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A NOTE ON W_3-MANIFOLDS

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ABSTRACT. Almost contact metric structures on hypersurfaces in W_3-manifolds are studied.

1. Introduction

We consider the class W_3 of almost Hermitian manifolds. This class is also known as the class of special Hermitian manifolds. We remark that the class of W_3-manifolds has been studied not so detailed as other "small" Gray–Hervella classes [**14**] of almost Hermitian manifolds. It is evident that the classes of nearly Kählerian, almost Kählerian and locally conformal Kählerian manifolds (or W_1-, W_2- and W_4-manifolds, respectively) are the subject of a large set of profound works that characterize such manifolds from the points of view of geometry, topology and theoretical physics. Unfortunately, much fewer of articles are written about special Hermitian manifolds. We are sure that the main cause is the following: the class of W_3-manifolds is a subclass of the class of Hermitian manifolds, and the selection of the specific character of this class is not so easy.

We also note that the present paper is a continuation of researches of the authors in the area of Hermitian manifolds, mainly six-dimensional (see, for example, [**1**, **2**, **4**, **6**–**9**, **11**] and others).

2. Preliminaries

As it is known, an almost Hermitian manifold is an even-dimensional manifold M^{2n} with a Riemannian metric $g = \langle\cdot,\cdot\rangle$ and an almost complex structure J. Moreover, the condition $\langle JX, JY\rangle = \langle X, Y\rangle$, $X, Y \in \aleph(M^{2n})$, must hold, where $\aleph(M^{2n})$ is the module of smooth vector fields on M^{2n} [**14**, **15**].

We recall that the fundamental form of an almost Hermitian manifold is determined by the relation $F(X, Y) = \langle X, JY\rangle$, $X, Y \in \aleph(M^{2n})$. An almost Hermitian manifold is called Hermitian, if its almost complex structure is integrable. The following identity characterizes the Hermitian structure [**14**]:

$$\nabla_X(F)(Y,Z) - \nabla_{JX}(F)(JY,Z) = 0, \quad X,Y,Z \in \aleph(M^{2n}).$$

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The special Hermitian structure in addition must comply with the condition $\delta F = 0$, where δ is the codifferentiation operator [**14**].

The specification of an almost Hermitian structure on a manifold is equivalent to the setting of a G-structure, where G is the unitary group $U(n)$ [**15**]. Its elements are the frames adapted to the structure (or A-frames). These frames look as follows:

$$(p, \varepsilon_1, \ldots, \varepsilon_n, \varepsilon_{\hat{1}}, \ldots, \varepsilon_{\hat{n}}),$$

where ε_a are the eigenvectors corresponded to the eigenvalue $i = \sqrt{-1}$, and $\varepsilon_{\hat{a}}$ are the eigenvectors corresponded to the eigenvalue $-i$. Here the index a ranges from 1 to n, and we state $\hat{a} = a + n$.

Therefore, the matrices of the operator of the almost complex structure, of the Riemannian metric and of the fundamental form written in an A-frame look as follows, respectively:

$$\left(J_j^k\right) = \left(\begin{array}{c|c} iI_n & 0 \\ \hline 0 & -I_n \end{array}\right), \quad \left(g_{kj}\right) = \left(\begin{array}{c|c} 0 & I_n \\ \hline I_n & 0 \end{array}\right), \quad \left(F_{kj}\right) = \left(\begin{array}{c|c} 0 & iI_n \\ \hline -iI_n & 0 \end{array}\right),$$

where I_n is the identity matrix; $k, j = 1, \ldots, 2n$.

The first group of the Cartan structural equations of a Hermitian manifold written in an A-frame looks as follows:

$$\begin{aligned} d\omega^a &= \quad \omega_b^a \wedge \omega^b + B^{ab}{}_c \, \omega^c \wedge \omega_b, \\ d\omega_a &= -\omega_\alpha^b \wedge \omega_b + B_{ab}{}^c \omega_c \wedge \omega^b, \end{aligned}$$

where $\left\{B^{ab}{}_c\right\}$ and $\{B_{ab}{}^c\}$ are the components of the Kirichenko tensors of M^{2n} [**3**]; $a, b, c = 1, \ldots, n$.

