PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 100(114)) (2016), 251–257 DOI: 10.2298/PIM1614251N

HERMITE–HADAMARD INEQUALITIES FOR DIFFERENTIABLE p-CONVEX FUNCTIONS USING HYPERGEOMETRIC FUNCTIONS

Muhammad Aslam Noor, Muhammad Uzair Awan, Marcela V. Mihai, and Khalida Inayat Noor

Abstract. We derive some new integral identities for differentiable functions. Then using these auxiliary results, we obtain new Hermite–Hadamard type inequalities for differentiable p-convex functions. Some special cases are also discussed.

1. Introduction

Recently theory of convexity has received much attentions by many researchers. Consequently the classical concepts of convex sets and convex functions have been extended and generalized in several directions using novel and innovative ideas, see [**1**]. Zhang [**11**] introduced the notion of p-convex functions. It is worth to mention here that besides the classical convex functions, the class of p-convex functions also includes the class of harmonically convex functions introduced and studied by Iscan [**5**]. For some recent investigations on p-convex functions, see [**4**].

The interrelationship between theory of convex functions and theory of inequalities has attracted many researchers. One of the most extensively studied inequality for convex functions is the Hermite–Hadamard inequality. This inequality provides the necessary and sufficient condition for a function to be convex. For some recent investigation on Hermite–Hadamard type inequalities, see [**2**–**10**].

In this article, We consider the class of p-convex functions. We derive two new integral identities for differentiable functions. Using these results we establish our main results that are Hermite–Hadamard type inequalities for differentiable p-convex functions. We use hypergeometric functions to solve our integrals. It is expected that the ideas and techniques of this paper may stimulate further research in this area. This is the main motivation of this paper.

2010 *Mathematics Subject Classification:* 26D15; 26A51.

Key words and phrases: convex functions; p-convex functions; Hermite–Hadamard inequalities.

Communicated by Gradimir Milovanović.

2. Preliminaries and lemmas

In this section, we recall some previously known concepts and derive some new results which play an important role in the development of our main results.

Definition 2.1. [**11**] An interval I is said to be a p-convex set, if

$$M_p(x,y;t)=[tx^p+(1-t)y^p]^{\frac{1}{p}}\in I,$$

for all $x,y\in I, t\in[0,1]$, where $p=2k+1$ or $p=\frac{n}{m}, n=2r+1, m=2t+1$ and $k,r,t\in\mathbb{N}$.

Definition 2.2. [**11**] Let I be a p-convex set. A function $f\colon I\to\mathbb{R}$ is said to be p-convex function or belongs to the class $PC(I)$, if

$$f(M_p(x,y;t))\leqslant tf(x)+(1-t)f(y),\quad \forall x,y\in I,\ \ t\in[0,1].$$

It is obvious that for $p=1$, Definition 2.2 reduces to the definition for classical convex functions. Note that for $p=-1$, we have the definition of harmonically convex functions.

Definition 2.3. [**5**] A function $f\colon I\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ is said to be harmonically convex function, if

$$f\Big(\frac{xy}{(1-t)x+ty}\Big)\leqslant tf(x)+(1-t)f(y),\quad \forall x,y\in I,\ \ t\in[0,1].$$

Also note that for $t=\frac{1}{2}$ in Definition 2.2, we have Jensen p-convex functions or mid p-convex functions

$$f(M_p(x,y;1/2))\leqslant\frac{f(x)+f(y)}{2},\quad \forall x,y\in I,\ \ t\in[0,1].$$

Now we derive some new integral identities; I^0 will denote the interior of I.

Lemma 2.4. Let $f\colon I=[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on I^0 with $a<b$. If $f'\in L[a,b]$, then

$$\begin{aligned}\frac{f(a)+f(b)}{2}&-\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx\\&=\frac{b^p-a^p}{2p}\int_0^1 M_p^{-1}(a,b;t)(1-2t)f'(M_p(a,b;t))\,dt,\end{aligned}$$

where $M_p^{-1}(a,b;t)=[ta^p+(1-t)b^p]^{\frac{1}{p}-1}$.

