

## WEIGHTED BOUNDEDNESS FOR COMMUTATORS OF PARAMETERIZED LITTLEWOOD–PALEY OPERATORS AND AREA INTEGRALS

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ABSTRACT. We establish the boundedness for commutators of parameterized Littlewood–Paley operators and area integrals on weighted Lebesgue spaces  $L^p(\omega)$  when  $1 < p < \infty$ , where the kernel satisfies certain logarithmic type Lipschitz condition. Moreover, the weighted endpoint estimates when  $p = 1$  are also obtained.

### 1. Introduction

Suppose that  $S^{n-1}$  is the unit sphere of  $\mathbb{R}^n (n \geq 2)$  equipped with normalized Lebesgue measure. Let  $\Omega$  be a homogeneous function of degree zero and

$$(1.1) \quad \int_{S^{n-1}} \Omega(x') d\sigma(x') = 0.$$

The parameterized area integral  $\mu_{\Omega,S}^\rho$  and parameterized Littlewood–Paley operator  $\mu_\lambda^{*,\rho}$  are defined by

$$\mu_{\Omega,S}^\rho(f)(x) = \left( \iint_{\Gamma(x)} \left| \frac{1}{t^\rho} \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} f(z) dz \right|^2 \frac{dy dt}{t^{n+1}} \right)^{\frac{1}{2}},$$

$$\mu_\lambda^{*,\rho}(f)(x) = \left( \iint_{\mathbb{R}_+^{n+1}} \left( \frac{t}{t+|x-y|} \right)^{\lambda n} \left| \frac{1}{t^\rho} \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} f(z) dz \right|^2 \frac{dy dt}{t^{n+1}} \right)^{\frac{1}{2}},$$

respectively, where  $\rho > 0$ ,  $\lambda > 1$  and  $\Gamma(x) = \{(t, y) \in \mathbb{R}_+^{n+1} : |x - y| < t\}$ .

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Define the Hilbert spaces as follows.

$$\mathcal{H}_1 = \left\{ h : \|h\|_{\mathcal{H}_1} = \left( \int_0^\infty \int_{|y|<1} |h(t,y)|^2 \frac{dy dt}{t} \right)^{\frac{1}{2}} < \infty \right\},$$

$$\mathcal{H}_2 = \left\{ h : \|h\|_{\mathcal{H}_2} = \left( \int_0^\infty \int_{\mathbb{R}^n} \left( \frac{1}{1+|y|} \right)^{\lambda n} |h(t,y)|^2 \frac{dy dt}{t} \right)^{\frac{1}{2}} < \infty \right\},$$

where  $\lambda > 1$ . Then

$$\mu_{\Omega,S}^\rho(f)(x) = \|\phi_{t,y}(f)(x)\|_{\mathcal{H}_1}, \quad \mu_\lambda^{*,\rho}(f)(x) = \|\phi_{t,y}(f)(x)\|_{\mathcal{H}_2},$$

where  $\phi(x) = \frac{\Omega(x)}{|x|^{n-\rho}} \chi_{\{|x|<1\}}$ , and  $\phi_{t,y}(f)(x) = \int t^{-n} \phi\left(\frac{x-z}{t} - y\right) f(z) dz$ .

The commutators of  $\mu_{\Omega,S}^\rho$  and  $\mu_\lambda^{*,\rho}$  are defined by

$$\mu_{\Omega,S}^{\rho,b}(f)(x) = \left( \iint_{\Gamma(x)} \left| \frac{1}{t^\rho} \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} (b(x) - b(z)) f(z) dz \right|^2 \frac{dy dt}{t^{n+1}} \right)^{\frac{1}{2}},$$

$$\mu_{\lambda,b}^{*,\rho}(f)(x) = \left( \iint_{\mathbb{R}_+^{n+1}} \left( \frac{t}{t+|x-y|} \right)^{\lambda n} \right. \\ \left. \times \left| \frac{1}{t^\rho} \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} (b(x) - b(z)) f(z) dz \right|^2 \frac{dy dt}{t^{n+1}} \right)^{\frac{1}{2}},$$

respectively.

Denote  $f_Q = \frac{1}{|Q|} \int_Q f(y) dy$ . For  $\delta > 0$ , we define

$$M_\delta(f) = [M(|f|^\delta)]^{\frac{1}{\delta}}, \quad M_\delta^\sharp(f) = [M^\sharp(|f|^\delta)]^{\frac{1}{\delta}},$$

where  $M$  is the Hardy–Littlewood maximal operator and  $M^\sharp$  is the Fefferman–Stein sharp function. The corresponding dyadic maximal operators are denoted by  $M_\delta^\Delta$  and  $M_\delta^{\sharp,\Delta}$ , respectively.

