PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 100(114) (2016), 87–93 DOI: 10.2298/PIM1614087S

DISTRIBUTIVE LATTICES OF JACOBSON RINGS

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Abstract. We characterize the distributive lattices of Jacobson rings and prove that if a semiring is a distributive lattice of Jacobson rings, then, up to isomorphism, it is equal to the subdirect product of a distributive lattice and a Jacobson ring. Also, we give a general method to construct distributive lattices of Jacobson rings.

1. Introduction and preliminaries

A semigroup S is called *periodic* if each element of S has a finite order, where the order of $a \in S$ is the order of the cyclic subsemigroup of S generated by a. Periodic semigroups have been studied by many algebraists. Suppose that S is a periodic semigroup. For any $a \in S$ we all know that there exist the smallest positive integer m and the smallest positive integer r such that $a^m = a^{m+r}$. The positive integer m is referred to as the *index* and the positive integer r as the *period* of a. In particular, if the index of each $a \in S$ is equal to 1, then we call S a *strongly periodic semigroup*. Idempotent semigroups and Burnside semigroups satisfying $x^n \approx x$ are special cases of strongly periodic semigroups.

A ring $(R, +, \cdot)$ is a *Jacobson ring* if, for any $a \in R$, there exists $n \in \mathbb{N}$, $n \geqslant 2$ such that $a = a^n$. That is to say, its multiplicative reduct is a strongly periodic semigroup. It is obvious that Boolean rings are Jacobson rings. Following [**6**, Theorem 11], we have

Theorem 1.1. *Let R be a Jacobson ring. Then every element of R has finite additive order and R is commutative.*

We denote by **JR** the class of all Jacobson rings.

2010 *Mathematics Subject Classification:* Primary 16Y60, 08B05; Secondary 20M07.

Key words and phrases: semiring; distributive lattice; Jacobson ring; Mal'cev product; congruence.

The first author is supported by the Natural Science Foundation of Shannxi Province, Grant 2015JQ1020. The second author is Supported by the Ministry of Education, Science and Technological Development of Serbia, Grant 174018. The third author is Supported by the Ministry of Education, Science and Technological Development of Serbia, Grant 174026.

Communicated by Žarko Mijajlović.

Semirings are the natural generalization of rings and distributive lattices. All semirings $(S, +, \cdot)$ occurring in the literature satisfy at least the following axioms: $(S, +)$, the *additive reduct*, and $(S, \cdot)$, the *multiplicative reduct* of a semiring S are semigroups, and the multiplication distributes over addition from both sides, i.e.,

(SR1) $x + (y + z) \approx (x + y) + z$;

(SR2) $x(yz) \approx (xy)z$;

(SR3) $x(y + z) \approx xy + xz, \quad (x + y)z \approx xz + yz$.

It is, as well, often assumed that $(S, +)$ is commutative, i.e.,

(SR4) $x + y \approx y + x$.

We consider the semiring classes considered that satisfy this identity too.

By an idempotent semiring, we mean a semiring S which satisfies the additional identities $xx \approx x + x \approx x$. An idempotent semiring $(S, +, \cdot)$ is called a bisemilattice if both the multiplicative reduct $(S, \cdot)$ and the additive reduct $(S, +)$ are semilattices. Of course, a distributive lattice is a bisemilattice which satisfies the absorption law $x + xy \approx x$. The class of all distributive lattices is denoted by $\mathbf{D}$. The Mal'cev product of two classes $\mathbf{V}$ and $\mathbf{W}$ of semirings, denoted by $\mathbf{V} \circ \mathbf{W}$, we mean that the class of all semirings S on which there exists a congruence ρ such that $S/\rho \in \mathbf{W}$ and the ρ-classes which are subsemirings of S belong to $\mathbf{V}$. Thus, in this way, some classes of semirings can be constructed by considering the Mal'cev products of some given semirings.

For a semiring $(S, +, \cdot)$ we denote Green's $\mathcal{H}$ relation on the additive reduct $(S, +)$ by $\mathcal{H}^+$. Let $(S, +, \cdot)$ be a semiring whose additive reduct $(S, +)$ is a completely regular semigroup. By Theorem II.1.4 and Corollary II.1.5 in [**8**], $(S, +)$ is a commutative Clifford semigroup and $\mathcal{H}^+$ is the least semilattice congruence of the additive reduct $(S, +)$ of S, moreover, every $\mathcal{H}^+$-class is a maximal subgroup of $(S, +)$. For any $a \in S$ we denote by $\mathcal{H}^+_a$ the $\mathcal{H}^+$-class containing a and 0_a the identity of $\mathcal{H}^+_a$, respectively. It can be easily seen that $a\mathcal{H}^+b$ if and only if $0_a = 0_b$ for any $a, b \in S$.