Proof. It suffices to show that

$$\begin{aligned}\int_0^1 &M_p^{-1}(a,b;t)(1-2t)f'(M_p(a,b;t))\,dt\\&=\frac{f(a)+f(b)}{b^p-a^p}-\frac{2p}{b^p-a^p}\int_0^1 f([ta^p+(1-t)b^p]^{\frac{1}{p}})\,dt\\&=\frac{f(a)+f(b)}{b^p-a^p}-\frac{2p^2}{(b^p-a^p)^2}\int_a^b\frac{f(x)}{x^{1-p}}dx.\end{aligned}$$

Multiplying both sides of the above inequality by $\frac{b^p-a^p}{2p}$, we get the required result. □

Note that for $p=1$, Lemma 2.4 reduces to the following known integral identity by Dragomir et al. [**2**].

LEMMA 2.5. [**2**] Let $f\colon I=[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on I^0 with $a<b$. If $f'\in L[a,b]$, then, we have

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx=\frac{b-a}{2}\int_0^1(1-2t)f'(ta+(1-t)b)\,dt.$$

If $p=-1$, then Lemma 2.4 reduces to the following integral identity mainly due to Iscan [**5**].

LEMMA 2.6. [**5**] Let $f: I=[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on I^0 with $a<b$. If $f'\in L[a,b]$, then

$$\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\,dx$$
$$=\frac{ab(b-a)}{2}\int_0^1\frac{1-2t}{[tb+(1-t)a]^2}f'\Big(\frac{ab}{tb+(1-t)a}\Big)dt.$$

LEMMA 2.7. Let $f\colon I=[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on I^0 with $a<b$. If $f'\in L[a,b]$, then

$$\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx-f\Big(\Big[\frac{a^p+b^p}{2}\Big]^{\frac{1}{p}}\Big)$$
$$=\frac{b^p-a^p}{p}\int_0^1 M_p^{-1}(a,b;t)\vartheta(t)f'(M_p(a,b;t))\,dt,$$

where

$$\vartheta(t)=\begin{cases}t, & [0,\frac{1}{2}),\\ t-1, & [\frac{1}{2},1].\end{cases}$$

PROOF. A simple integration by parts completes the proof. □

For $p=1$, Lemma 2.7 reduces to Lemma 2.1 of [**6**]. For the reader's convenience we recall here the definitions of the Gamma and Beta functions

$$\Gamma(x)=\int_0^\infty e^{-x}t^{x-1}dt,\qquad \mathrm{B}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt.$$

It holds

$$\mathrm{B}(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

The integral form of the hypergeometric function is

$${}_2F_1(x,y;c;z)=\frac{1}{\mathrm{B}(y,c-y)}\int_0^1 t^{y-1}(1-t)^{c-y-1}(1-zt)^{-x}dt$$

for $|z|<1, c>y>0$.

3. Main results

In this section, we derive our main results.

THEOREM 3.1. *Let $f : I = [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I^0 with $a < b$ and $f' \in L[a, b]$. If $|f'|$ is p-convex function, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx\right| \leqslant b^{1-p}\cdot\frac{b^p-a^p}{2p}\{K_1|f'(a)| + K_2|f'(b)|\}.$$

where

$$K_1 = \frac{2}{3}\cdot {}_2F_1\Big(1-\frac{1}{p}, 3; 4; 1-\frac{a^p}{b^p}\Big) - \frac{1}{2}\cdot {}_2F_1\Big(1-\frac{1}{p}, 2; 3; 1-\frac{a^p}{b^p}\Big) + \frac{1}{12}\cdot {}_2F_1\Big(1-\frac{1}{p}, 2; 4; \frac{1}{2}\Big(1-\frac{a^p}{b^p}\Big)\Big), \tag{3.1}$$

$$K_2 = \frac{1}{3}\cdot {}_2F_1\Big(1-\frac{1}{p}, 2; 4; 1-\frac{a^p}{b^p}\Big) - \frac{1}{2}\cdot {}_2F_1\Big(1-\frac{1}{p}, 1; 3; 1-\frac{a^p}{b^p}\Big) + \frac{1}{2}\cdot {}_2F_1\Big(1-\frac{1}{p}, 1; 3; \frac{1}{2}\Big(1-\frac{a^p}{b^p}\Big)\Big) - \frac{1}{12}\cdot {}_2F_1\Big(1-\frac{1}{p}, 2; 4; \frac{1}{2}\Big(1-\frac{a^p}{b^p}\Big)\Big). \tag{3.2}$$