A function  $A : [0, \infty) \rightarrow [0, \infty)$  is said to be a Young function if it is continuous, convex and increasing satisfying  $A(0) = 0$  and  $A(t) \rightarrow \infty$  as  $t \rightarrow \infty$ . The complementary Young function  $\bar{A}(t)$  of the Young function  $A(t)$  is defined by

$$\bar{A}(s) = \sup_{0 \leq t < \infty} [st - A(t)], \quad 0 \leq s < \infty.$$

As an example,  $\Phi_m(t) = t(1 + \log^+ t)^m, 1 \leq m < \infty$ , is a Young function with its complementary  $\bar{\Phi}_m(t) \approx e^{t^{1/m}}$ . If  $A$  is a Young function, then the Luxembury norm of  $f$  on a cube  $Q \subset \mathbb{R}^n$  is defined by

$$\|f\|_{A,Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q A\left(\frac{|f(y)|}{\lambda}\right) dy \leq 1 \right\}.$$

If  $A(t) = \Phi_1(t)$ , we denote

$$\|f\|_{L \log L, Q} = \|f\|_{\Phi_1, Q}, \quad \|f\|_{expL, Q} = \|f\|_{\bar{\Phi}_1, Q}, \quad M_{L \log L} f(x) = \sup_{Q \ni x} \|f\|_{L \log L, Q}.$$

For the Luxembury norm, there is the following generalized Hölder inequality.

$$\frac{1}{|Q|} \int_Q |f(y)g(y)|dy \leq \|f\|_{A,Q} \|g\|_{\bar{A},Q}.$$

Let us recall the definition of  $A_p$  weight class. A locally integrable nonnegative function  $\omega$  is said to belong to  $A_p(1 < p < \infty)$ , if there is a constant  $C > 0$  such that

$$\sup_{Q \subset \mathbb{R}^n} \left( \frac{1}{|Q|} \int_Q \omega(x) dx \right) \left( \frac{1}{|Q|} \int_Q \omega(x)^{-1/(p-1)} dx \right)^{p-1} \leq C,$$

where  $Q$  denotes a cube in  $\mathbb{R}^n$ . The smallest constant  $C$  such that the above inequality holds is called the  $A_p$  constant of  $\omega$  and denoted by  $[\omega]_{A_p}$ . A weight  $\omega$  is said to be in the class  $A_1$  if there is a positive constant  $C$  such that  $M\omega(x) \leq C\omega(x)$ , a.e.  $x \in \mathbb{R}^n$ . We denote by  $[\omega]_{A_1}$  the infimum of all these  $C$ . A weight  $\omega$  is in the class  $A_\infty$  if there are positive constants  $C, \epsilon$  such that

$$\frac{\omega(E)}{\omega(Q)} \leq C \left( \frac{|E|}{|Q|} \right)^\epsilon,$$

for all cubes  $Q$  and all measurable sets  $E \subset Q$ . We denote by  $[\omega]_{A_\infty}$  the infimum of all these  $C$ .

Inspired by Hörmander’s work [7] on the parameterized Marcinkiewicz integral, the parameterized Littlewood–Paley  $g_\lambda^*$  function  $\mu_\lambda^{*,\rho}$  and parameterized area integral  $\mu_{\Omega,S}^\rho$  were discussed by Sakamoto and Yabuta [12] in 1999. In [12], the authors studied the  $L^p(1 < p < \infty)$  boundedness with kernel satisfying the  $Lip_\alpha$  condition. In 2002, Ding, Lu and Yabuta [2] proved the  $L^p(2 \leq p < \infty)$  boundedness of  $\mu_\lambda^{*,\rho}$  and  $\mu_{\Omega,S}^\rho$  with kernel satisfying a weaker  $Llog^+L(S^{n-1})$  condition.

Torchinsky and Wang [14] considered the weighted  $L^p$  boundedness of  $\mu_\lambda^{*,\rho}$  and  $\mu_{\Omega,S}^\rho$  with kernel satisfying the  $Lip_\alpha$  condition. In 1999, Ding, Fan and Pan [1] improved Torchinsky and Wang’s result in [14] and gave the weighted  $L^p$  boundedness of  $\mu_\lambda^{*,\rho}$  and  $\mu_{\Omega,S}^\rho$  when  $\Omega \in L^q(S^{n-1})(q > 1)$ . In 2002, Duoandikoetxea and Seijo [5] studied the weighed  $L^p$  boundedness of  $\mu_\lambda^{*,\rho}$  and  $\mu_{\Omega,S}^\rho$  with rough kernel. In 2004, Xue [15] proved the weighed  $L^p$  boundedness of  $\mu_\lambda^{*,\rho}$  and  $\mu_{\Omega,S}^\rho$  with  $\Omega$  satisfying the  $L^2$ -Dini condition.

