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PROPERTIES OF THE NEARLY KÄHLER $\mathbb{S}^3 \times \mathbb{S}^3$

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Abstract. We show how the metric, the almost complex structure and the almost product structure of the homogeneous nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$ can be recovered from a submersion $\pi : \mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3$. On $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$ we have the maps obtained either by changing two coordinates, or by cyclic permutations. We show that these maps project to maps from $\mathbb{S}^3 \times \mathbb{S}^3$ to $\mathbb{S}^3 \times \mathbb{S}^3$ and we investigate their behavior.

1. Introduction

Nearly Kähler manifolds are almost Hermitian manifolds with almost complex structure J for which the tensor field $\tilde{\nabla}J$ is skew-symmetric, where $\tilde{\nabla}$ is the Levi Civita connection of the metric. The systematic study of these manifolds and their relation with 3-symmetric spaces has been initiated by Gray [**7, 8**]. Recently it has been shown by Butruille [**4**] that the only homogeneous 6-dimensional nearly Kähler manifolds are the nearly Kähler 6-sphere $\mathbb{S}^6$, $\mathbb{S}^3 \times \mathbb{S}^3$, the projective space $\mathbb{C}P^3$ and the flag manifold $SU(3)/U(1) \times U(1)$. One should remark that only very recently, the first complete non homogeneous nearly Kähler structures were discovered on $\mathbb{S}^6$ and $\mathbb{S}^3 \times \mathbb{S}^3$ in [**6**]. As far as their submanifolds are concerned, strict 6-dimensional nearly Kähler manifolds have the surprising property that their Lagrangian submanifolds are always minimal (see [**10, 13**]).

In this paper we show how the nearly Kähler metric g and the almost complex structure J of $\mathbb{S}^3 \times \mathbb{S}^3$ can be recovered in a natural way by looking at a submersion $\pi : \mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3$. Note that this is also more or less implicitly present in [**8, 9, 12, 11**]. We also show how the almost product structure P defined in [**3**] can be introduced using the submersion π from $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$. In this way we actually obtain three different almost product structures

$$P_l = \cos(\tfrac{2\pi l}{3})P - \sin(\tfrac{2\pi l}{3})JP, \text{ for } l = 1, 2, 3.$$

We show in the final section that these are precisely the three possible almost product structures which preserve the basic equations for $\mathbb{S}^3 \times \mathbb{S}^3$ derived in [**3**].

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We also show how the maps which interchange the components of $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$ give rise to isometries of $\mathbb{S}^3 \times \mathbb{S}^3$. We call $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) the isometry corresponding to interchanging the first two coordinates (resp. interchanging the first and third coordinate). We show that both these isometries preserve up to sign the almost complex structure. And even though they do not preserve the almost product structures individually, they do preserve the set of almost product structures $\{P_1, P_2, P_3\}$. This is of course the reason why in several classification theorems for Lagrangian submanifolds, see for example [**1, 2, 14**], one often has 3 isometric examples with slightly different properties of the almost product structure P. These examples are precisely obtained one from another by applying the isometries $\mathcal{F}_1$ and $\mathcal{F}_2$. The only exception so far to this is the classification of non totally geodesic Lagrangian submanifolds with constant sectional curvature in [**5**]. This is due to the special property of the angle functions (which determine P) of these last examples.

2. Preliminaries

2.1. The nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$. In this section, we recall the homogeneous nearly Kähler structure of $\mathbb{S}^3 \times \mathbb{S}^3$ and mention some known results from [**5, 14**]. First, we can identify the 3-sphere $\mathbb{S}^3$ with the set of all the unit quaternions in $\mathbb{H}$, i.e., $\mathbb{S}^3 = \{p \in \mathbb{H} \mid \langle p, p \rangle = 1\}$, where the metric $\langle \cdot, \cdot \rangle$ is induced from the Euclidean metric on $\mathbb{R}^4$. Let i, j, k denote the standard imaginary unit quaternions. Then the vector fields X_1, X_2, X_3 given by

$$\begin{aligned} X_1(p) &= p\,i = (-x_2, x_1, x_4, -x_3), \\ X_2(p) &= p\,j = (-x_3, -x_4, x_1, x_2), \\ X_3(p) &= p\,k = (-x_4, x_3, -x_2, x_1), \end{aligned}$$

where $p = x_1 + x_2\, i + x_3\, j + x_4\, k \in \mathbb{S}^3$, form a basis of the tangent bundle $T\mathbb{S}^3$. Hence, the tangent space of $\mathbb{S}^3$ is defined by $T_p\mathbb{S}^3 = \{p\alpha \mid \alpha \in \mathrm{Im}\mathbb{H}\}$.