PROOF. Using Lemma 2.4 and the fact that $|f'|$ is a p-convex function, we have

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx\right| \\ &= \left|\frac{b^p-a^p}{2p}\int_0^1 M_p^{-1}(a,b;t)(1-2t)f'(M_p(a,b;t))\,dt\right| \\ &\leqslant \frac{b^p-a^p}{2p}\int_0^1 \frac{|1-2t|}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}\left|f'([ta^p+(1-t)b^p]^{\frac{1}{p}})\right|dt \\ &\leqslant \frac{b^p-a^p}{2p}\int_0^1 \frac{|1-2t|}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}\big[t|f'(a)|+(1-t)|f'(b)|\big]dt \\ &= \frac{b^p-a^p}{2p}(|f'(a)|I_1 + |f'(b)|I_2), \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} I_1 &= \int_0^1 \frac{|1-2t|t}{[ta^p+(1-t)b^p]^{1-1/p}}dt = b^{1-p}\Big[\frac{2}{3}\cdot {}_2F_1\Big(1-\frac{1}{p}, 3; 4; 1-\frac{a^p}{b^p}\Big) \\ &\quad - \frac{1}{2}\cdot {}_2F_1\Big(1-\frac{1}{p}, 2; 3; 1-\frac{a^p}{b^p}\Big) + \frac{1}{12}\cdot {}_2F_1\Big(1-\frac{1}{p}, 2; 4; \frac{1}{2}\Big(1-\frac{a^p}{b^p}\Big)\Big)\Big], \end{aligned} \tag{3.4}$$

$$\begin{aligned} I_2 &= \int_0^1 \frac{|1-2t|(1-t)}{[ta^p+(1-t)b^p]^{1-1/p}}dt \\ &= b^{1-p}\Big[\frac{1}{3}\cdot {}_2F_1\Big(1-\frac{1}{p}, 2; 4; 1-\frac{a^p}{b^p}\Big) - \frac{1}{2}\cdot {}_2F_1\Big(1-\frac{1}{p}, 1; 3; 1-\frac{a^p}{b^p}\Big) \\ &\quad + \frac{1}{2}\cdot {}_2F_1\Big(1-\frac{1}{p}, 1; 3; \frac{1}{2}\Big(1-\frac{a^p}{b^p}\Big)\Big) - \frac{1}{22}\cdot {}_2F_1\Big(1-\frac{1}{p}, 2; 4; \frac{1}{2}\Big(1-\frac{a^p}{b^p}\Big)\Big)\Big]. \end{aligned} \tag{3.5}$$

Introducing relations (3.4) and (3.5) in (3.3) completes the proof. □

THEOREM 3.2. *Let $f\colon I=[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on I^0 with $a<b$ and $f'\in L[a,b]$. If $|f'|^q$ is p-convex function where $q\geqslant 1$, then*

$$\Big|\frac{f(a)+f(b)}{2}-\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx\Big|\\ \leqslant b^{1-p}\cdot\frac{b^p-a^p}{2p}H^{1-\frac{1}{q}}\big\{K_1|f'(a)|^q+K_2|f'(b)|^q\big\}^{\frac{1}{q}}.$$

where K_1, K_2 are given by (3.1) *and* (3.2) *and*

$$H={}_2F_1\Big(1-\frac{1}{p},2;3;1-\frac{a^p}{b^p}\Big)-{}_2F_1\Big(1-\frac{1}{p},1;2;1-\frac{a^p}{b^p}\Big)\\ +{}_2F_1\Big(1-\frac{1}{p},1;3;\frac{1}{2}\Big(1-\frac{a^p}{b^p}\Big)\Big).$$