Let N be an oriented hypersurface in a Hermitian manifold M^{2n} and let σ be the second fundamental form of the immersion of N into M^{2n}. As it is well-known [**12, 17**], the almost Hermitian structure on M^{2n} induces an almost contact metric structure on N. We recall also that an almost contact metric structure on an odd-dimensional manifold N is defined by the system of tensor fields $\{\Phi,\ \xi,\ \eta,\ g\}$ on this manifold, where ξ is a vector field, η is a covector field, Φ is a tensor of the type (1, 1) and $\langle\cdot,\cdot\rangle$ is the Riemannian metric [**13, 15**]. Moreover, the following conditions are fulfilled:

$$\begin{gathered} \eta(\xi) = 1, \quad \Phi(\xi) = 0, \quad \eta \circ \Phi = 0, \quad \Phi^2 = -id + \xi \otimes \eta, \\ \langle \Phi X, \Phi Y \rangle = \langle X, Y \rangle - \eta(X)\eta(Y), \quad X, Y \in \aleph(N), \end{gathered}$$

where $\aleph(N)$ is the module of the smooth vector fields on N. As an example of an almost contact metric structure we can consider the cosymplectic structure that is characterized by the conditions $\nabla\eta = 0$, $\nabla\Phi = 0$ [**13**].

It has been proved that the manifold, admitting the cosymplectic structure, is locally equivalent to the product $M \times R$, where M is a Kählerian manifold [**15**]. We note that the cosymplectic structures have many remarkable properties and play a fundamental role in contact geometry and mathematical physics [**15**].

As it was mentioned, the almost contact metric structures are closely connected to the almost Hermitian structures. For instance, if $(N,\ \{\Phi, \xi, \eta, g\})$ is an almost contact metric manifold, then an almost Hermitian structure is induced on the product $N \times R$ [**15**, **17**]. If this almost Hermitian structure is integrable, then the input almost contact metric structure is called normal. As it is known, a normal contact metric structure is called Sasakian. On the other hand, we can characterize the Sasakian structure by the condition

$$\nabla_X(\Phi)Y = \langle X, Y\rangle\xi - \eta(Y)X, \quad X, Y \in \aleph(N). \tag{2.1}$$

For example, Sasakian structures are induced on totally umbilical hypersurfaces in a Kählerian manifold [**15**]. As it is well known, the Sasakian structures have also many important properties. In 1972 Katsuei Kenmotsu introduced a new class of almost contact metric structures, defined by

$$\nabla_X(\Phi)Y = \langle \Phi X, Y\rangle\xi - \eta(Y)\Phi X, \quad X, Y \in \aleph(N). \tag{2.2}$$

The Kenmotsu manifolds are normal and integrable, but they are not contact manifolds, consequently, they can not be Sasakian. In spite of the fact that conditions (2.1) and (2.2) are similar, the properties of Kenmotsu manifolds are to some extent antipodal to the Sasakian manifolds properties [**15**]. We remark that the fundamental monograph by Gh. Pitiş contains a detailed description of Kenmotsu manifolds as well as a collection of examples of such manifolds [**16**].

At the end of this section, note that when we give a Riemannian manifold and its submanifold (in particular, its hypersurface), the rank of the determined second fundamental form is called the type number.