Let $(S, +, \cdot)$ be a semiring whose additive reduct is a Clifford semigroup. We can define the natural partial order on $(S, +)$ by

$$a \leqslant_+ b \Leftrightarrow (\exists e \in E_+(S))\ a = b + e$$

for $a, b \in S$, where $E_+(S)$ is the set of idempotents of the additive reduct $(S, +)$ of S.

The Mal'cev product of the class of Jacobson rings and the class of distributive lattices is denoted by $\mathbf{JR} \circ \mathbf{D}$. A semiring S is called a distributive lattice of Jacobson rings if it is in $\mathbf{JR} \circ \mathbf{D}$. In the following we shall study the semirings which are distributive lattices of Jacobson rings.

Some authors have studied the distributive lattices of rings(see [**1, 2, 7**]). In particular, [**1**] and [**2**] characterized the subdirect product of rings and distributive lattice, respectively. If a semiring $(S, +, \cdot)$ is isomorphic to a subdirect product of a ring and a distributive lattice, then the additive reduct $(S, +)$ of S is a sturdy semilattice of abelian groups, which means that $(S, +)$ is E-unitary. The following example shows that, in general, distributive lattices of rings are not the subdirect product of a ring and a distributive lattice.

Example 1.1. Consider a five element semiring A_5 with operations given by

$+$	a	b	c	d	e
a	a	b	a	b	e
b	b	a	b	a	e
c	a	b	c	d	e
d	b	a	d	c	e
c	e	e	e	e	e

$.$	a	b	c	d	e
a	a	a	c	c	a
b	a	a	c	c	a
c	c	c	c	c	c
d	c	c	c	c	c
e	a	a	c	c	e

It is easy to see that $\mathcal{H}^+$ is a distributive lattice congruence and $\mathcal{H}_a^+ = \{a, b\}$, $\mathcal{H}_c^+ = \{c, d\}$ and $\mathcal{H}_e^+ = \{e\}$ are subrings of A_5. Since $e \leqslant_+ a$, $e \leqslant_+ b$, this means that $(A_5, +)$ is not E-unitary. In [**2**] and [**4**], it was proved that a distributive lattice of Boolean rings is isomorphic to subdirect product of a Boolean ring and a distributive lattice. In this paper we prove that if a semiring is distributive lattice of Jacobson rings, then, up to isomorphism, it is equal to the subdirect product of a distributive lattice and a Jacobson ring, which generalize and enrich some results from [**2, 4, 10**]. Also, we shall give a general method to construct distributive lattices of Jacobson rings.

2. Main results

Lemma 2.1. *A semiring S is a distributive lattice of Jacobson rings, i.e., $S \in$ $\mathbf{JR} \circ \mathbf{D}$, if and only if $\mathcal{H}^+$ is the least distributive lattice congruence on S and every $\mathcal{H}^+$-class is a Jacobson ring.*

Proof. Let a semiring S be a distributive lattice of Jacobson rings. Then there exists a semiring congruence ρ on S such that S/ρ is a distributive lattice and every ρ-class is a Jacobson ring. This also implies that ρ is a semilattice congruence on $(S, +)$. Since the additive reduct $(S, +)$ of S is a Clifford semigroup, $\mathcal{H}^+$ is the least semilattice congruence on $(S, +)$. This leads to $\mathcal{H}^+ \subseteq \rho$. On the other hand, since ρ_u (the ρ-class containing u) is a Jacobson ring for any $u \in S$, the additive reduct of ρ_u is an abelian subgroup of $(S, +)$. Thus $\rho_u \subseteq \mathcal{H}_u^+$, furthermore, $\rho \subseteq \mathcal{H}^+$. We have now shown that $\rho = \mathcal{H}^+$. That is to say that $\mathcal{H}^+$ is a distributive lattice congruence of semiring S and every $\mathcal{H}^+$-class is a Jacobson ring.