PROOF. Using Lemma 2.4, the fact that $|f'|$ is a p-convex function and power mean's inequality, we have

$$\Big|\frac{f(a)+f(b)}{2}-\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx\Big|\\ =\Big|\frac{b^p-a^p}{2p}\int_0^1 M_p^{-1}(a,b;t)(1-2t)f'(M_p(a,b;t))\,dt\Big|\\ \leqslant\frac{b^p-a^p}{2p}\Big(\int_0^1\frac{|1-2t|}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt\Big)^{1-\frac{1}{q}}\\ \times\Big(\int_0^1\frac{|1-2t|}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}|f'([ta^p+(1-t)b^p]^{\frac{1}{p}})|^q dt\Big)^{\frac{1}{q}}\\ \leqslant\frac{b^p-a^p}{2p}\Big(\int_0^1\frac{|1-2t|}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt\Big)^{1-\frac{1}{q}}\\ \times\Big(\int_0^1\frac{|1-2t|}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}[t|f'(a)|^q+(1-t)|f'(b)|^q]dt\Big)^{\frac{1}{q}}\\ =b^{1-p}\cdot\frac{b^p-a^p}{2p}H^{1-\frac{1}{q}}\big\{K_1|f'(a)|^q+K_2\||f'(b)|^q\big\}^{\frac{1}{q}}.$$ □

THEOREM 3.3. *Let $f\colon I=[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on I^0 with $a<b$ and $f'\in L[a,b]$. If $|f'|$ is p-convex function, then*

$$\Big|\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx-f\Big(\Big[\frac{a^p+b^p}{2}\Big]^{\frac{1}{p}}\Big)\Big|\\ \leqslant\frac{b^{1-p}(b^p-a^p)}{p}\big[\{C_1+C_2-C_3\}|f'(a)|+\{C_4+C_5-C_6-C_7\}|f'(b)|\big],$$

where

$$C_1=\frac{1}{6}{}_2F_1\Big(1-\frac{1}{p},2;4;1-\frac{a^p}{b^p}\Big),$$

$$C_2 = \frac{1}{12}{}_2F_1\Big(1-\frac{1}{p},3;4;\frac{1}{2}\Big(1-\frac{a^p}{b^p}\Big)\Big),$$
$$C_3 = \frac{1}{48}{}_2F_1\Big(1-\frac{1}{p},2;3;\frac{1}{2}\Big(1-\frac{a^p}{b^p}\Big)\Big),$$
$$C_4 = \frac{1}{3}{}_2F_1\Big(1-\frac{1}{p},1;4;1-\frac{a^p}{b^p}\Big),$$
$$C_5 = \frac{3}{8}{}_2F_1\Big(1-\frac{1}{p},2;3;\frac{1}{2}\Big(1-\frac{a^p}{b^p}\Big)\Big),$$
$$C_6 = \frac{1}{12}{}_2F_1\Big(1-\frac{1}{p},3;4;\frac{1}{2}\Big(1-\frac{a^p}{b^p}\Big)\Big),$$
$$C_7 = \frac{1}{2}{}_2F_1\Big(1-\frac{1}{p},1;2;\frac{1}{2}\Big(1-\frac{a^p}{b^p}\Big)\Big).$$

