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PROFESSOR MILEVA PRVANOVIĆ – HER CONTRIBUTION TO THE THEORY OF PSEUDOSYMMETRY TYPE MANIFOLDS

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Dedicated to the memory of Professor Mileva Prvanović

Abstract. A part of research activity of Professor Mileva Prvanović is related to the theory of pseudosymmetry type manifolds. We present her contribution to this theory.

1. Warped product manifolds satisfying some curvature conditions

A part of research activity of Professor Mileva Prvanović is related to the theory of pseudosymmetry type manifolds. Her results on this subject are contained in [**5**, **8**, **15**–**18**, **25**–**27**]. In this paper we present comments and remarks on results contained in [**5**, **8**, **25**–**27**]. We mention that in [**2**, **23**] (see also [**1**]) some surveys on research results of Professor Prvanović are given.

Let (M, g), $n = \dim M \geqslant 3$, be a semi-Riemannian manifold. We denote by ∇, R, S, κ and C the Levi-Civita connection, the Riemann–Christoffel curvature tensor, the Ricci tensor, the scalar curvature and the Weyl conformal curvature tensor of (M, g), respectively. We refer to [**5**, **11**, **14**, **17**] for precise definitions of the symbols used. Let $(\overline{M}, \overline{g})$ and $(\widetilde{N}, \widetilde{g})$, $\dim \overline{M} = p$, $\dim \widetilde{N} = n - p$, $1 \leqslant p < n$, be semi-Riemannian manifolds and F a positive smooth function on $\overline{M}$. We denote by $\overline{M} \times_F \widetilde{N}$ the *warped product manifold* of $(\overline{M}, \overline{g})$ and $(\widetilde{N}, \widetilde{g})$, see, e.g., [**3**, **24**, **27**].

It is well known that if a semi-Riemannian manifold (M, g), $n \geqslant 3$, is locally symmetric, then $\nabla R = 0$ on M. This implies the following integrability condition $\mathcal{R}(X, Y) \cdot R = 0$, in short $R \cdot R = 0$. Semi-Riemannian manifolds satisfying the last condition are called *semisymmetric*, see, e.g., [**28**]. Semisymmetric manifolds form a subclass of the class of pseudosymmetric manifolds. A semi-Riemannian manifold (M, g), $n \geqslant 3$, is said to be *pseudosymmetric* if the tensors $R \cdot R$ and

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$Q(g,R)$ are linearly dependent at every point of M, see, e.g., [**29**]. This is equivalent to $R\cdot R = L_R Q(g,R)$ on $\mathcal{U}_R = \{x \in M \mid R - (\kappa/(2(n-1)n)\, g \wedge g \neq 0 \text{ at } x\}$, where L_R is some function on this set. It seems that the Schwarzschild spacetime, the Kottler spacetime, the Reissner–Nordstrøm spacetime, as well as some Friedmann–Lemaître–Robertson–Walker spacetimes are the "oldest" examples of non-semisymmetric pseudosymmetric manifolds, see, e.g., [**11**,**14**]. A semi-Riemannian manifold (M,g), $n \geqslant 3$, is called *Ricci-pseudosymmetric* if the tensors $R\cdot S$ and $Q(g,S)$ are linearly dependent at every point of M, see, e.g., [**11**,**13**,**19**]. This is equivalent to $R\cdot S = L_S\, Q(g,S)$ on $\mathcal{U}_S = \{x \in M \mid S - (\kappa/n)\, g \neq 0 \text{ at } x\}$, where L_S is some function on this set. Every pseudosymmetric manifold is Ricci-pseudosymmetric. The converse statement is not true, see, e.g., [**13**,**19**].

A semi-Riemannian manifold (M,g), $n \geqslant 3$, is said to be *recurrent*, resp., *birecurrent*, if $\nabla R = \Phi \otimes R$, resp., $\nabla^2 R = \Psi \otimes R$, on the set U of all points of M at which the Riemann–Christoffel curvature tensor R is nonzero, where Φ is an 1-form and Ψ a 2-form on U. In [**24**] Professor Mileva Prvanović determined necessary and sufficient conditions for a warped product manifold to be a recurrent manifold. A few years later, results of [**24**] were used in investigation of birecurrent, conformally symmetric ($\nabla C = 0$), conformally recurrent ($\nabla C = \Phi \otimes C$) and conformally birecurrent ($\nabla^2 C = \Psi \otimes C$) warped product manifolds [**9**,**21**,**22**]. In particular, some result contained in [**21**, pp. 21–22], we can present as follows: if the warped product manifold $\overline{M} \times_F \widetilde{N}$, $\dim \overline{M} \geqslant 1$, $\dim \widetilde{N} \geqslant 3$, is a semisymmetric manifold, then the fiber $(\widetilde{N}, \widetilde{g})$ is a pseudosymmetric manifold.

In [**27**] Professor Prvanović presented a survey of results on warped product manifolds. In particular, Section 4.4 of that paper is closely related to pseudosymmetric and Ricci-pseudosymmetric manifolds. We mention that necessary and sufficient conditions for a warped product manifold to be pseudosymmetric, resp., Ricci-pseudosymmetric, are given in [**10**], resp., [**13**,**19**].

