3 \# \mathbb{C}P^3$ [**14**].
- All six dimensional rationally elliptic manifolds with $b_2 = 3$ have the rational homotopy groups of $S^2 \times S^2 \times S^2$ [**23**],

The first result on the real cohomology structure of the geometrically formal rationally elliptic six-manifolds for which $b_2 \leqslant 2$ is as follows [**23**]:

Proposition 3.1. *All geometrically formal six-dimensional rationally elliptic manifolds with $b_2 \leqslant 2$ have the real cohomology of a symmetric space.*

Corollary 3.1. *The manifolds $SU(3)/T^2$ and $\mathbb{C}P^3 \# \mathbb{C}P^3$ are not geometrically formal.*

We discuss here the question of geometric formality for some simply-connected six-dimensional biquotients for which $b_2 = 3$.

Let us recall some notions and results on general six-dimensional biquotients. The biquotient $G /\!\!/ H$ is said to be reduced if G is simply-connected, H is connected and no simple factor of H acts transitively on any simple factor of G. By the result of Totaro [**25**] any compact simply-connected biquotient is diffeomorphic to reduced ones. The biquotient is said to be decomposable if it can be obtained as the total space of $G_1 /\!\!/ H_1$ bundle over $G_2 /\!\!/ H_2$. It is proved [**4**] that a reduced compact simply connected six-dimensional biquotient $G /\!\!/ H$ satisfies one of the following statements:

(1) it is diffeomorphic to a homogeneous space or Eschenburg inhomogeneous flag manifold $SU(3)/\!\!/T^2$;
(2) it is decomposable;
(3) it is diffeomorphic to $S^5 \times_{T^2} S^3$ or $(S^3)^3/\!\!/T^3$.

The only irreducible homogeneous space of dimension 6 which does not have the cohomology of a symmetric space is $SU(3)/T^2$ and it is not geometrically formal. The Eschenburg inhomogeneous flag manifold $SU(3)/\!\!/T^2$ is neither geometrically formal as it is proved in [**17**].

We analyze now the following decomposable biquotients: three $\mathbb{C}P^2$ bundles over S^2 and infinitely many S^2 bundles with base a 4-dimensional biquotient: $\mathbb{C}P^2$, $S^2 \times S^2$, $\mathbb{C}P^2\#\mathbb{C}P^2$, $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$. Any bundle from the infinite families of the considered bundles has the second Betti number equal to 3.

LEMMA 3.1. *All three $\mathbb{C}P^2$ bundles over S^2 have the real cohomology of $\mathbb{C}P^2 \times S^2$, that is of a symmetric space.*

PROOF. Any $\mathbb{C}P^2$-bundle E over S^2 is obtained as the projectivisation of rank three complex vector bundle over S^2. Therefore, the integral cohomology of its total space M is generated by two generators x and y of degree 2 subject to the relations

$$x^2 = 0, \ \ y^3 + c_1xy^2 = 0.$$

If we put $y_1 = y + \frac{c_1}{3}x$ then x and y_1 generate the real cohomology ring of M and satisfy the relations $x^2 = 0$, $y_1^2 \neq 0$, $y_1^3 = 0$ and $xy_1^2 = y^2x \neq 0$. □

Note that the cohomology structure can not be obstruction for geometric formality of any of these bundles. The trivial bundle $S^2 \times \mathbb{C}P^2$ is geometrically formal, while for the other two bundles we can remark that if some of them admits a formal metric it admits a symplectic structure as well.

It is proved in [**17**] that any of the infinitely many S^2 bundle over $\mathbb{C}P^2$ is geometrically formal if and only if it is a trivial bundle $S^2 \times \mathbb{C}P^2$. Applying the same argument as it is done in [**17**] for these family of bundles, we prove the following:

THEOREM 3.1. *None of the infinitely many non-trivial S^2-bundles over $\mathbb{C}P^2\#\mathbb{C}P^2$ is geometrically formal.*

