

## ON MATSUMOTO CHANGE OF $m$ -th ROOT FINSLER METRICS

Akbar Tayebi and Mohammad Shahbazi Nia

**ABSTRACT.** We consider Matsumoto change of Finsler metrics. First, we find a condition under which the Matsumoto change of a Finsler metric is projectively related to it. Then, considering the subspace of  $m$ -th root Finsler metrics, if  $\bar{F}$  is the Matsumoto change of  $F$ , we prove that  $\bar{F}$  is locally projectively flat if and only if it is locally dually flat. In this case,  $F$  and  $\bar{F}$  reduce to locally Minkowskian metrics.

### 1. Introduction

Let  $(M, F)$  be a Finsler space. In 1984, C. Shibata studied the properties of Finsler space  $(M, \bar{F})$  whose fundamental metric function  $\bar{F}$  is obtained from  $F$  by the relation  $F(x, y) \rightarrow \bar{F}(x, y) = f(F, \beta)$ , where  $\beta(x, y) = b_i(x)y^i$  is a 1-form on  $M$  and  $f = f(F, \beta)$  is a positively homogeneous function of  $F$  and  $\beta$ . This change of Finsler metric function has been called a  $\beta$ -change. He studied some geometrical properties of tensors being invariant by  $\beta$ -change of the metric [9]. If  $\|\beta\|_F := \sup_{F(x,y)=1} |\beta| < 1$ , then  $\bar{F}$  is again a Finsler metric.

There is a special case of  $\beta$ -change, namely

$$(1.1) \quad \bar{F}(x, y) = \frac{F^2}{F - \beta}$$

which is called the Matsumoto change of  $F$ . If  $F$  reduces to a Riemannian metric  $\alpha$ , then  $\bar{F}$  reduces to the Matsumoto metric  $F = \frac{\alpha^2}{\alpha - \beta}$ . Due to this reason, transformation (1.1) is called the Matsumoto change of Finsler metrics. The Matsumoto metric is an important metric in the Finsler geometry which is the Matsumoto's slope-of-a-mountain metric. This metric was introduced by Matsumoto as a realization of Finsler's idea "a slope measure of a mountain with respect to a time measure".

Two Finsler metrics  $F$  and  $\bar{F}$  on a manifold  $M$  are called projectively related if any geodesic of the first is also geodesic for the second and vice versa. In this case,

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there is a scalar function  $P = P(x, y)$  defined on  $TM_0$  such that  $\bar{G}^i = G^i + Py^i$ , where  $\bar{G}^i$  and  $G^i$  are the geodesic spray coefficients of  $\bar{F}$  and  $F$ , respectively. In this paper, we find a condition under which the Matsumoto change of a Finsler metric is projectively related to it. Let  $(M, F)$  be a Finsler manifold and  $\beta = b_i(x)y^i$  a 1-form on  $M$ . Put  $r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i})$ ,  $s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i})$ ,  $r_{00} = r_{ij}y^i y^j$ , where  $|$  denotes the horizontal derivation with respect to the Berwald connection of  $F$ . Then, we have the following.

**THEOREM 1.1.** *Let  $(M, F)$  be a Finsler manifold. Suppose that  $\bar{F} = F^2/(F - \beta)$  be the Matsumoto change of  $F$ . Then  $\bar{F}$  is projectively related  $F$  if and only if  $\beta$  satisfies*

$$s_{ij} = [\mathcal{A}_j r_{ik} - \mathcal{A}_i r_{jk}]y^k, \quad \text{where } \mathcal{A}_i := \frac{2(F_i - b_i)F - 2F_i(F - \beta)}{F(F - \beta)}.$$

In this case, the projective factor is given by  $P = \frac{1}{2(F - \beta)} r_{00}$ .