Let $Z_{(p,q)}$ be a tangent vector of $\mathbb{S}^3 \times \mathbb{S}^3$ at (p, q). From the known natural identification $T_{(p,q)}(\mathbb{S}^3 \times \mathbb{S}^3) \cong T_p\mathbb{S}^3 \oplus T_q\mathbb{S}^3$, we write $Z_{(p,q)} = (pU_{(p,q)}, qV_{(p,q)})$ or simply $Z = (pU, qV)$, where U and V are imaginary quaternions. Now, we define the vector fields on $\mathbb{S}^3 \times \mathbb{S}^3$ as

$$\begin{aligned} \tilde{E}_1(p,q) &= (p\,i, 0), \quad & \tilde{F}_1(p,q) &= (0, q\,i), \\ \tilde{E}_2(p,q) &= (p\,j, 0), \quad & \tilde{F}_2(p,q) &= (0, q\,j), \\ \tilde{E}_3(p,q) &= (p\,k, 0), \quad & \tilde{F}_3(p,q) &= (0, q\,k), \end{aligned}$$

which are mutually orthogonal with respect to the usual Euclidean product metric on $\mathbb{S}^3 \times \mathbb{S}^3$. The Lie brackets are $[\tilde{E}_i, \tilde{E}_j] = 2\varepsilon_{ijk}\tilde{E}_k$, $[\tilde{F}_i, \tilde{F}_j] = 2\varepsilon_{ijk}\tilde{F}_k$ and $[\tilde{E}_i, \tilde{F}_j] = 0$, where

$$\varepsilon_{ijk} = \begin{cases} 1, & \text{if } (ijk) \text{ is an even permutation of } (123), \\ -1, & \text{if } (ijk) \text{ is an odd permutation of } (123), \\ 0, & \text{if otherwise.} \end{cases}$$

The almost complex structure J on $\mathbb{S}^3 \times \mathbb{S}^3$ is defined by

$$J(pU, qV)_{(p,q)} = \frac{1}{\sqrt{3}} \left(p(2V - U), q(-2U + V) \right),$$

for U, V imaginary quaternions and therefore $(pU, qV) \in T_{(p,q)}(\mathbb{S}^3 \times \mathbb{S}^3)$ (see [**4**]). The nearly Kähler metric on $\mathbb{S}^3 \times \mathbb{S}^3$ with which we choose to work is the Hermitian metric associated to the usual Euclidean product metric on $\mathbb{S}^3 \times \mathbb{S}^3$:

$$\begin{aligned} g(Z, Z') &= \frac{1}{2} \left(\langle Z, Z' \rangle + \langle JZ, JZ' \rangle \right) \\ &= \frac{4}{3} \left(\langle U, U' \rangle + \langle V, V' \rangle \right) - \frac{2}{3} \left(\langle U, V' \rangle + \langle U', V \rangle \right), \end{aligned}$$

where $Z = (pU, qV)$ and $Z' = (pU', qV')$. In the first line $\langle \cdot, \cdot \rangle$ stands for the usual Euclidean product metric on $\mathbb{S}^3 \times \mathbb{S}^3$ and in the second line $\langle \cdot, \cdot \rangle$ stands for the usual Euclidean metric on $\mathbb{S}^3$. From the definition, it can be seen that the almost complex structure J is compatible with the metric g.

Let $G := \tilde{\nabla} J$. Then G is skew-symmetric and it satisfies

$$G(X, JY) = -JG(X, Y), \quad g(G(X, Y), Z) + g(G(X, Z), Y) = 0,$$

for any vector fields X, Y, Z tangent to $\mathbb{S}^3 \times \mathbb{S}^3$. Therefore, $\mathbb{S}^3 \times \mathbb{S}^3$ equipped with g and the almost complex structure J, becomes a nearly Kähler manifold. Moreover, we introduce the almost product structure P, defined in [**3**] as

$$P(pU, qV)_{(p,q)} = (pV, qU)$$

for U, V imaginary quaternions and therefore $(pU, qV) \in T_{(p,q)}(\mathbb{S}^3 \times \mathbb{S}^3)$. It satisfies the following properties:

$$P^2 = Id, \text{ i.e.,} P \text{ is involutive,} \tag{2.1}$$

$$PJ = -JP, \text{ i.e.,} P \text{ and } J \text{ anti-commute,} \tag{2.2}$$

$$g(PZ, PZ') = g(Z, Z'), \text{ i.e.,} P \text{ is compatible with } g, \tag{2.3}$$

$$g(PZ, Z') = g(Z, PZ'), \text{ i.e.,} P \text{ is symmetric.} \tag{2.4}$$

Furthermore, the almost product structure P and its covariant derivative $\tilde{\nabla} P$ admit the following properties.

LEMMA 2.1. [**3**] *For tangent vector fields X, Y on $(\mathbb{S}^3 \times \mathbb{S}^3, g, J)$ the following equations hold:*

$$PG(X, Y) + G(PX, PY) = 0, \tag{2.5}$$

$$(\tilde{\nabla}_X P)Y = \tfrac{1}{2} J(G(X, PY) + PG(X, Y)), \tag{2.6}$$

The Riemannian curvature tensor $\tilde{R}$ on $(\mathbb{S}^3 \times \mathbb{S}^3, g, J)$ is given by

$$\begin{aligned} \tilde{R}(U,V)W &= \frac{5}{12}\big(g(V,W)U - g(U,W)V\big) \\ &\quad + \frac{1}{12}\big(g(JV,W)JU - g(JU,W)JV - 2g(JU,V)JW\big) \\ &\quad + \frac{1}{3}\big(g(PV,W)PU - g(PU,W)PV \\ &\qquad + g(JPV,W)JPU - g(JPU,W)JPV\big). \end{aligned} \tag{2.7}$$