PROOF. Let M be the total space of a S^2-bundle over $\mathbb{C}P^2\#\mathbb{C}P^2$. Then M is the unit sphere bundle in the associated rank 3 vector bundle and it is obtained by the projectivisation of rank 2 complex vector bundle E. Therefore the integral cohomology of M is given by $H^*(M) = H^*(\mathbb{C}P^2\#\mathbb{C}P^2, \mathbb{Z})[y]$ subject to the relation

$$y^2 + c_1(E)y + c_2(E) = 0, \tag{3.1}$$

where $c_1(E)$ and $c_2(E)$ are the pull backs of the first and second Chern classes from $H^*((\mathbb{C}P^2\#\mathbb{C}P^2, \mathbb{Z})$. The cohomology ring $H^*(\mathbb{C}P^2\#\mathbb{C}P^2, \mathbb{Z})$ has two generators x_1, x_2 of degree 2 satisfying $x_1^2 = x_2^2$, $x_1x_2 = 0$ and $x_1^3 = 0$. Relation (3.1) reads

$$y^2 + (ax_1 + bx_2)y + cx_1^2 = 0 \text{ for } a, b, c \in \mathbb{Z}.$$

Let $z = y + \frac{a}{2}x_1 + \frac{b}{2}x_2$, then $z^2 = y^2 + (ax_1 + bx_2)y + \frac{a^2+b^2}{4}x_1^2$. It follows that

$$z^2 + dx_1^2 = 0, \text{ where } d = c - \frac{a^2 + b^2}{4} \tag{3.2}$$

and x_1, x_2, z are the cohomology generators for the real cohomology ring $H^*(M, \mathbb{R})$. We obtain that $z^2x_1 = z^2x_2 = 0$ and $z^3 = -dzx_1^2 = -dzx_2^2$, what implies that zx_1^2 is top degree cohomology class.

Assume that M is geometrically formal. Let ω_1 and η be the harmonic representatives for x_1 and z respectively. Since $\omega_1^3 = 0$ it follows that the kernel foliation of ω_1 is at least two-dimensional. Let v_1 and v_2 be linearly independent vector fields of this foliation. From (3.2), it follows $i_{v_1}(\eta^2) = 2(i_{v_1}\eta)\eta = 0$.

If $d \neq 0$ then $\eta^3 = -d\eta\omega_1^2$ is a volume form on M. But, $i_{v_1}(\eta^3) = 3(i_{v_1}\eta)\eta^2 = 0$, which is a contradiction.

If $d = 0$ then $4c = a^2 + b^2$, what implies that the integers a and b are even. It further implies that $w_2(V) = c_1(V) \pmod 2 = 0$ and $p_1(V) = c_1^2(E) - 4c_2(E) = (a^2 + b^2 - 4c)x_1^2 = 0$. Therefore, by [**22, 5**] the bundle M is trivial that is $M = S^2 \times (\mathbb{C}P^2 \# \mathbb{C}P^2)$. The connected sum $\mathbb{C}P^2 \# \mathbb{C}P^2$ is not geometrically formal, since it is known not to admit a symplectic structure. It follows by Lemma 2.1 that no product metric on M is formal. □

THEOREM 3.2. *None of the infinitely many S^2-bundles over $S^2 \times S^2$ which does not have the real cohomology of $(S^2)^3$ is geometrically formal.*

PROOF. As previously, the bundle M is obtained by the projectivisation of rank 2 complex vector bundle E. The integral cohomology of M is given by $H^*(M) = H^*(S^2 \times S^2, \mathbb{Z})[y]$ subject to the relation:

$$y^2 + (ax_1 + bx_2)y + cx_1x_2 = 0 \text{ for } a, b, c \in \mathbb{Z}, \tag{3.3}$$

where x_1, x_2 are the pull backs of the generators of the cohomology ring $H^*(S^2 \times S^2, \mathbb{Z})$ and they satisfy relations $x_1^2 = x_2^2 = 0$. Let $z = y + \frac{a}{2}x_1 + \frac{b}{2}x_2$. Then x_1, x_2 and z represent the generators for $H^*(M, \mathbb{R})$ and in terms of these generators the relation (3.3) reads

$$z^2 + qx_1x_2 = 0, \tag{3.4}$$

where $q = c - \frac{ab}{2}$. Since $z^2x_1 = z^2x_2 = 0$ we conclude that x_1x_2z is non-zero top-degree cohomology class on M.

Assume that M is geometrically formal and let ω, η_1 and η_2 be the harmonic representatives for z, x_1 and x_2. We have that $\eta_1^2 = \eta_2^2 = 0$, which implies that there exist linearly independent vector fields v_1 and v_2 in the intersection of the kernel foliations for η_1 and η_2. It follows from (3.4) that $i_{v_1}i_{v_2}\omega^2 = 0$, so $\omega^2\eta_1$ and $\omega^2\eta_2$ can not be the volume forms on M. Thus, the volume form must be $\omega\eta_1\eta_2$.