Let  $M$  be an  $n$ -dimensional  $C^\infty$  manifold,  $TM$  its tangent bundle. Let  $F = \sqrt[m]{A}$  be a Finsler metric on  $M$ , where  $A := a_{i_1 \dots i_m}(x)y^{i_1}y^{i_2} \dots y^{i_m}$  with  $a_{i_1 \dots i_m}$  symmetric in all its indices. Then  $F$  is called an  $m$ -th root Finsler metric [10–18]. The special  $m$ -th root metric in the form  $F = \sqrt[m]{y^1 y^2 \dots y^m}$  is called the Berwald–Moór metric [1–3, 5, 6].

A Finsler metric is said to be locally projectively flat if at any point there is a local coordinate system in which the geodesics are straight lines as point sets. It is known that a Finsler metric  $F(x, y)$  on an open domain  $U \subset \mathbb{R}^n$  is locally projectively flat if and only if  $G^i = Py^i$ , where  $P = P(x, y)$  is called the projective factor and is a  $C^\infty$  scalar function on  $TM_0$  satisfying  $P(x, \lambda y) = \lambda P(x, y)$  for all  $\lambda > 0$ .

**THEOREM 1.2.** *Let  $F = \sqrt[m]{A}$  ( $m > 2$ ), be an  $m$ -th root Finsler metric on an open subset  $U \subset \mathbb{R}^n$ . Suppose that  $\bar{F} = F^2/(F - \beta)$  be the Matsumoto change of  $F$ . Then  $\bar{F}$  is locally projectively flat if and only if  $A_{x^i} = 0$  and  $b_i = \text{constant}$ .*

A Finsler metric  $F$  on a manifold  $M$  is said to be locally dually flat if at any point there is a coordinate system  $(x^i)$  in which the spray coefficients are in the form  $G^i = -\frac{1}{2}g^{ij}H_{y^j}$ , where  $H = H(x, y)$  is a positively homogeneous scalar function on  $TM_0 = TM \setminus \{0\}$  [8].

**THEOREM 1.3.** *Let  $F = \sqrt[m]{A}$  ( $m > 2$ ), be an  $m$ -th root Finsler metric on an open subset  $U \subset \mathbb{R}^n$ . Suppose that  $\bar{F} = F^2/(F - \beta)$  be the Matsumoto change of  $F$ . Then  $\bar{F}$  is a locally dually flat Finsler metric if and only if  $A_{x^i} = 0$  and  $b_i = \text{constant}$ .*

## 2. Proof of Theorem 1.1

In this section, we are going to prove Theorem 1.1. To this aim we first prove the following result.

**LEMMA 2.1** (Rapcsák [7]). *Let  $F$  and  $\bar{F}$  be two Finsler metrics on a manifold  $M$ . Then  $\bar{F}$  is projectively related to  $F$  if and only if  $\bar{F}|_{k,l}y^k - \bar{F}|_l = 0$ ,*

where  $|$  denotes the horizontal derivation with respect to the Berwald connection of  $F$ . In this case, the spray coefficients are related by  $\bar{G}^i = G^i + Py^i$ , where

$$(2.1) \quad P = \frac{\bar{F}_{|k}y^k}{2\bar{F}}.$$

The  $P = P(x, y)$  is called the projective factor of  $F(x, y)$ .

Throughout this paper, we use the Berwald connection and the  $h$ - and  $v$ -covariant derivatives of a Finsler tensor field are denoted by " $|$ " and " $\cdot$ ", respectively. Now, let  $(M, F)$  be a Finsler manifold and  $\beta = b_i(x)y^i$  a 1-form on  $M$ . Put

$$\begin{aligned} r_{ij} &:= \frac{1}{2}(b_{i|j} + b_{j|i}), & r_{i0} &:= r_{ij}y^j, & r_{00} &:= r_{ij}y^i y^j, \\ s_{ij} &:= \frac{1}{2}(b_{i|j} - b_{j|i}), & s_{i0} &:= s_{ij}y^j, \\ R_{ij} &:= \frac{1}{2}\left(\frac{\partial b_i}{\partial x^j} + \frac{\partial b_j}{\partial x^i}\right), & R_{0i} &:= R_{ji}y^j, & R_{00} &:= R_{ij}y^i y^j. \end{aligned}$$