Lee and Rim [8], in 2004, established the logarithmic type Lipschitz condition

$$(1.2) \quad |\Omega(y_1) - \Omega(y_2)| \leq \frac{C}{\left(\log \frac{2}{|y_2 - y_1|}\right)^\alpha},$$

for any  $y_1, y_2 \in S^{n-1}$ , where  $\alpha > 1$ , and proved the type  $(L^\infty, BMO)$  and  $(L^p, L^p)$  boundedness of the Marcinkiewicz integral with kernel satisfying the condition (1.2). In 2012, Lin, Liu and Gao [9] gave the endpoint estimate of  $\mu_\lambda^{*,\rho}$ .

**THEOREM 1.1.** [9] *Let  $n \geq 2$ ,  $\Omega \in L^2(S^{n-1})$  be a homogeneous function of degree zero satisfying (1.1) and (1.2) with  $\alpha > \frac{3}{2}$ , then for  $\rho > n/2$ ,  $\lambda > 2$ , there exists a constant  $C > 0$ , such that for all  $\beta > 0$  and  $f \in L^1(\mathbb{R}^n)$ ,*

$$|\{x \in \mathbb{R}^n : |\mu_\lambda^{*,\rho}(f)(x)| > \beta\}| \leq C \|f\|_{L^1} / \beta.$$

In 2013, the authors in [10] discussed the operators  $\mu_\lambda^{*,\rho}$  and  $\mu_{\Omega,S}^\rho$  with kernel satisfying (1.2) on weak Hardy spaces. Recently the authors in [11] gave the weighted  $L^p$  boundedness of  $\mu_{\Omega,S}^\rho$  and  $\mu_\lambda^{*,\rho}$  with kernel satisfying (1.2).

**THEOREM 1.2.** [11] *Let  $\Omega \in L^2(S^{n-1})$  be a homogeneous function of degree zero satisfying (1.1) and (1.2) with  $\alpha > \frac{3}{2}$ . Suppose  $\omega \in A_p$ , then for  $\rho > n/2$ ,  $\lambda > 2$  and  $1 < p < \infty$ ,*

$$\|\mu_{\Omega,S}^\rho(f)\|_{p,\omega} \leq C\|f\|_{p,\omega}, \quad \|\mu_\lambda^{*,\rho}(f)\|_{p,\omega} \leq C\|f\|_{p,\omega}.$$

On the other hand, the boundedness of the commutator has also received increasing attentions. Torchinsky and Wang [14] in 1990 proved that if  $b \in BMO$ , then the commutator of the Marcinkiewicz integral  $[b, \mu_\Omega]$  is bounded on weighted spaces  $L^p(\omega)$  for  $1 < p < \infty$  and  $\omega \in A_p$ . In 2002, Ding, Lu and Yabuta [2] gave the weighted  $L^p$  boundedness of the higher order commutator  $\mu_{\Omega,b}^m$  for rough Marcinkiewicz integral. In 2004, Ding, Lu and Zhang [3] gave the endpoint weighted estimates for the higher order commutator  $\mu_{\Omega,b}^m$ . In 2007, Ding and Xue [4] gave the weighted boundedness and the weak  $L \log L$  estimates for the commutators of parameterized Littlewood–Paley operators and area integrals with kernel satisfying the  $L^2$ -Dini condition.

Inspired by the above results, in this paper, we will focus on the weighted  $L^p$  ( $1 < p < \infty$ ) boundedness and the weighted endpoint estimates ( $p = 1$ ) for the commutators  $\mu_{\Omega,S}^{\rho,b}$  and  $\mu_{\lambda,b}^{*,\rho}$ , where the kernel satisfies the logarithmic type Lipschitz condition (1.2).

## 2. Main results

Now, we state our main results as follows.