If $q \neq 0$ in (3.4) then it is easy to see that M does not have the real cohomology of $S^2 \times S^2 \times S^2$. The assumption that M is geometrically formal implies that ω^3 is a volume form on M as well, which is in contradiction with the fact $i_{v_1,v_2}\omega^2 = 0$.

If $q = 0$, then M has the real cohomology of $S^2 \times S^2 \times S^2$. In this case we have that $ab = 2c$, which implies that $p_1(V) = c_1^2(E) - 4c_2(E) = (ax_1 + bx_2)^2 - 4cx_1x_2 = 0$. Note that if the both integers a and b are even, then $w_2(V) = 0$ which implies that this bundle is trivial, that is $M = S^2 \times S^2 \times S^2$, which is geometrically formal symmetric space. □

THEOREM 3.3. *None of the infinitely many S^2-bundles over $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ which does not have the real cohomology of $(S^2)^3$ is geometrically formal.*

PROOF. Let M be the total space of a S^2-bundle over $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. The real cohomology ring for $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ is the same as for $S^2 \times S^2$. Therefore, as in the proof of previous theorem, we conclude that if M does not have the real cohomology of $S^2 \times S^2 \times S^2$ then M can not be geometrically formal.

Let $c_1(E) = ax_1 + bx_2$ and $c_2(E) = cx_1^2$ are the the pullbacks of the first and the second Chern classes for E, where x_1 and x_2 are the generators for $H^*(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, \mathbb{Z})$. Then, as previously, the real cohomology ring for M is also generated by x_1, x_2 and z such that $z^2 + dx_1^2 = 0$, where $d = c - \frac{a^2 - b^2}{4}$. It implies that M has the real cohomology of $(S^2)^3$ if and only if $4c = a^2 - b^2$. In this case $p_1(V) = 0$ and also the integers a and b are of the same parity. If both a and b are even, then $w_2(V) = 0$ and the bundle E is trivial, that is $M = S^2 \times (\mathbb{C}P^2 \# \overline{\mathbb{C}P^2})$. It is proved in [**16**] that $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ admits no formal metric, what implies that no product metric on M is formal. □

COROLLARY 3.2. *None of the biquotients from the infinite families of the six-dimensional biquotients of the form $(SU(2))^3 /\!\!/ T^3$ different from $S^2 \times S^2 \times S^2$ is geometrically formal.*

PROOF. The six-dimensional biquotients of the form $(SU(2))^3 /\!\!/ T^3$ are classified in [**4**]. They are parametrized by the three families of infinite matrices and four sporadic matrices. In the same paper it is established which of these biquotients that correspond to these matrices are diffeomorphic. For the biquotients considered in [**24**] which give one family of these biquotients and whose real cohomology ring has three generators x_1, x_2, x_3 subject to the relations $x_1^2 = 0$, $ax_1x_2 + x_2^2 + x_2x_3 = 0$, $bx_1x_3 + 2x_2x_3 + x_3^2 = 0$, it is proved in [**18**] that they are not geometrically formal. We want to remark that, as it is pointed in [**4**], these biquotients are decomposable meaning that any of them can be obtained as S^2 bundle over $S^2 \times S^2$ or as S^2 bundle over $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, so Theorem 3.2 and Theorem 3.3 provide the new proof that they are not geometrically formal. The other family of these biquotients has the real cohomology ring generated by x_1, x_2, x_3 subject to the relations $x_1^2 = 0$, $x_2^2 + bx_1x_2 = 0$, $x_3^2 + c_1x_1x_3 + c_2x_2x_x = 0$. This family is considered in [**23**], where it is proved that none of these biquotients which does not have the real cohomology of $(S^2)^3$ is not geometrically formal. The third family of the biquotients has the real cohomology generators x_1, x_2, x_3 subject to the relations $x_1^2 + 2x_1x_2 = 0$, $x_2^2 + x_1x_2 = 0$, $x_3^2 + c_1x_1x_3 + c_2x_2x_3 = 0$. For this family it is proved in [**4**] that they are decomposable meaning that they can be represented as S^2 bundles over $\mathbb{C}P^2 \# \mathbb{C}P^2$, so Theorem 3.1 proves that they are not geometrically formal. □

REMARK 3.1. Note that the biquotients from Corollary 3.2 belong to the third case in the description of six-dimensional biquotients that is given in [**4**]. This condition also describes the manifold $\mathbb{C}P^3 \# \mathbb{C}P^3$.