If μ is a distributive lattice congruence on S, then μ is a semilattice congruence on the additive reduct $(S, +)$. Since $\mathcal{H}^+$ is the least semilattice congruence on $(S, +)$, $\mathcal{H}^+ \subseteq \mu$, which implies that $\mathcal{H}^+$ is the least distributive lattice congruence on S.

Conversely, it is clear from definition that the semiring S is a distributive lattice of Jacobson rings since $\mathcal{H}^+$ is the least distributive lattice congruence on the semiring S and every $\mathcal{H}^+$-class is a Jacobson ring. $\square$

As a consequence of Lemma 2.1 we have the following result.

Corollary 2.1. *Let S be a semiring in $\mathbf{JR} \circ \mathbf{D}$. Then*

(i) *for any $a, b \in S$, $0_a + 0_b = 0_{a+b}$, $a0_b = 0_ba = 0_a0_b = 0_{ab}$, $a + a0_b = a$;*
(ii) *$E_+(S) = \{0_a \mid a \in S\}$ is a distributive lattice.*

Let S be a semiring in $\mathbf{JR} \circ \mathbf{D}$. Define a binary relation σ^+ on S by

$$(\forall a, b \in S)\, a\, \sigma^+ b \Leftrightarrow (\exists e \in E_+(S))\ a + e = b + e.$$

It follows from Proposition 5.3.2 in [**5**] that σ^+ is the least group congruence on the additive reduct $(S, +)$ of S. Thus we have

Lemma 2.2. *Suppose that S is a semiring in $\mathbf{JR} \circ \mathbf{D}$. Then σ^+ is the least Jacobson ring congruence on S.*

Proof. Assume that $a, b \in S$ and $a\, \sigma^+ b$. Then there exists $e \in E_+(S)$ such that $a + e = b + e$. For any $c \in S$ we have that $ca + ce = cb + ce$. By Corollary 2.1 it follows that $ce \in E_+(S)$ and so $ca\, \sigma^+ cb$. Dually, we can get $ac\, \sigma^+ bc$. Thus, σ^+ is a semiring congruence on S. Since $(S/\sigma^+, +)$ is an abelian group, $(S/\sigma^+, +, \cdot)$ is a ring. For any $a \in S$ we denote by σ^+_a the σ^+-class containing a. By Lemma 2.1 it yields that $\mathcal{H}^+_a$ is a Jacobson ring. Thus there exists a positive integer k such that $a^k = a$. Therefore, $(\sigma^+_a)^k = \sigma^+_{a^k} = \sigma^+_a$. Hence, $(S/\sigma^+, +, \cdot)$ is a Jacobson ring and so σ^+ is a Jacobson ring congruence on S.

Suppose that θ is a Jacobson ring congruence on S. If $a, b \in S$ and $a\, \sigma^+ b$, then there exists $f \in E_+(S)$ such that $a + f = b + f$. This yields $\theta_{a+f} = \theta_{b+f}$. Thus

$$\theta_a = \theta_a + \theta_f = \theta_b + \theta_f = \theta_b$$

since $(S/\theta, +)$ a group and and θ_f is the identity of $(S/\theta, +)$. This implies $a\, \theta\, b$ and so $\sigma^+ \subseteq \theta$. This shows that σ^+ is the least Jacobson ring congruence on S. □

Now we are able to obtain the decomposition theorem of distributive lattice of Jacobson rings.

Theorem 2.1. *Suppose that S is a semiring. Then S is a distributive lattice of Jacobson rings if and only if S is (isomorphic to) the subdirect product of a distributive lattice and a Jacobson ring.*

Proof. Suppose that $a \in S$, $e \in E_+(S)$ and $a + e \in E_+(S)$. Thus there is $f \in E_+(S)$ such that $a + e = f$. This yields $a + e + f = e + f$, and (left-)multiplying it by a, we have $a^2 + a(e + f) = a(e + f)$, which implies