**THEOREM 2.1.** *Suppose that  $\rho > n/2$ ,  $\lambda > 2$ ,  $\Omega \in L^2(S^{n-1})$  is a homogeneous function of degree zero satisfying (1.1) and (1.2) with  $\alpha > \frac{3}{2}$ . If  $b \in BMO$ , then for  $1 < p < \infty$  and  $\omega \in A_p$ , there exists a constant  $C > 0$  such that for any  $f \in L^p(\omega)$ ,*

$$\|\mu_{\Omega,S}^{\rho,b}(f)\|_{p,\omega} \leq C\|f\|_{p,\omega}, \quad \|\mu_{\lambda,b}^{*,\rho}(f)\|_{p,\omega} \leq C\|f\|_{p,\omega}.$$

**THEOREM 2.2.** *Suppose that  $\rho > n/2$ ,  $\lambda > 2$ ,  $\Omega \in L^2(S^{n-1})$  is a homogeneous function of degree zero satisfying (1.1) and (1.2) with  $\alpha > \frac{5}{2}$ . If  $b \in BMO$  and  $\omega \in A_1$ , then there exists a constant  $C > 0$  such that for any  $\beta > 0$  and each smooth function  $f$  with compact support, the following inequalities hold*

$$\omega(\{x \in \mathbb{R}^n : \mu_{\Omega,S}^{\rho,b}(f)(x) > \beta\}) \leq C \int_{\mathbb{R}^n} \frac{|f(x)|}{\beta} \left(1 + \log^+ \frac{|f(x)|}{\beta}\right) \omega(x) dx,$$

$$\omega(\{x \in \mathbb{R}^n : \mu_{\lambda,b}^{*,\rho}(f)(x) > \beta\}) \leq C \int_{\mathbb{R}^n} \frac{|f(x)|}{\beta} \left(1 + \log^+ \frac{|f(x)|}{\beta}\right) \omega(x) dx.$$

### 3. Some lemmas

In order to prove the main results, we need the following necessary lemmas.

LEMMA 3.1. [16] *Suppose  $f \in BMO$ . There exist constants  $C_1, C_2 > 0$ , depending only on the dimension  $n$ , such that for  $0 < C < \frac{C_2}{\|f\|_*}$ , every cube  $Q$  in  $\mathbb{R}^n$ , we have*

$$\int_Q e^{C|f(x)-f_Q|} dx \leq C_1 C \left( \frac{C_2}{\|f\|_*} - C \right)^{-1} |Q|.$$

LEMMA 3.2. *Let  $1 < p < \infty$  and  $\lambda' > 0$ . Then, when  $b(x) \in BMO$  with  $\|b\|_* < \min\{\frac{C_2}{\lambda'}, \frac{C_2(p-1)}{\lambda'}\}$ , where  $C_2$  is the constant in Lemma 3.1, we have  $e^{\lambda' b(x)} \in A_p$ .*

PROOF. We have

$$\begin{aligned} & \left( \frac{1}{|Q|} \int_Q e^{\lambda' b(x)} dx \right) \left( \frac{1}{|Q|} \int_Q \left( e^{\lambda' b(x)} \right)^{-\frac{1}{p-1}} dx \right)^{p-1} \\ &= \left( \frac{1}{|Q|} \int_Q e^{\lambda'(b(x)-b_Q)} dx \right) \left( \frac{1}{|Q|} \int_Q \left( e^{-\lambda'(b(x)-b_Q)} \right)^{\frac{1}{p-1}} dx \right)^{p-1} \\ &\leq \left( \frac{1}{|Q|} \int_Q e^{\lambda'|b(x)-b_Q|} dx \right) \left( \frac{1}{|Q|} \int_Q e^{\frac{\lambda'}{p-1}|b(x)-b_Q|} dx \right)^{p-1} := I_Q. \end{aligned}$$

Let  $\lambda_0 = \frac{\lambda'}{p-1}$ . If  $1 < p < 2$ , then  $\lambda_0 > \lambda'$ . By taking  $C = \lambda_0$  in Lemma 3.1, we have

$$\begin{aligned} I_Q &\leq \left( \frac{1}{|Q|} \int_Q e^{\lambda_0|b(x)-b_Q|} dx \right) \left( \frac{1}{|Q|} \int_Q e^{\lambda_0|b(x)-b_Q|} dx \right)^{p-1} \\ &= \left( \frac{1}{|Q|} \int_Q e^{\lambda_0|b(x)-b_Q|} dx \right)^p \leq \left( \frac{C_1 \lambda_0}{\frac{C_2}{\|b\|_*} - \lambda_0} \right)^p. \end{aligned}$$