3.1. On some hyperbolic six-dimensional manifolds. We show that none of the hyperbolic, closed, simply-connected six-dimensional manifolds for which $b_2(M) \leqslant 2$ and $b_3(M) = 0$ can be geometrically formal because of its cohomology structure. For such a manifold it is known [**14**] that it is rationally hyperbolic if and only if it has the real homotopy type of $(S^2\times S^4)\#\mathbb{C}P^3$ or $(S^2\times S^4)\#(S^2\times S^4)$.

PROPOSITION 3.2. *A manifold having real cohomology structure of $(S^2 \times S^4)\#$ $\mathbb{C}P^3$ or $(S^2 \times S^4)\#(S^2 \times S^4)$ can not be geometrically formal.*

PROOF. The manifold $(S^2\times S^4)\#\mathbb{C}P^3$ has three cohomology generators x, y, z, such that $\deg x = \deg z = 2$ and $\deg y = 4$ and $x^2 = 0$, $xz = 0$ and $yz = 0$. If assumed that this manifold is geometrically formal, we have harmonic forms α, β and γ representing the classes x, y, z respectively, which satisfy the same relations as these classes. Since $\alpha^2 = 0$ this form have four-dimensional kernel foliation. Denote by v_1, v_2, v_3, v_4 linearly independent vector fields of this foliation. Since $\alpha\gamma = 0$ we obtain that $0 = i_{v_i}i_{v_j}(\alpha\gamma) = \alpha\gamma(v_i, v_j)$, what implies $\gamma(v_i, v_j) = 0$. It further gives

$$i_{v_4}i_{v_3}i_{v_2}i_{v_1}(\gamma^3) = i_{v_4}i_{v_3}i_{v_2}(3i_{v_1}(\gamma)\gamma^2) = 3i_{v_4}i_{v_3}(\gamma(v_1,v_2)\gamma^2) - 2i_{v_1}(\gamma)i_{v_2}(\gamma)\gamma$$
$$= -6i_{v_4}(\gamma(v_1,v_3) - i_{v_1}(\gamma)(\gamma(v_2,v_3)\gamma - i_{v_2}(\gamma)i_{v_3}(\gamma)) = -6i_{v_4}(i_{v_1}\gamma i_{v_2}\gamma i_{v_3}\gamma) = 0.$$

This is in contradiction with the fact that γ^3 is a volume form.

The manifold $(S^2\times S^4)\#(S^2\times S^4)$ has four cohomology generators x_1, x_2, y_1, y_2 such that $\deg x_1 = \deg x_2 = 2$ and $\deg y_1 = \deg y_2 = 4$, which satisfy relations $x_1^2 = x_2^2 = 0$, $x_1x_2 = 0$ and $x_1y_2 = x_2y_1 = 0$. If this manifold is geometrically formal, we would have that the harmonic forms α_1 and α_2, which represent the cohomology classes x_1 and x_2, satisfy $\alpha_1^2 = \alpha_2^2 = 0$. Therefore, the kernel foliations for α_1 and α_2 are four-dimensional. We denote their basis by v_1, v_2, v_3, v_4 and u_1, u_2, u_3, u_4 respectively.

Let β_1 and β_2 be harmonic representatives for y_1 and y_2. Since $\alpha_1\beta_2 = 0$ and $\alpha_2\beta_1 = 0$ we obtain that $\beta_2(v_1, v_2, v_3, v_4) = 0$ and $\beta_1(u_1, u_2, u_3, u_4) = 0$.

The intersection of the kernel foliations for α_1 and α_2 is at least two-dimensional. Note that this kernel foliations can not coincide since this gives contradiction with the fact that $\alpha_1\beta_1$ and $\alpha_2\beta_2$ are volume forms.

Assume that the kernel intersection is two-dimensional and let $v_1 = u_1$ and $v_2 = u_2$ be a basis of this intersection. Since $\alpha_1\alpha_2 = 0$, we obtain that $0 = i_{v_3,v_4}(\alpha_1\alpha_2) = \alpha_2(v_3, v_4)\alpha_1$ what gives $\alpha_2(v_3, v_4) = 0$. Therefore, $(\alpha_2\beta_2)(u_1, u_2, u_3, u_4, v_3, v_4) = \alpha_2(v_3, v_4)\beta_2(u_1, u_2, u_3, u_4) = 0$, contradicting the fact that $\alpha_2\beta_2$ is a volume form.