PROOF OF THEOREM 1.1. The following relations hold

$$b_{i|j} = \frac{\partial b_i}{\partial x^j} - \Gamma_{il}^s b_s, \quad \Gamma_{il}^s = \Gamma_{li}^s,$$

where  $\Gamma_{jk}^i = \Gamma_{jk}^i(x, y)$  is the Christoffel symbols of the Berwald connection of  $F$ . Then we have

$$(2.2) \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}) = \frac{1}{2}\left(\frac{\partial b_i}{\partial x^j} - \frac{\partial b_j}{\partial x^i}\right),$$

$$(2.3) \quad r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}) = \frac{1}{2}\left(\frac{\partial b_i}{\partial x^j} + \frac{\partial b_j}{\partial x^i} - 2b_s \Gamma_{ij}^s\right) = R_{ij} - b_s \Gamma_{ij}^s.$$

By (2.2) and (2.3), we get

$$(2.4) \quad \beta_{|l} = b_{i|l}y^i = \left(\frac{\partial b_i}{\partial x^l} - \Gamma_{il}^s b_s\right)y^i,$$

$$(2.5) \quad \beta_{|l}y^l = \left(\frac{\partial b_i}{\partial x^l} - \Gamma_{il}^s b_s\right)y^i y^l = R_{00} - 2b_s \Gamma_{ij}^s = r_{00},$$

$$(2.6) \quad \beta_{|k,l}y^k = \left(\frac{\partial b_l}{\partial x^k} - b_s \Gamma_{lk}^s\right)y^k,$$

whence (2.4) and (2.6) imply  $\beta_{|k,l}y^k - \beta_{|l} = s_{lk}y^k = s_{l0}$ . For  $\bar{F} = F^2/(F - \beta)$ , we have

$$(2.7) \quad \begin{aligned} \bar{F}_{|l} &= \frac{\beta_{|l}F^2}{(F - \beta)^2}, & \bar{F}_{|k,l} &= \left[\frac{(2F_l\beta_{|k} + F\beta_{|k,l})(F - \beta) - 2(F_l - b_l)F\beta_{|k}}{(F - \beta)^3}\right]F, \\ \bar{F}_{|k,l}y^k &= \left[\frac{(2F_l r_{00} + F\beta_{|k,l}y^k)(F - \beta) - 2(F_l - b_l)F r_{00}}{(F - \beta)^3}\right]F. \end{aligned}$$

Then

$$\begin{aligned} \bar{F}_{|k,l}y^k - \bar{F}_{|l} &= \left[\frac{(2F_l r_{00} + F\beta_{|k,l}y^k - F^2\beta_{|l})(F - \beta) - 2(F_l - b_l)F r_{00}}{(F - \beta)^3}\right]F \\ &= \left[\frac{(2F_l r_{00} + F s_{l0})(F - \beta) - 2(F_l - b_l)F r_{00}}{(F - \beta)^3}\right]F. \end{aligned}$$

Put

$$\mathcal{A}_i := \frac{2(F_i - b_i)F - 2F_i(F - \beta)}{F(F - \beta)}.$$

Then by Lemma 2.1,  $\bar{F}$  is projectively related to  $F$  if and only if  $s_{l0} = \mathcal{A}_l r_{00}$ , which, taking a vertical derivation, yields

$$(2.8) \quad s_{li} = \mathcal{A}_{li} r_{00} + 2\mathcal{A}_l r_{0i}.$$

Since  $s_{li} = -s_{il}$ , then by (2.8) we get  $\mathcal{A}_{li} r_{00} + 2\mathcal{A}_l r_{0i} = -\mathcal{A}_{il} r_{00} - 2\mathcal{A}_i r_{0l}$  or

$$(2.9) \quad \mathcal{A}_{li} r_{00} = -\mathcal{A}_l r_{0i} - \mathcal{A}_i r_{0l}.$$

By (2.8) and (2.9), we get  $s_{ij} = \mathcal{A}_i r_{0j} - \mathcal{A}_j r_{0i}$ . Now, by (2.5) and (2.7), we have

$$\bar{F}_{|y} y^k = \frac{\beta_{|k} y^k F^2}{(F - \beta)^2} = \frac{F^2 r_{00}}{(F - \beta)^2}.$$