As it was proved in [**20**], on every hypersurface M, $\dim M \geqslant 3$, isometrically immersed in a semi-Riemannian space of constant curvature N, the tensors $R\cdot R$, $Q(S,R)$ and $Q(g,C)$ of M satisfy on M the following identity

$$R\cdot R = Q(S,R) + L\,Q(g,C), \tag{1.1}$$

where $L = (-(n-2)\widetilde{\kappa})/(n(n+1))$ and $\widetilde{\kappa}$ is the scalar curvature of the ambient space. Evidently, if N is a semi-Euclidean space, then (1.1) reduces to

$$R\cdot R = Q(S,R). \tag{1.2}$$

The necessary and sufficient conditions for a warped product manifold to be a manifold satisfying (1.1), resp., (1.2), were determined in [**8**], resp., in [**7**]. For instance, in [**8**, Theorem 4.1] it was stated that every manifold $\overline{M} \times_F \widetilde{N}$, $\dim \overline{M} = 1$, $\dim \widetilde{N} = 3$, satisfies (1.1), for some function L. Thus, in particular, every 4-dimensional generalized Robertson–Walker spacetime satisfies (1.1). Recently, in [**11**, Theorem 7.1 (i)] it was proved that warped product manifold $\overline{M} \times_F \widetilde{N}$, $\dim \overline{M} = \dim \widetilde{N} = 2$, as well as warped product manifold $\overline{M} \times_F \widetilde{N}$, with the fiber $(\widetilde{N}, \widetilde{g})$, $\dim \overline{M} = 2$, $\dim \widetilde{N} = n-2 \geqslant 3$, which is a space of constant curvature, satisfies (1.1). In the proof of that theorem results of [**8**] were applied.

The semi-Riemannian manifold (M, g), $n \geqslant 3$, is said to be a quasi-Einstein manifold if $\operatorname{rank}(S - \alpha\, g) = 1$ on $\mathcal{U}_S \subset M$, where α is some function on this set. Quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations and the investigation on quasi-umbilical hypersurfaces of conformally flat spaces, see e.g., [**4**] and references therein. Recently quasi-Einstein manifolds satisfying some pseudosymmetry type curvature conditions were investigated among others in [**5**, **11**]. There are different extensions of the class of quasi-Einstein manifolds. For instance we have the class of almost quasi-Einstein manifolds, see, e.g., [**4**], or the class of 2-quasi-Einstein manifolds, see, e.g., [**11**, **12**].

Let $(\widetilde{N}, \widetilde{g})$, $\dim \widetilde{N} = n-1 \geqslant 4$, be a not of constant curvature semi-Riemannian Einstein manifold, $\overline{M} = (a, b)$, $a < b$, $\overline{g}_{11} = \varepsilon = \pm 1$, and $F : (a, b) \to \mathbb{R}_+$ a smooth function. According to [**13**], the manifold $\overline{M} \times_F \widetilde{N}$, is a Ricci-pseudosymmetric manifold satisfying $R \cdot R = L_S Q(g, S)$ on $\mathcal{U}_S \subset \overline{M} \times_F \widetilde{N}$, with $L_S = \varepsilon((F'^2/4F^2) - (F''/2F))$, $F' = dF/dt$, $F'' = dF'/dt$ and $t \in (a, b)$. Further, on $\mathcal{U}_S$ we also have [**5**, Theorem 4.1]: $\operatorname{rank}(S - ((\kappa/(n-1)) - L_S)\, g) = 1$ and $(n-2)(R \cdot C - C \cdot R) = Q(S, R) - L_S Q(g, R)$. From this we obtain [**11**, Example 4.1] $(n-2)(R \cdot C - C \cdot R) = Q(S, C) - L_S Q(g, C)$.

Let $\overline{M} \times_F \widetilde{N}$ be a warped product manifold such that $(\widetilde{N}, \widetilde{g})$, $\dim \widetilde{N} = n-1 \geqslant 4$, is a semi-Riemannian non-Einstein manifold, $\overline{M} = (a, b)$, $a < b$, $\overline{g}_{11} = \varepsilon = \pm 1$, and $F : (a, b) \to \mathbb{R}_+$ a smooth function. In [**5**, Theorem 4.4] the necessary and sufficient conditions for such warped product to be a quasi-Einstein manifold are given.

As it was stated in Section 1, every warped product manifold $\overline{M} \times_F \widetilde{N}$, $\dim \overline{M} = 1$, $\dim \widetilde{N} = 3$, satisfies (1.1). We also have the following result related to manifolds: $\overline{M} \times_F \widetilde{N}$, $\dim \overline{M} = 1$, $\dim \widetilde{N} \geqslant 3$ [**11**, Theorem 4.3]. Let $\overline{M} \times_F \widetilde{N}$, $\dim \overline{M} = 1$, $\dim \widetilde{N} = n - 1 \geqslant 3$, be the manifold such that $(\widetilde{N}, \widetilde{g})$ is a quasi-Einstein manifold and let $(\widetilde{N}, \widetilde{g})$ is a conformally flat manifold, when $n \geqslant 5$. Then the manifold $\overline{M} \times_F \widetilde{N}$ satisfies (1.1) and $C \cdot C = L_C Q(g, C)$, for some function and L_C.