3. The main results

Let M^{2n} be an almost Hermitian manifold. As we have noted, an almost contact metric structure is induced on its oriented hypersurface N. The first group of the Cartan structural equations of such an almost contact metric structure looks as follows [**12**, **17**]:

$$\begin{aligned}
d\omega^\alpha &= \omega^\alpha_\beta \wedge \omega^\beta + B^{\alpha\beta}{}_\gamma\omega^\gamma \wedge \omega_\beta + B^{\alpha\beta\gamma}\omega_b \wedge \omega_\gamma + \big(\sqrt{2}B^{\alpha n}{}_\beta + i\sigma^\alpha_\beta\big)\omega^\beta \wedge \omega \\
&\quad + \Big(-\sqrt{2}\tilde{B}^{n\alpha\beta} - \frac{1}{\sqrt{2}}B^{\alpha\beta}{}_n - \frac{1}{\sqrt{2}}\tilde{B}^{\alpha\beta n} + i\sigma^{\alpha\beta}\Big)\omega_\beta \wedge \omega, \\
d\omega_\alpha &= -\omega^\beta_\alpha \wedge \omega_\beta + B_{\alpha\beta}{}^\gamma\omega_\gamma \wedge \omega^\beta + B_{\alpha\beta\gamma}\omega^\beta \wedge \omega^c + \big(\sqrt{2}B_{\alpha n}{}^\beta - i\sigma^\beta_\alpha\big)\omega_\beta \wedge \omega \\
&\quad + \Big(-\sqrt{2}\tilde{B}_{n\alpha\beta} - \frac{1}{\sqrt{2}}\tilde{B}_{\alpha\beta n} - \frac{1}{\sqrt{2}}B_{\alpha\beta}{}^n - i\sigma_{\alpha\beta}\Big)\omega^\beta \wedge \omega, \\
d\omega &= \sqrt{2}B_{n\alpha\beta}\omega^\alpha \wedge \omega^\beta + \sqrt{2}B^{n\alpha\beta}\omega_\alpha \wedge \omega_\beta \\
&\quad + \big(\sqrt{2}B^{n\alpha}{}_\beta - \sqrt{2}B_{n\beta}{}^\alpha - 2i\sigma^\alpha_\beta\big)\omega^\beta \wedge \omega_\alpha \\
&\quad + \big(\tilde{B}_{n\beta n} + B_{n\beta}{}^n + i\sigma_{n\beta}\big)\omega \wedge \omega^\beta + \big(\tilde{B}^{n\beta n} + B^{n\beta}{}_n - i\sigma^\beta_n\big)\omega \wedge \omega_\beta.
\end{aligned} \tag{3.1}$$

Here the indices α, β, γ range from 1 to $n-1$.

Knowing that an almost Hermitian structure is special Hermitian if and only if its Kirichenko tensors satisfy the conditions $B^{abc} = 0$, $B_{abc} = 0$, $B^{ab}{}_b = 0$,

${B_{ab}}^b = 0$, we obtain the Cartan structural equations of a special Hermitian structure:

$$
\begin{aligned}
d\omega^\alpha &= \omega^\alpha_\beta \wedge \omega^\beta + {B^{\alpha\beta}}_\gamma \omega^\gamma \wedge \omega_\beta + \big(\sqrt{2}{B^{\alpha n}}_\beta + i\sigma^\alpha_\beta\big)\omega^\beta \wedge \omega \\
&\quad + \Big(-\frac{1}{\sqrt{2}}{B^{\alpha\beta}}_n + i\sigma^{\alpha\beta}\Big)\omega_\beta \wedge \omega; \\
d\omega_\alpha &= -\omega^\beta_\alpha \wedge \omega_\beta + {B_{\alpha\beta}}^\gamma \omega_\gamma \wedge \omega^\beta + \big(\sqrt{2}{B_{\alpha n}}^\beta - i\sigma^\beta_\alpha\big)\omega_\beta \wedge \omega \\
&\quad + \Big(-\frac{1}{\sqrt{2}}{B_{\alpha\beta}}^n - i\sigma_{\alpha\beta}\Big)\omega^\beta \wedge \omega; \\
d\omega &= \big(\sqrt{2}{B^{n\alpha}}_\beta - \sqrt{2}{B_{n\beta}}^\alpha - 2i\sigma^\alpha_\beta\big)\omega^\beta \wedge \omega_\alpha + \big({B_{n\beta}}^n + i\sigma_{n\beta}\big)\omega \wedge \omega^\beta \\
&\quad + \big({B^{n\beta}}_n - i\sigma^\beta_n\big)\omega \wedge \omega_\beta.
\end{aligned}
\tag{3.2}
$$

So, we can state our first result.