By (2.1), it follows that

$$P = \frac{\bar{F}_{|k} y^k}{2\bar{F}} = \frac{r_{00}}{2(F - \beta)}. \quad \square$$

### 3. Proof of Theorem 1.2

It is known that a Finsler metric  $F(x, y)$  on  $U \subset R^n$  is projective if and only if its geodesic coefficients  $G^i$  are of the form  $G^i(x, y) = P(x, y)y^i$ , where  $P: TU = U \times R^n \rightarrow R$  is positively homogeneous of degree one with respect to  $y$ . In [4], Hamel showed that a Finsler metric  $F$  on  $U \subset R^n$  is projectively flat if and only if it satisfies  $F_{x^k y^l} y^k = F_{x^l}$ .

LEMMA 3.1. *Let  $F = \sqrt[m]{A}$  ( $m > 2$ ), be an  $m$ -th root Finsler metric on an open subset  $U \subset R^n$ . Suppose that the equation*

$$\Psi A^{\frac{2}{m}-1} + \Xi A^{\frac{2}{m}} + \Phi A^{\frac{1}{m}} + \Theta A^{\frac{1}{m}+1} + \Upsilon A^{\frac{1}{m}-1} + A^{\frac{1}{m}+2} \Omega + \Gamma = 0$$

*holds, where  $\Phi, \Psi, \Theta, \Upsilon, \Omega, \Xi$  are homogeneous polynomials in  $y$ . Then  $\Psi = \Xi = \Phi = \Theta = \Upsilon = \Omega = \Gamma = 0$ .*

For an  $m$ -th root metric  $F = \sqrt[m]{A}$ , put

$$A_i = \frac{\partial A}{\partial y^i}, \quad A_{ij} = \frac{\partial^2 A}{\partial y^j \partial y^i}, \quad A_{x^i} = \frac{\partial A}{\partial x^i}, \quad A_0 = A_{x^i} y^i, \quad A_{0l} = A_{x^k y^l} y^k = \frac{\partial^2 A}{\partial x^i \partial y^l} y^k.$$

Then we have the following.

PROOF OF THEOREM 1.2. For  $\bar{F} = \frac{\sqrt[m]{A^2}}{\sqrt[m]{A-\beta}}$ , we infer

$$(3.1) \quad [\bar{F}]_{x^l} = \frac{1}{m(\sqrt[m]{A-\beta})^2} \left[ A^{\frac{3}{m}-2} A_{x^l} - 2A^{\frac{2}{m}-1} A_{x^l} \beta + mA^{\frac{2}{m}} \beta_{x^l} \right],$$

$$[\bar{F}]_{x^k y^l} y^k = \frac{1}{m} \left[ \frac{(\frac{1}{m} - 2)A_0 A_l A^{\frac{4}{m}-3} + A_{0l} A^{\frac{4}{m}-2} + (2\beta_l A_0 - A_{0l} \beta + 2A_0 A_l \beta) A^{\frac{3}{m}-2}}{(A^{\frac{1}{m}} - \beta)^3} + \frac{(-2\beta A_{0l} - 2A_0 \beta_l) A^{\frac{3}{m}-1} + (2 - \frac{3}{m}) \beta A_0 A_l A^{\frac{3}{m}-3} + m\beta_{0l} A^{\frac{3}{m}}}{(A^{\frac{1}{m}} - \beta)^3} \right]$$

$$(3.2) \quad + \frac{(2\beta^2 A_{0l} - 2\beta A_l \beta_0 - 6\beta A_0 \beta_l) A^{\frac{2}{m}-1} + (\frac{4}{m} - 2)\beta^2 A_0 A_l A^{\frac{2}{m}-2}}{(A^{\frac{1}{m}} - \beta)^3} + \frac{(2m\beta_0 \beta_l - m\beta \beta_{0l}) A^{\frac{2}{m}}}{(A^{\frac{1}{m}} - \beta)^3},$$

where

$$\beta_{x^i} := \frac{\partial \beta}{\partial x^i}, \quad \beta_i := \frac{\partial \beta}{\partial y^i} = b_i, \quad \beta_0 := \beta_{x^i} y^i, \quad \beta_{0l} := \beta_{x^i l} y^i.$$