2.2. The space $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$. We consider $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$ with its usual structure induced from $\mathbb{H}^3$. For tangent vectors (g_1V_1, g_2V_2, g_3V_3) and (g_1W_1, g_2W_2, g_3W_3) at the point (g_1, g_2, g_3), we have that $V_1, V_2, V_3, W_1, W_2, W_3$ are imaginary quaternions and that the induced metric is given by

$$\begin{aligned} \langle (g_1V_1, g_2V_2, g_3V_3), (g_1W_1, g_2W_2, g_3W_3)\rangle &= \sum_{l=1}^{3} \mathrm{Re}(g_l V_l \bar{W}_l \bar{g}_l) \\ &= -\sum_{l=1}^{3} \mathrm{Re}(g_l V_l W_l \bar{g}_l) \\ &= \sum_{l=1}^{3} \mathrm{Re}(g_l(\langle V_l, W_l\rangle - V_l \times W_l)\bar{g}_l) \\ &= \sum_{l=1}^{3} \langle V_l, W_l\rangle. \end{aligned}$$

Notice that here we have identified the imaginary quaternions with the Euclidean 3-space for defining the inner and the cross products, that is $\langle\, , \,\rangle$ and $\times$. Next, we define the following vector fields on $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$ as

$$\begin{aligned} E_1(g_1,g_2,g_3) &= (g_1\bar{g}_3 i g_3, 0, 0), \quad & F_1(g_1,g_2,g_3) &= (0, g_2\bar{g}_3 i g_3, 0), \\ E_2(g_1,g_2,g_3) &= (g_1\bar{g}_3 j g_3, 0, 0), \quad & F_2(g_1,g_2,g_3) &= (0, g_2\bar{g}_3 j g_3, 0), \\ E_3(g_1,g_2,g_3) &= (g_1\bar{g}_3 k g_3, 0, 0), \quad & F_3(g_1,g_2,g_3) &= (0, g_2\bar{g}_3 k g_3, 0), \end{aligned}$$
$$\begin{aligned} G_1(g_1,g_2,g_3) &= (0,0,ig_3), \\ G_2(g_1,g_2,g_3) &= (0,0,jg_3), \\ G_3(g_1,g_2,g_3) &= (0,0,kg_3). \end{aligned}$$

Note that using the induced metric, it immediately follows that E_1, E_2, E_3, F_1, F_2, F_3, G_1, G_2, G_3 form an orthonormal basis of the tangent space. We also have that for any (g_1, g_2, g_3), γ_1 given by $\gamma_1(t) = (g_1\bar{g}_3 e^{it} g_3, g_2, g_3)$ is a curve in $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$ with initial conditions $\gamma_1(0) = (g_1, g_2, g_3)$ and $\gamma_1'(0) = E_1(g_1, g_2, g_3)$. Similarly we have that the curves $\gamma_2(t) = (g_1, g_2\bar{g}_3 e^{it} g_3, g_3)$ and $\gamma_3(t) = (g_1, g_2, e^{it} g_3)$ are curves in $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$ with initial conditions respectively

$$\begin{aligned} \gamma_2(0) &= (g_1,g_2,g_3), \qquad & \gamma_2'(0) &= F_1(g_1,g_2,g_3), \\ \gamma_3(0) &= (g_1,g_2,g_3), \qquad & \gamma_3'(0) &= G_1(g_1,g_2,g_3). \end{aligned}$$

By replacing i with j and k in the expressions of the curves γ_1, γ_2 and γ_3, we define similarly the corresponding curves for the other vectors in the basis.

We also have on each tangent space the natural linear applications:

$$\begin{aligned}\tilde{\tau}(g_1V_1, g_2V_2, g_3V_3) &= (g_1V_2, g_2V_3, g_3V_1),\\ \tilde{P}_1(g_1V_1, g_2V_2, g_3V_3) &= (g_1V_2, g_2V_1, g_3V_3),\\ \tilde{P}_2(g_1V_1, g_2V_2, g_3V_3) &= (g_1V_3, g_2V_2, g_3V_1),\\ \tilde{P}_3(g_1V_1, g_2V_2, g_3V_3) &= (g_1V_1, g_2V_3, g_3V_2).\end{aligned}$$

Note that these applications all preserve the induced metric. Moreover we have that $\tilde{\tau}^3 = I = \tilde{P}_1^2 = \tilde{P}_2^2 = \tilde{P}_3^2$, $\tilde{P}_3\tilde{P}_1 = \tilde{\tau}$. In terms of the previously induced vector fields, we have that

$$\begin{aligned}\tilde{\tau}E_l &= G_l, & \tilde{\tau}F_l &= E_l, & \tilde{\tau}G_l &= F_l,\\ \tilde{P}_1E_l &= F_l, & \tilde{P}_1F_l &= E_l, & \tilde{P}_1G_l &= G_l,\\ \tilde{P}_2E_l &= G_l, & \tilde{P}_2F_l &= F_l, & \tilde{P}_2G_l &= E_l,\\ \tilde{P}_3E_l &= E_l, & \tilde{P}_3F_l &= G_l, & \tilde{P}_3G_l &= F_l.\end{aligned}$$