$$a^2 + a + a(e + f) = a + a(e + f). \tag{1}$$

Since the $\mathcal{H}^+$-class containing $a(e + f)$ is a Jacobson ring, by Theorem 1.1, there exists a positive integer k such that $k \cdot (a(e + f)) = 0_{a(e+f)}$. Adding $(k - 1) \cdot a(e + f)$ to the both sides of (1), we get $a^2 + a + k \cdot a(e + f) = a + k \cdot a(e + f)$. This implies $a^2 + a + 0_{a(e+f)} = a + 0_{a(e+f)}$. By Corollary 2.1(i), it follows that $a^2 + a = a$, and, multiplying it by a, we have $a^3 + a^2 = a^2$, which implies $a + a^3 + a^2 = a + a^2$. Thus $a^3 + a = a$. By induction, it can be easily shown that $a^m + a = a$ for any positive integer $m \geqslant 2$. Since the $\mathcal{H}^+$-class containing a is also a Jacobson ring, there exists a positive integer $l \geqslant 2$ such that $a^l = a$. Thus, $a = a^l + a = a + a$, i.e., $a \in E_+(S)$. Therefore the additive reduct $(S, +)$ is E-unitary. By Proposition 5.9.1 in [**5**] we have $\sigma^+ \cap \mathcal{H}^+ = 1_S$, which, by Lemma I.4.18 in [**8**], implies that S is the subdirect product of $S/\mathcal{H}^+$ and S/σ^+.

The converse is trivial. □

Let $F_1, \ldots, F_k$ be a fixed list of finite fields with different characteristics $p_1, ..., p_k$ and respective sizes $q_1 = {p_1}^{n_1}, \ldots, q_k = {p_k}^{n_k}$, for some positive integers $n_1, \ldots, n_k$. Let $c = p_1 \cdots p_k$, and let n be a positive integer such that $n-1$ is the least common multiple of $q_1 - 1, \ldots, q_k - 1$. It was proved in [**10**] that the semiring variety $\mathbf{V} = \mathbf{HSP}\{B_2, F_1, \ldots, F_k\}$ generated by two-element distributive lattice B_2 and finite fields $F_1, \ldots, F_k$ is determined by (SR1-4) and the following identities:

(DFSR1) $(c+1)\cdot x \approx x$;	(DFSR4) $x + c \cdot xy \approx x$;
(DFSR2) $x^n \approx x$;	(DFSR5) $xy \approx yx$;
(DFSR3) $c \cdot x^2 \approx c \cdot x$;	(DFSR6) $\dfrac{c}{p_i} \cdot x^{q_i} \approx \dfrac{c}{p_i} \cdot x$ $(1 \leqslant i \leqslant k)$.

Suppose that S is a semiring in $\mathbf{V}$. From (SR4) and (DFSR1) we have that the additive reduct $(S, +)$ is a commutative Clifford semigroup. It follows by Theorem 2.1 in [**10**] that S is isomorphic to the subdirect product of the distributive lattice $S/\mathcal{H}^+$ and Jacobson ring S/σ^+. Thus, by Theorem 2.1, S belongs to $\mathbf{JR} \circ \mathbf{D}$ and so $\mathbf{V} \subseteq \mathbf{JR} \circ \mathbf{D}$. This shows that the above theorem generalizes Theorem 2.1 in [**10**].

In the rest of this section we give a method to construct distributive lattices of Jacobson rings. Assume that $(D, +, \cdot)$ is a distributive lattice. Define a binary relation $\leqslant$ on D by

$$(\forall \alpha, \beta \in D)\, \alpha \leqslant \beta \Leftrightarrow \alpha = \alpha + \beta.$$

It is easy to check that $\leqslant$ is a partial order on D. For any α, $\beta \in D$ it is easy to see that $\alpha + \beta \leqslant \alpha$. Similarly, we have $\alpha + \beta \leqslant \beta$. It is well known that $\alpha \leqslant \alpha\beta$, $\beta \leqslant \alpha\beta$ and $\alpha + \beta \leqslant \alpha\beta$.

In order to discuss the structure of S, we have to recall the following concept from [**9**] and [**11**].

Let $\{(S_\alpha, +, \cdot) \mid \alpha \in D\}$ be a family of disjoint semirings $(S_\alpha, +, \cdot)$ which are indexed by a distributive lattice D together with a family of monomorphisms $\varphi_{\alpha,\beta} \colon S_\alpha \to S_\beta (\beta \leqslant \alpha)$ satisfying the following conditions: for any $\alpha, \beta, \gamma \in D$,

(i) $\varphi_{\alpha,\alpha} = 1_{S_\alpha}$;
(ii) If $\gamma \leqslant \beta \leqslant \alpha$, then $\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} = \varphi_{\alpha,\gamma}$;
(iii) If $\gamma \leqslant \alpha + \beta$ then
$S_\alpha\varphi_{\alpha,\gamma} + S_\beta\varphi_{\beta,\gamma} \subseteq S_{\alpha+\beta}\varphi_{\alpha+\beta,\gamma}$.
$S_\alpha\varphi_{\alpha,\gamma} \cdot S_\beta\varphi_{\beta,\gamma} \subseteq S_{\alpha\beta}\varphi_{\alpha\beta,\gamma}$.