Since  $\bar{F}$  is locally projectively flat metric, we have  $[\bar{F}]_{x^k y^l} y^k - [\bar{F}]_{x^l} = 0$ . By substituting (3.1) and (3.2) into this, we get

$$(3.3) \quad \left(\frac{1}{m} - 2\right) A_0 A_l A^{\frac{4}{m}-3} + (A_{0l} - A_{x^l}) A^{\frac{4}{m}-2} \\ + [2\beta_l A_0 - \beta(A_{0l} - A_{x^l}) + 2A_0 A_l \beta] A^{\frac{3}{m}-2} \\ + [-2\beta(A_{0l} - A_{x^l}) - 2A_0 \beta_l] A^{\frac{3}{m}-1} + \left(2 - \frac{3}{m}\right) \beta A_0 A_l A^{\frac{3}{m}-3} \\ + m(\beta_{0l} - \beta_{x^l}) A^{\frac{3}{m}} + [2\beta^2(A_{0l} - A_{x^l}) - 2\beta A_l \beta_0 - 6\beta A_0 \beta_l] A^{\frac{2}{m}-1} \\ + \left(\frac{4}{m} - 2\right) \beta^2 A_0 A_l A^{\frac{2}{m}-2} + [2m\beta_0 \beta_l - m\beta(\beta_{0l} - \beta_{x^l})] A^{\frac{2}{m}} = 0.$$

Simplifying (3.3), it results that

$$(3.4) \quad \left(\frac{1}{m} - 2\right) A_0 A_l A^{\frac{2}{m}-1} + (A_{0l} - A_{x^l}) A^{\frac{2}{m}} \\ + [2\beta_l A_0 - \beta(A_{0l} - A_{x^l}) + 2A_0 A_l \beta] A^{\frac{1}{m}} \\ - [2\beta(A_{0l} - A_{x^l}) + 2A_0 \beta_l] A^{\frac{1}{m}+1} + \left(2 - \frac{3}{m}\right) \beta A_0 A_l A^{\frac{1}{m}-1} \\ + m(\beta_{0l} - \beta_{x^l}) A^{\frac{1}{m}+2} + m[2\beta_0 \beta_l - \beta(\beta_{0l} - \beta_{x^l})] A^2 \\ + 2\beta[\beta(A_{0l} - A_{x^l}) - A_l \beta_0 - 3A_0 \beta_l] A + \left(\frac{4}{m} - 2\right) \beta^2 A_0 A_l = 0.$$

According to Lemma 3.1, (3.4) reduces to the following

$$(3.5) \quad A_0 A_l = 0,$$

$$(3.6) \quad A_{0l} - A_{x^l} = 0,$$

$$2\beta_l A_0 - \beta(A_{0l} - A_{x^l}) + 2A_0 A_l \beta = 0,$$

$$\beta(A_{0l} - A_{x^l}) + A_0 \beta_l = 0,$$

$$(3.7) \quad \beta_{0l} - \beta_{x^l} = 0,$$

$$2\beta_0 \beta_l - \beta(\beta_{0l} - \beta_{x^l}) = 0,$$

$$(3.8) \quad \beta(A_{0l} - A_{x^l}) - A_l \beta_0 - 3A_0 \beta_l = 0.$$

By (3.6), we have

$$(3.9) \quad A_{x^l} - A_{0l} = 0.$$

The relations (3.5),  $A_l \neq 0$  and  $\beta \neq 0$  imply that

$$(3.10) \quad A_0 = 0.$$

Taking a vertical derivation of (3.10) yields

$$(3.11) \quad A_{x^l} + A_{0l} = 0.$$

By (3.9) and (3.11), we get  $A_{x^l} = 0$ . On the other hand, by substituting (3.9) and (3.10) in (3.8), we have  $\beta_0 = 0$ . Taking a vertical derivation of it implies that  $\beta_{0l} + \beta_{x^l} = 0$ . By considering (3.7), we get  $\beta_{x^l} = 0$ , which means that  $b_i$  are constants.  $\square$