THEOREM 3.1. *The Cartan structural equations of the almost contact metric structure on a hypersurface in a special Hermitian manifold look as* (3.2). *In addition, the Kirichenko virtual tensors of M^{2n} must comply with the following conditions: ${B^{ab}}_b = 0$, ${B_{ab}}^b = 0$.*

If the hypersurface is totally umbilical, then its second fundamental form complies with the condition $\sigma = \lambda g$, $\lambda -$ const. Knowing how the matrix of the contravariant metric tensor on N looks [**12**]:

$$
\big(g^{ps}\big) = \left(\begin{array}{c|c|c} 0 & \begin{matrix}0\\ \cdots\\ 0\end{matrix} & I_{n-1} \\ \hline 0\ldots0 & 1 & 0\ldots0 \\ \hline I_{n-1} & \begin{matrix}0\\ \cdots\\ 0\end{matrix} & 0 \end{array}\right)
$$

$$p, s = 1, \ldots, 2n-1,$$

we obtain that in this case the matrix of the second fundamental form looks as follows:

$$
\big(\sigma_{ps}\big) = \left(\begin{array}{c|c|c} 0 & \begin{matrix}0\\ \cdots\\ 0\end{matrix} & \lambda I_{n-1} \\ \hline 0\ldots0 & \lambda & 0\ldots0 \\ \hline \lambda I_{n-1} & \begin{matrix}0\\ \cdots\\ 0\end{matrix} & 0 \end{array}\right)
$$

$$p, s = 1, \ldots, 2n-1.$$

Evidently, when $\lambda = 0$ the hypersurface is totally geodesic and the matrix (σ_{ps}) vanishes. That is why we can rewrite the Cartan structural equations (3.2) for almost contact structures on totally umbilical and totally geodesic hypersurfaces in a special Hermitian manifold as follows:

$$
\begin{aligned}
d\omega^\alpha &= \omega^\alpha_\beta \wedge \omega^\beta + B^{\alpha\beta}{}_\gamma \omega^\gamma \wedge \omega_\beta + \big(\sqrt{2}B^{\alpha n}{}_\beta + i\sigma^\alpha_\beta\big)\omega^\beta \wedge \omega \\
&\quad + \Big(-\frac{1}{\sqrt{2}}B^{\alpha\beta}{}_n\Big)\omega_\beta \wedge \omega; \\
d\omega_\alpha &= -\omega^\beta_\alpha \wedge \omega_\beta + B_{\alpha\beta}{}^\gamma \omega_\gamma \wedge \omega^\beta + \big(\sqrt{2}B_{\alpha n}{}^\beta - i\sigma^\beta_\alpha\big)\omega_\beta \wedge \omega \\
&\quad + \Big(-\frac{1}{\sqrt{2}}B_{\alpha\beta}{}^n\Big)\omega^\beta \wedge \omega; \\
d\omega &= \big(\sqrt{2}B^{n\alpha}{}_\beta - \sqrt{2}B_{n\beta}{}^\alpha - 2i\sigma^\alpha_\beta\big)\omega^\beta \wedge \omega_\alpha + \big(B_{n\beta}{}^n + i\sigma_{n\beta}\big)\omega \wedge \omega^\beta \\
&\quad + \big(B^{n\beta}{}_n - i\sigma^\beta_n\big)\omega \wedge \omega_\beta;
\end{aligned} \tag{3.3}
$$