2. Ricci-generalized pseudosymmetric manifolds

According to [**6**], a semi-Riemannian manifold (M, g), $n = \dim M \geqslant 3$, is said to be *Ricci-generalized pseudosymmetric* if on M we have

$$R \cdot R = L Q(S, R), \tag{2.1}$$

where L is some function on M. In [**25**, **26**] Professor Mileva Prvanović presented some extension of the class of manifolds satisfying (2.1). Namely, in [**25**, **26**] SP-Sasakian manifolds satisfying the following curvature conditions were investigated

$$R \cdot R = L_p\, Q(S^p, R), \quad p = 0, 1, 2, \ldots, \tag{2.2}$$

$$R \cdot T = L_q\, Q(S^q, T), \quad q = 0, 1, 2, \ldots, \tag{2.3}$$

where L_p and L_q are some functions, the tensors S^0, S^1, S^2, $S^3, \ldots$, are defined by $S^0 = g$, $S^1 = S$, $S^2(X, Y) = S(\mathcal{S}X, Y)$, $S^3(X, Y) = S^2(\mathcal{S}X, Y), \ldots$, respectively, $\mathcal{S}$ is the Ricci operator, $g(\mathcal{S}X, Y) = S(X, Y)$, T is a generalized curvature tensor and X, Y are vector fields on M.

It is known that the Gödel spacetime satisfies among other things the conditions: (1.2), $S^2 = \kappa S$ and $\kappa = \mathrm{const} \neq 0$, see, e.g., [**11**, p. 14]. These relations yield $R \cdot R = (1/\kappa) Q(S^2, R)$. Thus the Gödel spacetime satisfies (2.2), with $p = 1$ and $L_1 = 1$, i.e., (1.2), as well as (2.2), with $p = 2$ and $L_2 = 1/\kappa$.

We define on an open connected and nonempty set $M \subset \mathrm{R}^4$ the metric g by

$$g_{ij} dx^i dx^j = f_1(x^1)\,(dx^1)^2 + f_2(x^1)\,(dx^2)^2 + f_3(x^1)\,(dx^3)^2 + f_4\,(x^2, x^3)\,(dx^4)^2,$$

where $f_1, \ldots, f_4$ are some positive smooth functions on M. We can check that at every point of M the tensors $R \cdot R$, $Q(g, C)$, $Q(S, R)$, $Q(S^2, R)$ and $Q(S^3, R)$ are linearly dependent.

Let $\overline{M} \times \widetilde{N}$ be the product manifold of a 3-dimensional Riemannian manifold $(\overline{M}, \overline{g})$ and an 1-dimensional Riemannian manifold $(\widetilde{N}, \widetilde{g})$. It is known that (1.2) holds on $\overline{M} \times \widetilde{N}$ [**7**, Corollary 3.2]. Moreover, we also have on $\overline{M} \times \widetilde{N}$: $S^4 + \alpha_3 S^3 + \alpha_2 S^2 + \alpha_1 S + \alpha_0 g = 0$, where $\alpha_0, \ldots, \alpha_3$ are some functions. Let $U \subset \overline{M} \times \widetilde{N}$ be the set of all points at which α_1 is nonzero. Thus $\alpha_1 R \cdot R = -\alpha_0 Q(g, R) - \alpha_2 Q(S^2, R) - \alpha_3 Q(S^3, R) - Q(S^4, R)$ on U.

Let M, $\dim M = n \geqslant 4$, is a hypersurface isometrically immersed in a Riemannian space of constant curvature N. The Ricci tensor S of the hypersurface M satisfies $S^n + \alpha_{n-1} S^{n-1} + \cdots + \alpha_2 S^2 + \alpha_1 S + \alpha_0 g = 0$, where $\alpha_0, \alpha_1, \ldots, \alpha_n$ are some functions on M. Let U be the set of all points of M at which α_1 is nonzero. Now, using (1.1), we can express the tensor $R \cdot R$ by a linear combination of the tensors $Q(g, C)$, $Q(g, R)$, $Q(S^2, R), \ldots, Q(S^n, R)$.

Using the above presented remarks, we can define, on a semi-Riemannian manifold of dimension $\geqslant 4$, the following curvature conditions:

(a) the tensor $R \cdot R$ is a linear combination of the tensors $Q(g, C)$, $Q(g, R)$, $Q(S^2, R), \ldots, Q(S^k, R)$, where $k = 0, 1, 2, \ldots$, and

(b) the tensor $R \cdot S$ is a linear combination of the tensors $Q(S^k, S^l)$, $k < l$, where $k = 0, 1, 2, \ldots$ and $l = 1, 2, \ldots$.

Some particular subcase of (b), the tensor $R \cdot S$ is a linear combination of the tensors $Q(g, S)$, $Q(g, S^2)$ and $Q(S, S^2)$, was investigated in [**11**, Section 6], see also [**12**, Section 4].

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