#### 4. Proof of Theorem 1.3

In [8], Shen proved that the Finsler metric  $F$  on an open subset  $U \subset \mathbb{R}^n$  is dually flat if and only if it satisfies  $(F^2)_{x^k y^l} y^k = 2(F^2)_{x^l}$ . Now, we are going to characterize locally dually flat Finsler metrics which is obtained by a Matsumoto change of  $m$ -th root metrics. First, we remark the following.

LEMMA 4.1. *Let  $F = \sqrt[m]{A}$  ( $m > 2$ ), be an  $m$ -th root Finsler metric on an open subset  $U \subset \mathbb{R}^n$ . Suppose that the equation*

$$\Psi A^{\frac{2}{m}+1} + \Xi A^{\frac{2}{m}} + \Phi A^{\frac{1}{m}} + \Theta A^{\frac{1}{m}+1} + \Upsilon A^{\frac{1}{m}+2} + \Omega = 0$$

*holds, where  $\Phi, \Psi, \Theta, \Upsilon, \Omega, \Xi$  are homogeneous polynomials in  $y$ . Then  $\Psi = \Xi = \Phi = \Theta = \Upsilon = \Omega = 0$ .*

PROOF OF THEOREM 1.3. The following holds

$$(4.1) \quad [\bar{F}^2]_{x^k} = \frac{2A^{\frac{4}{m}-1} [A^{\frac{1}{m}} A_{x^l} - 2A_{x^l} \beta + mA\beta_{x^l}]}{m(A^{\frac{1}{m}} - \beta)^3},$$

$$(4.2) \quad [\bar{F}^2]_{x^k y^l} y^k = \frac{2}{m} \left[ \frac{A_{0l} A^{\frac{6}{m}-1} + (\beta_l A_0 - 3A_{0l} \beta + A_l \beta_0) A^{\frac{5}{m}-1} + (\frac{2}{m} - 1) A_0 A_l A^{\frac{6}{m}-2}}{(A^{\frac{1}{m}} - \beta)^4} \right. \\ \left. + \frac{(\frac{3}{m} - 1) \beta A_0 A_l A^{\frac{5}{m}-2} + m\beta_{0l} A^{\frac{5}{m}} + (\frac{8}{m} - 2) \beta^2 A_0 A_l A^{\frac{4}{m}-2}}{(A^{\frac{1}{m}} - \beta)^4} \right. \\ \left. + \frac{(2\beta^2 A_{0l} - 4\beta A_0 \beta_l + 4\beta A_l \beta_0) A^{\frac{4}{m}-1} + m\beta \beta_{0l} A^{\frac{4}{m}}}{(A^{\frac{1}{m}} - \beta)^4} \right].$$

Since  $\bar{F}$  is a locally dually flat metric, then

$$(4.3) \quad [\bar{F}^2]_{x^k y^l} y^k - 2[\bar{F}^2]_{x^l} = 0.$$

By substituting (4.1) and (4.2) in (4.3), we infer:

$$(4.4) \quad (A_{0l} - 2A_{x^l}) A^{\frac{6}{m}-1} + [\beta_l A_0 - 3(A_{0l} - 2A_{x^l}) \beta + A_l \beta_0] A^{\frac{5}{m}-1} \\ + \left(\frac{2}{m} - 1\right) A_0 A_l A^{\frac{6}{m}-2} + \left(\frac{3}{m} - 1\right) \beta A_0 A_l A^{\frac{5}{m}-2} + m(\beta_{0l} - 2\beta_{x^l}) A^{\frac{5}{m}} \\ + 2\beta [\beta(A_{0l} - 2A_{x^l}) - 2A_0 \beta_l + 2A_l \beta_0] A^{\frac{4}{m}-1} + \left(\frac{8}{m} - 2\right) \beta^2 A_0 A_l A^{\frac{4}{m}-2} \\ + m\beta(\beta_{0l} - 2\beta_{x^l}) A^{\frac{4}{m}} = 0.$$