3. The nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$ as a Riemannian submersion

We look at the map

$$\pi : \mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3, \quad (g_1, g_2, g_3) \mapsto (g_1\bar{g}_3, g_2\bar{g}_3).$$

It follows immediately that

$$\pi(g_1, g_2, g_3) = \pi(g_1', g_2', g_3') \;\Leftrightarrow\; (g_1', g_2', g_3') = (g_1a, g_2a, g_3a),$$

where $a \in \mathbb{S}^3$ is a unit quaternion. We have that

$$\begin{aligned}d\pi(E_1(g_1, g_2, g_3)) &= \tfrac{d}{dt}\pi(g_1\bar{g}_3e^{it}g_3, g_2, g_3)|_{t=0} = \tfrac{d}{dt}(g_1\bar{g}_3e^{it}, g_2\bar{g}_3)|_{t=0}\\ &= (g_1\bar{g}_3i, 0) = \tilde{E}_1(g_1\bar{g}_3, g_2\bar{g}_3) = \tilde{E}_1(\pi(g_1, g_2, g_3)).\end{aligned}$$

By similar computations we obtain that

$$\begin{aligned}d\pi(E_l(g_1, g_2, g_3)) &= \tilde{E}_l(\pi(g_1, g_2, g_3)),\\ d\pi(F_l(g_1, g_2, g_3)) &= \tilde{F}_l(\pi(g_1, g_2, g_3)),\\ d\pi(G_l(g_1, g_2, g_3)) &= -\tilde{E}_l(\pi(g_1, g_2, g_3)) - \tilde{F}_l(\pi(g_1, g_2, g_3)).\end{aligned}$$

This implies that $d\pi$ is surjective (and hence π is a submersion). We also see that the space of vertical vectors $\mathcal{V}$ is given by

$$\mathcal{V} = \operatorname{span}\{E_1 + F_1 + G_1, E_2 + F_2 + G_2, E_3 + F_3 + G_3\}.$$

Therefore, we have the space of horizontal vector fields $\mathcal{H}$ spanned by

$$\{\tfrac{1}{3}(2E_l - F_l - G_l), \tfrac{1}{3}(-E_l + 2F_l - G_l)\},$$

for $l = 1, 2, 3$. It also follows that

$$\begin{aligned}d\pi(\tfrac{1}{3}(2E_l - F_l - G_l))(g_1, g_2, g_3) &= \tilde{E}_l(\pi(g_1, g_2, g_3)),\\ d\pi(\tfrac{1}{3}(-E_l + 2F_l - G_l))(g_1, g_2, g_3) &= \tilde{F}_l(\pi(g_1, g_2, g_3)).\end{aligned}$$

Note that

$$\langle \tfrac{1}{3}(2E_l - F_l - G_l), \tfrac{1}{3}(2E'_l - F'_l - G'_l)\rangle = \tfrac{2}{3}\delta_{ll'}$$
$$\langle \tfrac{1}{3}(2E_l - F_l - G_l), \tfrac{1}{3}(-E'_l + 2F'_l - G'_l)\rangle = -\tfrac{1}{3}\delta_{ll'}$$
$$\langle \tfrac{1}{3}(-E_l + 2F_l - G_l), \tfrac{1}{3}(-E'_l + 2F'_l - G'_l)\rangle = \tfrac{2}{3}\delta_{ll'}$$

Moreover, as the right-hand sides are independent of the point (g_1, g_2, g_3) for which $\pi(g_1, g_2, g_3) = (p, q)$, we see from the above formulas that we can define the canonical metric, g_s on $\mathbb{S}^3 \times \mathbb{S}^3$, of the submersion π by

$$g_s(\tilde{E}_l, \tilde{E}_l) = g_s(\tilde{F}_l, \tilde{F}_l) = -2g_s(\tilde{E}_l, \tilde{F}_l) = \tfrac{2}{3},$$

and such that all other components vanish. Note that $g = 2g_s$ and therefore the nearly Kähler metric is twice the metric induced by the submersion.

Theorem 3.1. *The map*

$$\pi : \mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3 : (g_1, g_2, g_3) \mapsto (g_1\bar{g}_3, g_2\bar{g}_3)$$

is a submersion. Moreover there exists a canonical metric g_s on $\mathbb{S}^3 \times \mathbb{S}^3$ such that the submersion becomes a Riemannian submersion. This metric is related to the nearly Kähler metric by $g = 2g_s$.

Next we look at what happens with the applications $\tilde{\tau}$, $\tilde{P}_1$, $\tilde{P}_2$ and $\tilde{P}_3$. In order to do so we will use the following lemma.

Lemma 3.1. *Let $\tilde{A}$ be a linear application on the tangent space of $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$. Suppose that*

(1) *$\tilde{A}$ maps vertical vector fields to vertical vector fields*
(2) *$\tilde{A}$ preserves the metric*
(3) *If v, w are horizontal tangent vectors at resp. (g_1, g_2, g_3) and $(g_1 a, g_2 a, g_3 a)$ such that $d\pi(v) = d\pi(w)$, then we also have that $d\pi(\tilde{A}v) = d\pi(\tilde{A}w)$, i.e., $\tilde{A}$ preserves the fibers of π.*

Then there exists a linear isometry A of the tangent space of $\mathbb{S}^3 \times \mathbb{S}^3$, such that

$$AZ(p, q) = d\pi(\tilde{A}\tilde{Z}(g_1, g_2, g_3)),$$

where (g_1, g_2, g_3) is any point such that $\pi(g_1, g_2, g_3) = (p, q)$ and $\tilde{Z}(g_1, g_2, g_3)$ is the unique horizontal tangent vector such that $d\pi(\tilde{Z}(g_1, g_2, g_3)) = Z(p, q)$.