PROOF. Using Lemma 2.7 and the fact that $|f'|$ is a p-convex function, we have

$$\begin{aligned}
&\left|\frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx - f\Big(\Big[\frac{a^p+b^p}{2}\Big]^{\frac{1}{p}}\Big)\right| \\
&= \bigg|\frac{b^p-a^p}{p}\int_0^{\frac{1}{2}} tM_p^{-1}(a,b;t)f'(M_p(a,b;t))\,dt \\
&\quad + \frac{b^p-a^p}{p}\int_{\frac{1}{2}}^1 (t-1)M_p^{-1}(a,b;t)f'(M_p(a,b;t))\,dt\bigg| \\
&\leqslant \frac{b^p-a^p}{p}\bigg[\int_0^{\frac{1}{2}} tM_p^{-1}(a,b;t)|f'(M_p(a,b;t))|dt \\
&\quad + \int_{\frac{1}{2}}^1 |t-1|M_p^{-1}(a,b;t)|f'(M_p(a,b;t))|dt\bigg] \\
&\leqslant \frac{b^{1-p}(b^p-a^p)}{p}\bigg[\bigg\{\frac{1}{6}{}_2F_1\Big(1-\frac{1}{p},2;4;1-\frac{a^p}{b^p}\Big) \\
&\quad + \frac{1}{12}{}_2F_1\Big(1-\frac{1}{p},3;4;\frac{1}{2}\Big(1-\frac{a^p}{b^p}\Big)\Big) \\
&\quad - \frac{1}{48}{}_2F_1\Big(1-\frac{1}{p},2;3;\frac{1}{2}\Big(1-\frac{a^p}{b^p}\Big)\Big)\bigg\}|f'(a)| \\
&\quad + \bigg\{\frac{1}{3}{}_2F_1\Big(1-\frac{1}{p},1;4;1-\frac{a^p}{b^p}\Big) + \frac{3}{8}{}_2F_1\Big(1-\frac{1}{p},2;3;\frac{1}{2}\Big(1-\frac{a^p}{b^p}\Big)\Big) \\
&\quad - \frac{1}{12}{}_2F_1\Big(1-\frac{1}{p},3;4;\frac{1}{2}\Big(1-\frac{a^p}{b^p}\Big)\Big) \\
&\quad - \frac{1}{2}{}_2F_1\Big(1-\frac{1}{p},1;2;\frac{1}{2}\Big(1-\frac{a^p}{b^p}\Big)\Big)\bigg\}|f'(b)|\bigg].
\end{aligned}$$

This completes the proof. □

Acknowledgement. The authors would like to thank anonymous referee for his/her valuable comments and suggestions.

References

1. G. Cristescu, L. Lupsa, *Non-connected Convexities and Applications*, Kluwer, Dordrecht, 2002.
2. S. S. Dragomir, R. P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. Lett. **11** (1998), 91–95.
3. S. S. Dragomir, C. E. M. Pearce, *Selected topics on Hermite–Hadamard inequalities and applications*, Victoria University, 2000.
4. Z. B. Fang, R. Shi, *On the (p,h)-convex function and some integral inequalities*, J. Inequal. Appl. **2014** (2014), Article ID 45, 16 p.
5. I. Iscan, *Hermite–Hadamard type inequalities for harmonically convex functions*, Hacettepe J. Math. Stat. **43**(6) (2014), 935–942.
6. U. S. Kirmaci, *Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula*, Appl. Math. Comput. **147**, (2004), 137–146.
7. M. V. Mihai, *New Hermite–Hadamard type inequalities obtained via Riemann-Liouville fractional calculus*, An. Univ. Oradea, Fasc. Mat. **XX**(2) (2013), 127–132.
8. ———, *New inequalities for co-ordinated convex functions via riemann-liouville fractional calculus*, Tamkang J. Math. **45**(3) (2014), 285–296.
9. M. V. Mihai, F.-C. Mitroi, *Hermite–Hadamard type inequalities obtained via Riemann-Liouville fractional calculus*, Acta Math. Univ. Comen., New Ser. **83**(2) (2014), 209–215.
10. M. A. Noor, G. Cristescu, M. U. Awan, *Generalized fractional Hermite–Hadamard inequalities for twice differentiable s-convex functions*, Filomat **29**(4) (2015), 807–815.
11. K. S. Zhang, J. P. Wan, *p-convex functions and their properties.* Pure Appl. Math. **23**(1) (2007), 130–133.

Department of Mathematics
COMSATS Institute of Information Technology
Islamabad, Pakistan
noormaslam@gmail.com
khalidanoor@hotmail.com

(Received 07 12 2014)

Department of Mathematics
Government College University
Faisalabad, Pakistan
awan.uzair@gmail.com

Department scientific-methodical sessions
Romanian Mathematical Society-branch Bucharest
Bucharest, Romania
marcelamihai58@yahoo.com