On the set $S = \bigcup_{\alpha \in D} S_\alpha$ define addition and multiplication by

$$a + b = a\varphi_{\alpha,\alpha+\beta} + b\varphi_{\beta,\alpha+\beta},$$
$$a \cdot b = (a\varphi_{\alpha,\alpha+\beta} b\varphi_{\beta,\alpha+\beta})\varphi^{-1}_{\alpha\beta,\alpha+\beta},$$

for any $a \in S_\alpha$, $b \in S_\beta$. Then we can check that $(S, +, \cdot)$ is a semiring, denoted by $S = [D; S_\alpha, \varphi_{\alpha,\beta}]$. We call the constructed semiring $S = [D; S_\alpha, \varphi_{\alpha,\beta}]$ the *sturdy distributive lattice* of semirings S_α.

If all semirings S_α are in a class of semirings $\mathbf{C}$, we call $S = [D; S_\alpha, \varphi_{\alpha,\beta}]$ the sturdy distributive lattice of $\mathbf{C}$-semirings.

Theorem 2.2. *Suppose that S is a semiring. Then S is a distributive lattice of Jacobson rings if and only if S is a sturdy distributive lattice of Jacobson rings.*

Proof. Let a semiring S belong to $\mathbf{JR} \circ \mathbf{D}$. By Lemma 2.1, we can assume that S is a distributive lattice D of Jacobson rings R_α's, where $D \cong S/\mathcal{H}^+$ and each R_α is an $\mathcal{H}^+$-class of S. For convenience, for any $\alpha \in D$ we denote by 0_α the unique idempotent of abelian group $(R_\alpha, +)$. Thus, $E_+(S) = \{0_\alpha \mid \alpha \in D\}$. From Lemma 2.2 we have that σ^+ is the least Jacobson ring congruence on S, which means that $(S/\sigma^+, +, \cdot)$ is a Jacobson ring. By Theorem 2.1 it follows that S is isomorphic to the subdirect product of the distributive lattice $S/\mathcal{H}^+$ and Jacobson ring S/σ^+. This implies that the additive reduct $(S, +)$ of S is isomorphic to the subdirect product of the semilattice $(S/\mathcal{H}^+, +)$ and abelian group $(S/\sigma^+, +)$. Thus, $(S, +)$ is a sturdy semilattice $(D, +)$ of abelian groups $(R_\alpha, +)(\alpha \in D)$. Then, by Theorems IV.1.3, IV.1.6 and IV.1.7 in [**8**], we can express $(S, +) = [(D, +);\ (R_\alpha, +); \varphi_{\alpha,\beta}]$ as a sturdy semilattice of additive abelian groups $R_\alpha(\alpha \in D)$, where $(D, +)[(R_\alpha, +)]$ denotes the additive semigroup of distributive lattice D [of Jacobson rings R_α] and $\varphi_{\alpha,\beta}$ is defined by

$$(\forall a \in R_\alpha)\, a\varphi_{\alpha,\beta} = a + 0_\beta.$$

From $(S, +) = [(D, +); (R_\alpha, +); \varphi_{\alpha,\beta}]$ we have that $\varphi_{\alpha,\beta}(\beta \leqslant \alpha)$ is a group monomorphism from $(R_\alpha, +)$ to $(R_\beta, +)$. In the following, we are going to show that $\varphi_{\alpha,\beta}(\beta \leqslant \alpha)$ is a semiring homomorphism.

For $a, b \in R_\alpha$, we have $a\varphi_{\alpha,\beta} = a + 0_\beta$ and $b\varphi_{\alpha,\beta} = b + 0_\beta$. Then, by Corollary 2.1, we have

$$(ab)\varphi_{\alpha,\beta} = ab + 0_\beta = a + a0_\beta + b0_\beta + 0_\beta = (a + 0_\beta)(b + 0_\beta) = (a\varphi_{\alpha,\beta})(b\varphi_{\alpha,\beta}).$$

This shows that $\varphi_{\alpha,\beta}$ is a semigroup homomorphism from $(R_\alpha, \cdot)$ to $(R_\beta, \cdot)$ and so $\varphi_{\alpha,\beta}$ is a semiring monomorphism.