If  $p \geq 2$ , then  $\lambda_0 \leq \lambda'$ . By taking  $C = \lambda'$  in Lemma 3.1, we have

$$\begin{aligned} I_Q &\leq \left( \frac{1}{|Q|} \int_Q e^{\lambda'|b(x)-b_Q|} dx \right) \left( \frac{1}{|Q|} \int_Q e^{\lambda'|b(x)-b_Q|} dx \right)^{p-1} \\ &= \left( \frac{1}{|Q|} \int_Q e^{\lambda'|b(x)-b_Q|} dx \right)^p \leq \left( \frac{C_1 \lambda'}{\frac{C_2}{\|b\|_*} - \lambda'} \right)^p. \end{aligned}$$

The two cases imply the desired result.  $\square$

REMARK 3.1. It follows from Lemma 3.2 that if  $1 < p < \infty$ ,  $\lambda' > 0$  and  $a(x), b(x) \in BMO$  with  $\|a\|_* \leq \|b\|_* < \min\{\frac{C_2}{\lambda'}, \frac{C_2(p-1)}{\lambda'}\}$ , then  $e^{\lambda' a(x)} \in A_p$  and the  $A_p$  constant of  $e^{\lambda' a(x)}$  satisfies

$$\left[ e^{\lambda' a(x)} \right]_{A_p} \leq \begin{cases} \left( \frac{C_1 \lambda_0}{\frac{C_2}{\|b\|_*} - \lambda_0} \right)^p, & 1 < p < 2, \\ \left( \frac{C_1 \lambda'}{\frac{C_2}{\|b\|_*} - \lambda'} \right)^p, & 2 \leq p < \infty. \end{cases}$$

LEMMA 3.3. *Suppose  $b \in BMO$ . There is a positive constant  $C$  such that for all ball  $B \subset \mathbb{R}^n$ ,*

$$\|b - b_B\|_{expL, B} \leq C \|b\|_*.$$

PROOF. When  $C > \frac{1}{C_2}$ , by Lemma 3.1, we have

$$\frac{1}{|B|} \int_B e^{\frac{|b(x)-b_B|}{C\|b\|_*}} dx \leq C_1 \frac{1}{C\|b\|_*} \left( \frac{C_2}{\|b\|_*} - \frac{1}{C\|b\|_*} \right)^{-1} = \frac{C_1}{C_2 C - 1}.$$

Taking  $C \geq \frac{C_1+1}{C_2}$ , we have  $\frac{C_1}{C_2 C - 1} \leq 1$ , then  $\frac{1}{|B|} \int_B e^{\frac{|b(x)-b_B|}{C\|b\|_*}} dx \leq 1$ . By the definition of  $\|b - b_B\|_{expL, B}$ , there is  $\|b - b_B\|_{expL, B} \leq C \|b\|_*$ .  $\square$

LEMMA 3.4. [6] *Let  $0 < r < l < \infty$ . For each function  $f$ , define*

$$\|f\|_{WL^l} = \sup_{t>0} t |\{x : |f(x)| > t\}|^{\frac{1}{l}}, N_{l,r}(f) = \sup_E \frac{\|f\chi_E\|_r}{\|\chi_E\|_s}, \frac{1}{s} = \frac{1}{r} - \frac{1}{l},$$

where the supremum is taken over all the measurable sets  $E$  with  $0 < |E| < \infty$ . Then

$$\|f\|_{WL^l} \leq N_{l,r}(f) \leq \left( \frac{l}{l-r} \right)^{\frac{1}{r}} \|f\|_{WL^l}.$$

LEMMA 3.5. [3] (1) *Let  $M^\Delta, M^{\sharp, \Delta}$  be the dyadic Hardy–Littlewood maximal operator and the dyadic sharp function, respectively. If  $\omega \in A_\infty$ , then there exists a positive dimensional constant  $C$  for which the following good- $\lambda$  inequality holds. For all  $\lambda, \varepsilon > 0$ ,*

$$\begin{aligned} &\omega(\{y \in \mathbb{R}^n : M^\Delta f(y) > \lambda, M^{\sharp, \Delta} f(y) \leq \varepsilon \lambda\}) \\ &\leq C [\omega]_{A_\infty} \varepsilon \omega(\{y \in \mathbb{R}^n : M^\Delta f(y) > \lambda/2\}). \end{aligned}$$

(2) *If  $\varphi : (0, \infty) \rightarrow (0, \infty)$  is a doubling function and  $\delta > 0$ , then there exists a positive constant  $C$  such that*

$$\sup_{\lambda>0} \varphi(\lambda) \omega(\{y \in \mathbb{R}^n : M_\delta^\Delta f(y) > \lambda\}) \leq C [\omega]_{A_\infty} \sup_{\lambda>0} \varphi(\lambda) \omega(\{y \in \mathbb{R}^n : M_\delta^{\sharp, \Delta} f(y) > \lambda\})$$

for all functions  $f$  such that the left side is finite.