Proof. As $\tilde{A}$ maps vertical vector fields to vertical vector fields and preserves the metric, $\tilde{A}$ also maps horizontal vector fields to horizontal vector fields. The third condition then implies that the map $AZ(p, q) = d\pi(\tilde{A}\tilde{Z}(g_1, g_2, g_3))$, is well defined and is an isometry. □

Note that the maps $\tilde{\tau}$, $\tilde{P}_1$, $\tilde{P}_2$, $\tilde{P}_3$ satisfy the conditions of the above lemma. Therefore we get the corresponding maps at the tangent space of a point $(p, q) \in \mathbb{S}^3 \times \mathbb{S}^3$ given by τ, P_1, P_2 and P_3. In terms of the vector fields $\tilde{E}_l$ and $\tilde{F}_l$, the map τ can be described by

$$\tau(\tilde{E}_l) = d\pi(\tilde{\tau}(\tfrac{1}{3}(2E_l - F_l - G_l)) = d\pi(\tfrac{1}{3}(2G_l - E_l - F_l))$$

$$= -d\pi(\tfrac{1}{3}(2E_l - F_l - G_l)) - d\pi(\tfrac{1}{3}(-E_l + F_l - G_l)) = -\tilde{E}_l - \tilde{F}_l,$$
$$\tau(\tilde{F}_l) = d\pi(\tilde{\tau}(\tfrac{1}{3}(-E_l + 2F_l - G_l)) = d\pi(\tfrac{1}{3}(2E_l - F_l - G_l)) = \tilde{E}_l.$$

It now follows, by straightforward computations, that $(\frac{2}{\sqrt{3}}(\tau + \frac{1}{2}I))^2 = -I$ and that the almost complex structure is given by

$$J = \tfrac{2}{\sqrt{3}}(\tau + \tfrac{1}{2}I).$$

In particular

$$J\tilde{E}_l = \tfrac{2}{\sqrt{3}}(-\tfrac{1}{2}\tilde{E}_l - \tilde{F}_l) = \tfrac{1}{\sqrt{3}}(-\tilde{E}_l - 2\tilde{F}_l,$$
$$J\tilde{F}_l = \tfrac{2}{\sqrt{3}}(\tilde{E}_l + \tfrac{1}{2}\tilde{F}_l) = \tfrac{1}{\sqrt{3}}(2\tilde{E}_l + \tilde{F}_l).$$

Using similar computations, for the maps P_1, P_2 and P_3, we obtain the following lemma.

LEMMA 3.2. *We have that*

$$P_1 = P, \quad P_2 = -\tfrac{1}{2}P - \tfrac{\sqrt{3}}{2}JP, \quad P_3 = -\tfrac{1}{2}P + \tfrac{\sqrt{3}}{2}JP.$$

4. The role of the almost product structure P

The tensor P appears in the basic equations of $\mathbb{S}^3 \times \mathbb{S}^3$ in (2.1)–(2.7).

We call a tensor P^* satisfying the above conditions a nearly productlike structure on $\mathbb{S}^3 \times \mathbb{S}^3$. In order to determine all nearly productlike structures on $\mathbb{S}^3 \times \mathbb{S}^3$, we have the following lemmas.

LEMMA 4.1. *Let P^* be a structure which satisfies* (2.1), (2.2), (2.3), (2.4), *and* (2.7). *Then there exists an angle θ such that $P^* = \cos\theta P + \sin\theta JP$. The converse is also true.*

PROOF. We use (2.7) and we take an arbitrary vector $U = X$. We take $V = Y$ orthogonal to X, JX, P^*PX and JP^*PX. We take $W = P^*Y$. Then we have that

$$\begin{aligned}
&\big(g(P^*Y,P^*Y)P^*X - g(P^*X,P^*Y)P^*V \\
&\qquad + g(JP^*Y,P^*Y)JP^*X - g(JP^*X,P^*Y)JP^*V\big) \\
&\quad = \big(g(PY,P^*Y)PX - g(PX,P^*Y)PY \\
&\qquad + g(JPY,P^*Y)JPX - g(JPX,P^*Y)JPY\big).
\end{aligned} \tag{4.1}$$

Using the properties of P and P^*, we see that the left hand side of (4.1) reduces to $g(Y,Y)P^*X$, whereas the right hand side reduces to $g(PY,P^*Y)PX + g(JPY,P^*Y)JPX$. Hence for any X there exists an angle $\theta(X)$ such that

$$P^*X = \cos(\theta(X))PX + \sin(\theta(X))JPX.$$

Using the properties of P and P^* we deduce that

$$P^*JX = -JP^*X = \cos(\theta(X))PJX + \sin(\theta(X))JPJX.$$

Hence $\theta(JX) = \theta(X)$. By linearity the same is now true for any linear combination of X and JX. Take now a vector field Y, orthogonal to X and JX, such that