For any $\alpha, \beta \in D$, since R_α and R_β are $\mathcal{H}^+$-classes and $\mathcal{H}^+$ is a distributive lattice congruence, $R_\alpha \cdot R_\beta \subseteq R_{\alpha\beta}$. Thus, for any $a \in S_\alpha, b \in S_\beta$, we have

$$a\varphi_{\alpha,\alpha+\beta} = a + 0_{\alpha+\beta}, \quad b\varphi_{\beta,\alpha+\beta} = b + 0_{\alpha+\beta}, \quad (ab)\varphi_{\alpha\beta,\alpha+\beta} = ab + 0_{\alpha+\beta}.$$

By Corollary 2.1 we have

$$\begin{aligned} ab + 0_{\alpha+\beta} &= a + a0_{\alpha+\beta} + b0_{\alpha+\beta} + 0_{\alpha+\beta} \\ &= (a + 0_{\alpha+\beta})(b + 0_{\alpha+\beta}) = (a\varphi_{\alpha,\alpha+\beta})(b\varphi_{\beta,\alpha+\beta}). \end{aligned}$$

Thus, $(ab)\varphi_{\alpha\beta,\alpha+\beta} = (a\varphi_{\alpha,\alpha+\beta})(b\varphi_{\beta,\alpha+\beta})$.

Let $\gamma \in D$ and $\gamma \leqslant \alpha + \beta$. Since $\varphi_{\alpha+\beta,\gamma}$ is a semiring homomorphism, we have

$$\begin{aligned} a\varphi_{\alpha,\gamma} b\varphi_{\beta,\gamma} &= a\varphi_{\alpha,\alpha+\beta}\varphi_{\alpha+\beta,\gamma} b\varphi_{\beta,\alpha+\beta}\varphi_{\alpha+\beta,\gamma} \\ &= (a\varphi_{\alpha,\alpha+\beta} b\varphi_{\beta,\alpha+\beta})\varphi_{\alpha+\beta,\gamma} = (ab)\varphi_{\alpha\beta,\alpha+\beta}\varphi_{\alpha+\beta,\gamma} = (ab)\varphi_{\alpha\beta,\gamma}. \end{aligned}$$

This shows $R_\alpha\varphi_{\alpha,\gamma} \cdot R_\beta\varphi_{\beta,\gamma} \subseteq R_{\alpha\beta}\varphi_{\alpha\beta,\gamma}$. Hence,

$$ab = ((ab)\varphi_{\alpha\beta,\alpha+\beta})\varphi^{-1}_{\alpha\beta,\alpha+\beta} = (a\varphi_{\alpha,\alpha+\beta} b\varphi_{\beta,\alpha+\beta})\varphi^{-1}_{\alpha\beta,\alpha+\beta}.$$

Since $a + b = a\varphi_{\alpha,\alpha+\beta} + b\varphi_{\beta,\alpha+\beta}$ is evident, by the above definition, S is a sturdy distributive lattice D of Jacobson rings R_α's, where $D \cong S/\mathcal{H}^+$ and each R_α is a $\mathcal{H}^+$-class of semiring S.

Conversely, if the semiring S is a sturdy distributive lattice D of Jacobson rings R_α $(\alpha \in D)$, then $S = [D; R_\alpha, \varphi_{\alpha,\beta}]$. Define a binary relation η on S by

$$(a, b \in S)\, a\, \eta\, b \Leftrightarrow (\exists \alpha \in D)\, a, b \in R_\alpha.$$

It is a routine matter to verify that η is a distributive lattice congruence and that every η-class is a Jacobson ring. That is to say, $S \in \mathbf{JR} \circ \mathbf{D}$. $\square$

By Theorems 2.1 and 2.2 the following corollary is directly obtained.

COROLLARY 2.2. *Let S be a semiring. Then the following statements are equivalent:*

(i) *S is a distributive lattice of Jacobson rings;*
(ii) *S is the subdirect product of a distributive lattice and a Jacobson ring;*
(iii) *S is a sturdy distributive lattice of Jacobson rings.*

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(Received 31 08 2015)
(Revised 21 01 2016)

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