LEMMA 3.6. [3] *There exists a constant  $C > 0$  such that for any weight  $\omega$  and all  $\beta > 0$ ,*

$$\omega(\{y : M^{m+1} f(y) > \beta\}) \leq C \int_{\mathbb{R}^n} \frac{|f(y)|}{\beta} \left( 1 + \log^+ \frac{|f(y)|}{\beta} \right)^m M\omega(y) dy$$

for every locally integrable function  $f$ .

LEMMA 3.7. *As for  $|y - z| \geq 4r$ , there is*

$$\int_{|y-z|}^\infty \frac{(\log \frac{t}{r})^{4+2\epsilon}}{t^{2\rho-n+1}} dt \leq C \frac{[\log(\frac{|y-z|}{r})]^{4+2\epsilon}}{(|y-z|)^{2\rho-n}},$$

where  $r > 0, 0 < \epsilon < \rho - \frac{n}{2}$  and  $\rho > \frac{n}{2}$ .

The proof of this lemma is similar to that of Lemma 2.1.2 in [15], so we omit the details.

LEMMA 3.8. [11] *There exists a constant  $C > 0$  such that for any  $z \in (8B^*)^c$ ,  $|y - z| \geq 6r$ ,*

$$\left| \frac{\Omega(y - z)}{|y - z|^{n-\rho}} - \frac{\Omega(\omega - x_0 + y - z)}{|\omega - x_0 + y - z|^{n-\rho}} \right| \leq \frac{C(1 + |\Omega(y - z)|)}{|y - z|^{n-\rho} \left( \log \frac{|y-z|}{r} \right)^\alpha},$$

where  $\Omega \in L^2(S^{n-1})$  is a homogeneous function of degree zero satisfying (1.1) and (1.2) with  $\alpha > \frac{3}{2}$ ;  $B$  is a ball with center at  $\bar{x}$  and radius  $r_0$ ;  $B^*$  is a ball with center at  $\bar{x}$  and radius  $r = 2r_0$  and  $x_0, \omega \in B$ .

LEMMA 3.9. *Let  $b \in BMO$ ,  $0 < \delta < l < 1$ . Suppose that  $\rho > n/2$ ,  $\lambda > 2$ ,  $\Omega \in L^2(S^{n-1})$  is a homogeneous function of degree zero satisfying (1.1) and (1.2) with  $\alpha > \frac{5}{2}$ . Then for any smooth function with compact support  $f$ , there exists a positive constant  $0 < C = C_\delta$  such that*

$$M_\delta^{\sharp, \Delta}(\mu_{\lambda, b}^{*, \rho}(f))(x) \leq C \|b\|_* (M_l^\Delta(\mu_\lambda^{*, \rho}(f))(x) + M^2(f)(x)).$$

PROOF. Given  $x \in \mathbb{R}^n$ , let  $Q = Q(\bar{x}, r_1)$  be a dyadic cube centered at  $\bar{x}$  with half side length  $r_1$  and  $x \in Q$ . Let  $B$  be a ball centered at  $\bar{x}$  and with radius  $r_0 = \sqrt{n}r_1$ , and  $B^* = B(\bar{x}, r)$  with  $r = 2r_0$ . Decompose  $f = f\chi_{8B^*} + f(1 - \chi_{8B^*}) := f_1 + f_2$ . Take  $C_Q = \frac{1}{|Q|} \int_Q \mu_\lambda^{*, \rho}[(b - b_{B^*})f_2](u) du$ ; then

$$\begin{aligned} & |\mu_{\lambda, b}^{*, \rho}(f)(u) - C_Q| \\ & \leq |b(u) - b_{B^*}| \mu_\lambda^{*, \rho}(f)(u) + \mu_\lambda^{*, \rho}((b - b_{B^*})f_1)(u) + |\mu_\lambda^{*, \rho}((b - b_{B^*})f_2)(u) - C_Q|, \end{aligned}$$

and

$$\begin{aligned} & \left( \frac{1}{|Q|} \int_Q \left| |\mu_{\lambda, b}^{*, \rho}(f)(u)|^\delta - |C_Q|^\delta \right| du \right)^{\frac{1}{\delta}} \\ & \leq C_\delta \left( \frac{1}{|Q|} \int_Q |b(u) - b_{B^*}| \mu_\lambda^{*, \rho}(f)(u)^\delta du \right)^{\frac{1}{\delta}} \\ & \quad + C_\delta \left( \frac{1}{|Q|} \int_Q |\mu_\lambda^{*, \rho}((b - b_{B^*})f_1)(u)|^\delta du \right)^{\frac{1}{\delta}} \\ & \quad + C_\delta \left( \frac{1}{|Q|} \int_Q |\mu_\lambda^{*, \rho}((b - b_{B^*})f_2)(u) - C_Q|^\delta du \right)^{\frac{1}{\delta}} \\ & := C_\delta (I_1 + I_2 + I_3). \end{aligned}$$