$\|Y\| = \|X\|$. For any angle α, we can now compute $\psi_\alpha = \theta(\cos\alpha X + \sin\alpha Y)$. On the one hand we have that

$$\begin{aligned} P^*(\cos\alpha X + \sin\alpha Y) &= \cos\alpha(\cos(\theta(X))PX + \sin(\theta(X))JPX) \\ &\quad + \sin\alpha(\cos(\theta(Y))PY + \sin(\theta(Y))JPY), \end{aligned}$$

while on the other hand

$$\begin{aligned} P^*(\cos\alpha X + \sin\alpha Y) &= \cos\psi_\alpha(\cos\alpha PX + \sin\alpha PY) \\ &\quad + \sin\psi_\alpha(\cos\alpha JPX + \sin\alpha JPY)). \end{aligned}$$

As the above formula is valid for any angle α and the vector fields X, JX, Y and JY are mutually orthogonal (and therefore independent) we deduce that $\theta(Y) = \theta(X) = \psi_\alpha$. Hence $\theta(X) = \theta$ is constant. The converse can be verified by a straightforward computation. □

LEMMA 4.2. *Let P^* be as in Lemma* 4.1. *Then P^* satisfies* (2.5) *if and only if θ is a multiple of $\frac{2\pi}{3}$, i.e., if and only if P^* is either P_1, P_2 or P_3. Moreover, in that case* (2.6) *is trivially satisfied.*

PROOF. We write $P^*(X) = \cos\theta PX + \sin\theta JPX$. It then follows that

$$\begin{aligned} G(P^*X, P^*Y) &= \cos^2\theta G(PX, PY) + \sin^2\theta G(JPX, JPY) \\ &\quad + \cos\theta\sin\theta(G(PX, JPY) + G(JPX, PY)) \\ &= \cos 2\theta G(PX, PY) - \sin 2\theta JG(PX, PY) \\ &= -\cos 2\theta PG(X, Y) + \sin 2\theta JPG(X, Y). \end{aligned}$$

On the other hand, we have that

$$-P^*G(X, Y) = -\cos\theta PG(X, Y) - \sin\theta JPG(X, Y).$$

As $PG(X, Y)$ and $JPG(X, Y)$ are mutually orthogonal, we see that equality holds if and only if $\cos 2\theta = \cos\theta = \cos(-\theta)$ and $\sin 2\theta = -\sin\theta = \sin(-\theta)$. Hence, if and only if, 3θ is a multiple of 2π.

In order to show that $P^\star$ now satisfies also (2.6), it is sufficient to consider the case that $P^* = -\frac{1}{2}P + \frac{\sqrt{3}}{2}\varepsilon JP$ where $\varepsilon = \pm 1$. On the one hand we get that

$$\begin{aligned} (\tilde\nabla_X P^*)Y &= \tilde\nabla_X P^*Y - P^*\tilde\nabla_X Y \\ &= -\tfrac{1}{2}(\tilde\nabla_X P)Y + \varepsilon\tfrac{\sqrt{3}}{2}\tilde\nabla_X JPY - \varepsilon\tfrac{\sqrt{3}}{2}JP\tilde\nabla_X Y \\ &= -\tfrac{1}{2}(\tilde\nabla_X P)Y + \varepsilon\tfrac{\sqrt{3}}{2}(G(X, PY) + J\tilde\nabla_X PY - JP\tilde\nabla_X Y) \\ &= -\tfrac{1}{2}(\tilde\nabla_X P)Y + \varepsilon\tfrac{\sqrt{3}}{2}(G(X, PY) + J(\tilde\nabla_X P)Y) \\ &= -\tfrac{1}{4}J(G(X, PY) + PG(X, Y)) \\ &\quad + \varepsilon\tfrac{\sqrt{3}}{4}(2G(X, PY) - G(X, PY) - PG(X, Y)) \\ &= -\tfrac{1}{4}J(G(X, PY) + PG(X, Y)) + \varepsilon\tfrac{\sqrt{3}}{4}(G(X, PY) - PG(X, Y)). \end{aligned}$$

On the other hand we get that

$$\tfrac{1}{2}J(G(X, P^*Y) + P^*G(X, Y)) = -\tfrac{1}{4}(JG(X, PY) + JPG(X, Y))$$

$$+\,\varepsilon\tfrac{\sqrt{3}}{2}(JG(X,JPY)-PG(X,Y))$$
$$=-\tfrac{1}{4}(JG(X,PY)+JPG(X,Y))$$
$$+\,\varepsilon\tfrac{\sqrt{3}}{4}(G(X,PY)-PG(X,Y)).$$

Comparing now both right-hand sides completes the proof of the lemma. □

Combining the previous lemmas, we deduce that the only nearly productlike structures on $\mathbb{S}^3 \times \mathbb{S}^3$ are $P_1 = P$, P_2 and P_3.