If the kernel intersection is three dimensional, let $v_1 = u_1, v_2 = u_2$, $u_3 = v_3$ and denote by $v_4 \in \mathrm{Ker}(\alpha_1), v_4 \notin \mathrm{Ker}(\alpha_2)$ and $u_4 \in \mathrm{Ker}(\alpha_1), u_4 \in \mathrm{Ker}(\alpha_2), u_4 \notin \mathrm{Ker}(\alpha_1)$. Then from $\alpha_1\beta_2 = 0$ it follows that $i_{u_4}\alpha_1\beta_2 + \alpha_1 i_{u_4}\beta_2 = 0$ and from $\alpha_1\alpha_2 = 0$ it follows that $i_{u_4}\alpha_1 i_{v_4}\alpha_2 = 0$. Further, there exists vector field x orthogonal to the sum of these foliations $\mathrm{Ker}(\alpha_1) \oplus \mathrm{Ker}(\alpha_2)$. We obtain that $\alpha_1(u_4, x)i_{v_4}\alpha_2 - \alpha_2(v_4, x)i_{u_4}\alpha_1 = 0$. Note that $\alpha_1(u_4, x), \alpha_2(v_4, x) \neq 0$ since, say, for $\alpha_2(v_4, x) = 0$ we would have $\alpha_2\beta_2(u_1, u_2, u_3, u_4, v_4, x) = \alpha_2(v_4, x)\beta_2(u_1, u_2, u_3, u_4)$ $= 0$, which is in contradiction with $\alpha_2\beta_2$ being volume form. Therefore, $i_{u_4}\alpha_1 = \frac{\alpha_1(u_4,x)}{\alpha_2(v_4,x)} i_{v_4}\alpha_2$ which, together with previous, implies $\frac{\alpha_1(u_4,x)}{\alpha_2(v_4,x)} i_{v_4}\alpha_2\beta_2 + \alpha_1 i_{u_4}\beta_2 = 0$.

Therefore, we obtain that $i_{v_4}\alpha_2 i_{u_4}\beta_2 = 0$ contradicting that $\alpha_2\beta_2$ is a volume form. $\square$

4. Seven-dimensional rationally elliptic manifolds

It is proved in [**14**] that a closed simply-connected seven-dimensional manifold is rationally elliptic if and only if it has the real homotopy type of one of the following manifolds: S^7, $S^2 \times S^5$, $\mathbb{C}P^2 \times S^3$, $S^3 \times S^4$, N^7, $S^3 \times (\mathbb{C}P^2 \# \mathbb{C}P^2)$ or $S^3 \times (\mathbb{C}P^2 \# \overline{\mathbb{C}P^2})$. Here the manifold N^7 is a homogeneous space $(SU(2))^3/T^2$, where the embedding $T^2 \subset (SU(2))^3$ is given by

$$\left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}, \begin{pmatrix} w & 0 \\ 0 & w^{-1} \end{pmatrix}, \begin{pmatrix} zw & 0 \\ 0 & (zw)^{-1} \end{pmatrix} \right\}.$$

The manifolds S^7, $S^2 \times S^5$, $\mathbb{C}P^2 \times S^3$ and $S^3 \times S^4$ are obviously geometrically formal. On the other side, not all manifolds having the real homotopy types of these manifolds are geometrically formal. The Alloff–Wallach spaces $SU(3)/T^1$ have the real cohomology of $S^2 \times S^5$, but the normal homogeneous metrics on these spaces are not formal [**18**]. This result, as we already mentioned, is recently strengthened in [**1**], where it is proved that none of the homogeneous metrics on Alloff-Wallach spaces can be geometrically formal.

The real cohomology algebra for N^7 is as follows:

$$\mathbb{R}[x_1, x_2] \otimes \wedge(y_1, y_2, y_3), dx_1 = dx_2 = 0, dy_1 = x_1^2, dy_2 = x_2^2, dy_3 = (x_1 + x_2)^2,$$

where $\deg x_1 = \deg x_2 = 2$. It follows that N^7 is not Cartan pair homogeneous space and, thus, not formal in the sense of rational homotopy theory [**20**]. Therefore, it can not be geometrically formal.

The product metric on any of manifolds $S^3 \times (\mathbb{C}P^2 \# \mathbb{C}P^2)$ and $S^3 \times (\mathbb{C}P^2 \# \overline{\mathbb{C}P^2})$ can not be formal since, otherwise, it would by Lemma 2.1 imply that the connected sums $\mathbb{C}P^2 \# \mathbb{C}P^2$ and $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ are geometrically formal manifolds which is, as we already noted, not the case.

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