$$
\begin{aligned}
d\omega^\alpha &= \omega^\alpha_\beta \wedge \omega^\beta + B^{\alpha\beta}{}_\gamma \omega^\gamma \wedge \omega_\beta + \sqrt{2}B^{\alpha n}{}_\beta \omega^\beta \wedge \omega \\
&\quad + \Big(-\frac{1}{\sqrt{2}}B^{\alpha\beta}{}_n\Big)\omega_\beta \wedge \omega; \\
d\omega_\alpha &= -\omega^\beta_\alpha \wedge \omega_\beta + B_{\alpha\beta}{}^\gamma \omega_\gamma \wedge \omega^\beta + \sqrt{2}B_{\alpha n}{}^\beta \omega_\beta \wedge \omega \\
&\quad + \Big(-\frac{1}{\sqrt{2}}B_{\alpha\beta}{}^n\Big)\omega^\beta \wedge \omega; \\
d\omega &= \big(\sqrt{2}B^{n\alpha}{}_\beta - \sqrt{2}B_{n\beta}{}^\alpha\big)\omega^\beta \wedge \omega_\alpha + B_{n\beta}{}^n \omega \wedge \omega^\beta + B^{n\beta}{}_n \omega \wedge \omega_\beta.
\end{aligned} \tag{3.4}
$$

So, we state our second result.

THEOREM 3.2. *The Cartan structural equations of the almost contact metric structure on a totally umbilical and on a totally geodesic hypersurface in a special Hermitian manifold look as* (3.3) *and* (3.4), *respectively. In addition, the Kirichenko virtual tensors of M^{2n} must comply with the conditions $B^{ab}{}_b = 0$ and $B_{ab}{}^b = 0$.*

If the type number of the hypersurface is equal to 1, then the matrix of the second fundamental form of the immersion of N into M^{2n} looks as follows [**14**]:

$$
(\sigma_{ps}) = \left(\begin{array}{c|c|c}
0 & \begin{matrix}0\\ \cdots \\ 0\end{matrix} & 0 \\ \hline
0\ldots 0 & \sigma_{nn} & 0\ldots 0 \\ \hline
0 & \begin{matrix}0\\ \cdots \\ 0\end{matrix} & 0
\end{array}\right)
$$

$$p, s = 1, \ldots, 2n-1.$$

It is easy to see that Cartan structural equations (3.2) of the almost contact metric structure on a 1-type hypersurface in a special Hermitian manifold look precisely as (3.4). That is why we conclude that all the characteristics of the almost contact metric structures on totally geodesic and 1-type hypersurfaces in a special Hermitian manifold are also identical. This fact permits to state our next result.

THEOREM 3.3. *The properties of almost contact metric structures on totally geodesic hypersurfaces and* 1*-type hypersurfaces in a special Hermitian manifold are identical.*

This statement gives an exhaustive explanation of two incomprehensible results from [**5**, **8**] on special Hermitian manifolds. We remind that an almost Hermitian manifold M^{2n} complies with the 1-cosymplectic hypersurfaces axiom (respectively, G-cosymplectic hypersurfaces axiom), if a cosymplectic hypersurface with type number one (respectively, a totally geodesic cosymplectic hypersurface) passes through every point of M^{2n}.

THEOREM 3.4. [**5**] *If an arbitrary special Hermitian manifold complies with the G-cosymplectic hypersurfaces axiom, then it is a Kählerian manifold.*

THEOREM 3.5. [**8**] *If an arbitrary special Hermitian manifold complies with the 1-cosymplectic hypersurfaces axiom, then it is a Kählerian manifold.*

From Theorem 3.3 we get directly such a reasonable explanation. Namely, if the almost contact metric structure on totally geodesic and 1-type hypersurfaces in a manifolds are identical, then the influence of so-called G-cosymplectic and 1-cosymplectic hypersurfaces axioms on the ambient special Hermitian manifold is also identical.

Taking into account that every Gray–Hervella class of almost Hermitian manifolds contains all Kählerian manifolds, we remark that our theorems generalize some results obtained for almost contact metric hypersurfaces in Kählerian manifolds [**10**].

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