5. Some isometries of $\mathbb{S}^3 \times \mathbb{S}^3$ and their properties

We look at the maps $\widetilde{\mathcal{F}_{abc}}$, $\widetilde{\mathcal{F}_1}$ and $\widetilde{\mathcal{F}_2}$ of $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$ defined respectively by

$$\widetilde{\mathcal{F}_{abc}}(g_1,g_2,g_3)=(ag_1,bg_2,cg_3),$$
$$\widetilde{\mathcal{F}_1}(g_1,g_2,g_3)=(g_2,g_1,g_3),$$
$$\widetilde{\mathcal{F}_2}(g_1,g_2,g_3)=(g_3,g_2,g_1),$$

where a,b,c are unitary quaternions. We could, of course, consider

$$\widetilde{\mathcal{F}}_3(g_1,g_2,g_3)=\widetilde{\mathcal{F}}_1\circ\widetilde{\mathcal{F}}_2\circ\widetilde{\mathcal{F}}_1(g_1,g_2,g_3),$$

as well the cyclic permutations $\widetilde{\mathcal{F}}_1 \circ \widetilde{\mathcal{F}}_2$ and $(\widetilde{\mathcal{F}}_1 \circ \widetilde{\mathcal{F}}_2)^2$. Their properties follow immediately from those of $\widetilde{\mathcal{F}}_1$ and $\widetilde{\mathcal{F}}_2$.

An elementary computation shows that $\widetilde{\mathcal{F}_{abc}}$, $\widetilde{\mathcal{F}_1}$ and $\widetilde{\mathcal{F}_2}$ are isometries of $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$. Note that these isometries all have the property that for any unit quaternion d we have that

$$\pi\widetilde{\mathcal{F}_{abc}}(g_1d,g_2d,g_3d)=\pi(ag_1d,bg_2d,cg_3d)=(ag_1\bar{g}_3\bar{c},bg_2\bar{g}_3\bar{c}),$$
$$\pi\widetilde{\mathcal{F}_1}(g_1d,g_2d,g_3d)=\pi(g_2d,g_1d,g_3d)=(g_2\bar{g}_3,g_1\bar{g}_3),$$
$$\pi\widetilde{\mathcal{F}_2}(g_1d,g_2d,g_3d)=\pi(g_3d,g_2d,g_1d)=(g_3\bar{g}_1,g_2\bar{g}_1)$$

are independent of the unit quaternion d. Therefore we can define the applications $\mathcal{F}_{abc}$, $\mathcal{F}_1$ and $\mathcal{F}_2$ of $\mathbb{S}^3 \times \mathbb{S}^3$ such that

$$\pi\circ\widetilde{\mathcal{F}_{abc}}=\mathcal{F}_{abc}\circ\pi,\quad \pi\circ\widetilde{\mathcal{F}_1}=\mathcal{F}_1\circ\pi,\quad \pi\circ\widetilde{\mathcal{F}_2}=\mathcal{F}_2\circ\pi.$$

As $\widetilde{\mathcal{F}_{abc}}$, $\widetilde{\mathcal{F}_1}$ and $\widetilde{\mathcal{F}_2}$ are isometries of $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$ and the nearly Kähler metric is a constant multiple of the metric of the Riemannian submersion, it follows that $\mathcal{F}_{abc}$, $\mathcal{F}_1$ and $\mathcal{F}_2$ are isometries of the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$. The same remains of course valid for all compositions of these applications. Note that these applications are given by

$$\mathcal{F}_{abc}(p,q)=(ap\bar{c},bq\bar{c}),\quad \mathcal{F}_1(p,q)=(q,p),\quad \mathcal{F}_2(p,q)=(\bar{p},q\bar{p}).$$

As indicated in [**3**], the isometries $\mathcal{F}_{abc}$ also preserve both the almost complex structure J and the almost product structure P. As we will see in the next theorems, this is no longer true for the isometries $\mathcal{F}_1$ and $\mathcal{F}_2$.

In order to investigate the behavior of J, P_1, P_2 and P_3 under the maps $\mathcal{F}_1$ and $\mathcal{F}_2$, we write an arbitrary tangent vector at a point (p,q) by $X(p,q)=(p\alpha,q\beta)$, where α and β are imaginary quaternions. This is a tangent vector to a curve $\delta(t) = (\delta_1(t), \delta_2(t))$ in $\mathbb{S}^3 \times \mathbb{S}^3$ with initial conditions

$$\delta_1(0) = p, \quad \delta_2(0) = q, \quad \delta_1'(0) = p\alpha, \quad \delta_2'(0) = q\beta.$$

It now follows that

$$\begin{aligned} d\mathcal{F}_1(p\alpha, q\beta) &= d\mathcal{F}_1(X(p,q)) = \tfrac{d}{dt}\mathcal{F}_1(\delta(t))|_{t=0} \\ &= \tfrac{d}{dt}(\delta_2(t), \delta_1(t))|_{t=0} = (q\beta, p\alpha), \\ d\mathcal{F}_2(p\alpha, q\beta) &= d\mathcal{F}_2(X(p,q)) = \tfrac{d}{dt}\mathcal{F}_2(\delta(t))|_{t=0} \\ &= \tfrac{d}{dt}(\bar{\delta}_1(t), \delta_2(t)\bar{\delta}_1(t))|_{t=0} = (\bar{\alpha}\bar{p}, q(\beta + \bar{\alpha})\bar{p}) \\ &= (\bar{p}(p(-\alpha)\bar{p}), q\bar{p}(p(\beta - \alpha)\bar{p})). \end{aligned}$$