As for  $I_1$ , we can choose  $1 < r < \min\{\frac{l}{\delta}, \frac{1}{1-\delta}\}$ , then by Hölder's inequality,

$$\begin{aligned} I_1 & \leq \left( \frac{1}{|Q|} \int_Q |b(u) - b_{B^*}|^{\delta r'} du \right)^{\frac{1}{\delta r'}} \left( \frac{1}{|Q|} \int_Q |\mu_\lambda^{*, \rho}(f)(u)|^{\delta r} du \right)^{\frac{1}{\delta r}} \\ & \leq C \|b\|_* \left( \frac{1}{|Q|} \int_Q |\mu_\lambda^{*, \rho}(f)(u)|^l du \right)^{\frac{1}{l}} \leq C \|b\|_* M_l^\Delta(\mu_\lambda^{*, \rho}(f))(x). \end{aligned}$$

As for  $I_2$ , applying Lemma 3.4 with  $\frac{1}{s} = \frac{1}{\delta} - 1$ , Theorem 1.1, Hölder's generalized inequality and Lemma 3.3, we have

$$\begin{aligned}
 (3.1) \quad I_2 &\leq \left(\frac{1}{|Q|}\right)^{\frac{1}{\delta}} \left(\frac{1}{1-\delta}\right)^{\frac{1}{\delta}} \|\mu_\lambda^{*,\rho}((b-b_{B^*})f_1)\|_{WL^1} \|\chi_Q\|_s \\
 &\leq \frac{C}{|8B^*|} \int_{8B^*} |b(u) - b_{B^*}| |f(u)| du \\
 &\leq \|b - b_{B^*}\|_{\text{exp}L, 8B^*} \|f\|_{L \log L, 8B^*} \\
 &\leq C \|b\|_* M_{L \log L}(f)(x).
 \end{aligned}$$

Note that  $M^2(f)(x) \approx M_{L \log L}(f)(x)$ , we have  $I_2 \leq C \|b\|_* M^2(f)(x)$ .

Finally, let us estimate  $I_3$ . Since  $f \in L^p$ , and  $\mu_\lambda^{*,\rho}$  is  $L^p$  bounded for  $1 < p < \infty$  by Theorem 1.2, then

$$\int_Q |\mu_\lambda^{*,\rho}(f_2)(u)| du \leq |Q|^{\frac{1}{p'}} \left( \int_Q |\mu_\lambda^{*,\rho}(f_2)(u)|^p du \right)^{\frac{1}{p}} \leq C |Q|^{\frac{1}{p'}} \left( \int_{\mathbb{R}^n} |f(u)|^p du \right)^{\frac{1}{p}}.$$

This fact shows that  $\mu_\lambda^{*,\rho}(f_2)(u) < \infty$  a.e. on  $Q$ , so except a subset  $E$  with measure zero, for all  $u \in Q \setminus E$ ,  $\mu_\lambda^{*,\rho}(f_2)(u) < \infty$ . Thus,

$$\begin{aligned}
 I_3 &\leq \frac{1}{|Q|} \int_Q |\mu_\lambda^{*,\rho}((b-b_{B^*})f_2)(u) - (\mu_\lambda^{*,\rho}((b-b_{B^*})f_2))_Q| du \\
 &\leq \frac{1}{|Q|^2} \int_{Q \setminus E} \int_{Q \setminus E} |\mu_\lambda^{*,\rho}((b-b_{B^*})f_2)(u) - \mu_\lambda^{*,\rho}((b-b_{B^*})f_2)(v)| dv du.
 \end{aligned}$$