Simplifying (4.4) implies that

$$\begin{aligned}
 (4.5) \quad & (A_{0l} - 2A_{x^l})A^{\frac{2}{m}+1} + [\beta_l A_0 - 3(A_{0l} - 2A_{x^l})\beta + A_l \beta_0]A^{\frac{1}{m}+1} \\
 & + \left(\frac{2}{m} - 1\right)A_0 A_l A^{\frac{2}{m}} + \left(\frac{3}{m} - 1\right)\beta A_0 A_l A^{\frac{1}{m}} + m(\beta_{0l} - 2\beta_{x^l})A^{\frac{1}{m}+2} \\
 & + m\beta(\beta_{0l} - 2\beta_{x^l})A^2 + 2\beta[\beta(A_{0l} - 2A_{x^l}) - 2A_0\beta_l + 2A_l\beta_0]A \\
 & + \left(\frac{8}{m} - 2\right)\beta^2 A_0 A_l = 0.
 \end{aligned}$$

By Lemma 4.1, (4.5) reduces to

$$(4.6) \quad A_0 A_l = 0,$$

$$(4.7) \quad A_{0l} - 2A_{x^l} = 0,$$

$$2\beta(\beta_{0l} - 2\beta_{x^l}) = 0,$$

$$(4.8) \quad 3\beta(A_{0l} - 2A_{x^l}) - A_0\beta_l - A_l\beta_0 = 0,$$

$$(4.9) \quad \beta_{0l} - 2\beta_{x^l} = 0,$$

$$\beta(A_{0l} - 2A_{x^l}) + 2(\beta_l A_0 + \beta_0 A_l) = 0.$$

By (4.6), we have  $A_0 = 0$ . Taking a vertical derivation of it implies that  $A_{x^l} + A_{0l} = 0$ . Then by (4.7), it follows that  $A_{x^l} = 0$ . In this case, (4.8) reduces to  $\beta_0 = 0$ . Taking a vertical derivation of it implies that  $\beta_{0l} + \beta_{x^l} = 0$ . Then by (4.9), we get  $\beta_{x^l} = 0$  which means that  $b_i$  are constants. This completes the proof.  $\square$

**COROLLARY 4.1.** *Let  $F = \sqrt[m]{A}$  ( $m > 2$ ) be an  $m$ -th root Finsler metric on an open subset  $U \subset \mathbb{R}^n$ . Suppose that  $\bar{F} = F^2/(F - \beta)$  be the Matsumoto change of  $F$ . Then  $\bar{F}$  is locally projectively flat if and only if it is locally dually flat. In this case,  $F$  and  $\bar{F}$  are Berwald–Moór metrics.*

**PROOF.** By Theorem 1.2 and 1.3,  $\bar{F}$  is locally projectively flat if and only if it is locally dually flat. Since  $A_{x^i} = 0$ , then  $a_{i_1 \dots i_m}(x) = c$  is a constant. In this case, we get

$$F = \sqrt[m]{c y^1 y^2 \dots y^m}$$

which is a locally Minkowskian metric. Since  $b_i = \text{constant}$  and  $\bar{F} = F^2/(F - \beta)$ , then  $\bar{F}$  is locally Minkowskian, too.  $\square$

### 5. Conclusion

Every locally Minkowskian metric is locally projectively flat and locally dually flat metric. In this paper, we study the Matsumoto change of a Finsler metric and prove that the Matsumoto change of an  $m$ -th root metric is locally projectively flat if and only if it is locally dually flat if and only if it is locally Minkowskian. The study of this Finslerian change on an  $m$ -th root metric will enhance our understanding of the geometric meaning of the class of  $m$ -th root metrics.

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Department of Mathematics  
 Faculty of Science  
 University of Qom  
 Qom, Iran  
 akbar.tayebi@gmail.com  
 m.shahbazinia@gmail.com

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