On the other hand, we recall that

$$\begin{aligned} J(p\alpha, q\beta) &= \tfrac{1}{\sqrt{3}}(p(2\beta - \alpha), q(-2\alpha + \beta)), \\ P_1(p\alpha, q\beta) &= P(p\alpha, q\beta) = (p\beta, q\alpha), \end{aligned}$$

wherefrom we deduce that

$$\begin{aligned} P_2(p\alpha, q\beta) &= -\tfrac{1}{2}P(p\alpha, q\beta) - \tfrac{\sqrt{3}}{2}JP((p\alpha, q\beta)) \\ &= -\tfrac{1}{2}((p\beta, q\alpha)) - \tfrac{1}{2}(p(2\alpha - \beta), q(-2\beta + \alpha)) \\ &= (-p\alpha, q(\beta - \alpha)) \\ P_3(p\alpha, q\beta) &= -\tfrac{1}{2}P(p\alpha, q\beta) + \tfrac{\sqrt{3}}{2}JP((p\alpha, q\beta)) \\ &= -\tfrac{1}{2}((p\beta, q\alpha)) + \tfrac{1}{2}(p(2\alpha - \beta), q(-2\beta + \alpha)) \\ &= (p(\alpha - \beta), q(-\beta)). \end{aligned}$$

Using the above formulas, if necessary at different points and for different tangent vectors, we now can prove the following two theorems. Notice first that P_1, P_2, P_3, given by Lemma 3.2, are precisely the three possible almost product structures on $\mathbb{S}^3 \times \mathbb{S}^3$ which preserve the basic equations. We will see that even though the maps $\mathcal{F}_1$ and $\mathcal{F}_2$ are isometries of $\mathbb{S}^3 \times \mathbb{S}^3$, which do not necessarily preserve the almost product structure P, they do preserve the triple of almost product structures $\{P_1, P_2, P_3\}$.

Theorem 5.1. *The differential of the isometry $\mathcal{F}_1$ anticommutes with J, i.e., $d\mathcal{F}_1 \circ J = -J \circ d\mathcal{F}_1$. For the almost product structures P_1, P_2 and P_3 we have*

$$\begin{aligned} d\mathcal{F}_1 \circ P_1 &= P_1 \circ d\mathcal{F}_1, \\ d\mathcal{F}_1 \circ P_2 &= P_3 \circ d\mathcal{F}_1, \\ d\mathcal{F}_1 \circ P_3 &= P_2 \circ d\mathcal{F}_1. \end{aligned}$$

Proof. We have

$$\begin{aligned} d\mathcal{F}_1(JX) &= \tfrac{1}{\sqrt{3}}(q(-2\alpha + \beta), p(2\beta - \alpha)), \\ Jd\mathcal{F}_1(X) &= J_{(q,p)}(q\beta, p\alpha) = \tfrac{1}{\sqrt{3}}(q(2\alpha - \beta), p(-2\beta + \alpha)), \end{aligned}$$

from which the first claim follows. The other claims follow from comparing

$$\begin{aligned} d\mathcal{F}_1(P_1X) &= (q\alpha, p\beta), & P_1 d\mathcal{F}_1(X) &= (q\alpha, p\beta),\\ d\mathcal{F}_1(P_2X) &= (q(\beta-\alpha), p(-\alpha)), & P_2 d\mathcal{F}_1(X) &= (q(-\beta), p(\alpha-\beta)),\\ d\mathcal{F}_1(P_3X) &= (q(-\beta), p(\alpha-\beta)), & P_3 d\mathcal{F}_1(X) &= (q(\beta-\alpha), p(-\alpha)). \end{aligned} \qquad \square$$

Theorem 5.2. *The differential of the isometry $\mathcal{F}_2$ anticommutes with J, i.e., $d\mathcal{F}_2 \circ J = -J \circ d\mathcal{F}_2$. For the almost product structures P_1, P_2 and P_3 we have*

$$\begin{aligned} d\mathcal{F}_2 \circ P_1 &= P_3 \circ d\mathcal{F}_2,\\ d\mathcal{F}_2 \circ P_2 &= P_2 \circ d\mathcal{F}_2,\\ d\mathcal{F}_2 \circ P_3 &= P_1 \circ d\mathcal{F}_2. \end{aligned}$$

We omit the proof of this theorem as it is similar to the one of Theorem 5.1. From the above two results we see that J is preserved up to sign by $\mathcal{F}_1$ and $\mathcal{F}_2$ (and therefore preserved by the composition of the two). On the other hand, by a suitable composition of $\mathcal{F}_1$ and $\mathcal{F}_2$, we see that we can switch between $P = P_1$, P_2 and P_3.

Of course, by applying the isometries $\mathcal{F}_1$ and $\mathcal{F}_2$ allows us to switch between these structures and therefore from an isometrical point of view these can not be distinguished. As a consequence, in many classification theorems of submanifolds, there will appear 3 isometrical examples with slightly different tensors P (see [**1, 2, 14**]).

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