Next we will prove the following fact. For any  $x_0, w \in Q \setminus E$ ,

$$\begin{aligned}
 (3.2) \quad J &= |\mu_\lambda^{*,\rho}((b-b_{B^*})f_2)(x_0) - \mu_\lambda^{*,\rho}((b-b_{B^*})f_2)(w)| \\
 &\leq Cr^\varepsilon \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}| |f(z)|}{|z - x_0|^{n+\varepsilon}} dz + Cr^{\varepsilon/2} \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}| |f(z)|}{|z - x_0|^{n+\varepsilon/2}} dz \\
 &\quad + Cr^{\rho - \frac{n}{2}} \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}| |f(z)|}{|z - x_0|^{\frac{n}{2} + \rho}} dz \\
 &\quad + C \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}| |f(z)|}{|z - x_0|^n (\log \frac{|z - x_0|}{r})^{2+\varepsilon}} dz \\
 &:= T_1 + T_2 + T_3 + T_4.
 \end{aligned}$$

write

$$\begin{aligned}
 J &= \left| \|\phi_{t,y}((b-b_{B^*})f_2)(x_0)\|_{\mathcal{H}_2} - \|\phi_{t,y}((b-b_{B^*})f_2)(w)\|_{\mathcal{H}_2} \right| \\
 &\leq \|\phi_{t,y}((b-b_{B^*})f_2)(x_0) - \phi_{t,y}((b-b_{B^*})f_2)(w)\|_{\mathcal{H}_2} \\
 &\leq \left( \int_0^\infty \int_{|y|<1} \left(\frac{1}{1+|y|}\right)^{\lambda n} \left| \int t^{-n} \left( \phi\left(\frac{x_0-z}{t} - y\right) - \phi\left(\frac{w-z}{t} - y\right) \right) \right. \right. \\
 &\quad \left. \left. \times (b(z) - b_{B^*})f_2(z) dz \right|^2 \frac{dy dt}{t} \right)^{\frac{1}{2}}
 \end{aligned}$$



$$\begin{aligned}
& + \left( \int_0^\infty \int_{|y| \geq 1} \left( \frac{1}{1+|y|} \right)^{\lambda n} \left| \int t^{-n} \left( \phi \left( \frac{x_0 - z}{t} - y \right) - \phi \left( \frac{\omega - z}{t} - y \right) \right) \right. \right. \\
& \quad \left. \left. \times (b(z) - b_{B^*}) f_2(z) dz \right|^2 \frac{dy dt}{t} \right)^{\frac{1}{2}} := J_1 + J_2.
\end{aligned}$$

Since  $\left( \frac{1}{1+|y|} \right)^{\lambda n} \leq 1$ , there is

$$\begin{aligned}
J_1 & \leq \left( \int_0^\infty \int_{|y| < 1} \left| \int_{\substack{|\frac{x_0 - z}{t} - y| < 1 \\ |\frac{\omega - z}{t} - y| \geq 1}} t^{-n} \phi \left( \frac{x_0 - z}{t} - y \right) (b(z) - b_{B^*}) f_2(z) dz \right|^2 \frac{dy dt}{t} \right)^{\frac{1}{2}} \\
& \quad + \left( \int_0^\infty \int_{|y| < 1} \left| \int_{\substack{|\frac{x_0 - z}{t} - y| \geq 1 \\ |\frac{\omega - z}{t} - y| < 1}} t^{-n} \phi \left( \frac{\omega - z}{t} - y \right) \right. \right. \\
& \quad \left. \left. \times (b(z) - b_{B^*}) f_2(z) dz \right|^2 \frac{dy dt}{t} \right)^{\frac{1}{2}} \\
& \quad + \left( \int_0^\infty \int_{|y| < 1} \left| \int_{\substack{|\frac{x_0 - z}{t} - y| < 1 \\ |\frac{\omega - z}{t} - y| < 1}} t^{-n} \left( \phi \left( \frac{x_0 - z}{t} - y \right) - \phi \left( \frac{\omega - z}{t} - y \right) \right) \right. \right. \\
& \quad \left. \left. \times (b(z) - b_{B^*}) f_2(z) dz \right|^2 \frac{dy dt}{t} \right)^{\frac{1}{2}}.
\end{aligned}$$

Use the transform  $y \rightarrow \frac{x_0 - y'}{t}$  (we still use  $y$  instead of  $y'$ ), then

$$\begin{aligned}
J_1 & \leq \left( \int_0^\infty \int_{|x_0 - y| < t} \left| \int_{\substack{|y - z| < t \\ |\omega - x_0 + y - z| \geq t}} \frac{\Omega(y - z)}{|y - z|^{n - \rho}} (b(z) - b_{B^*}) f_2(z) dz \right|^2 \frac{dy dt}{t^{n + 2\rho + 1}} \right)^{\frac{1}{2}} \\
& \quad + \left( \int_0^\infty \int_{|x_0 - y| < t} \left| \int_{\substack{|y - z| \geq t \\ |\omega - x_0 + y - z| < t}} \frac{\Omega(\omega - x_0 + y - z)}{|\omega - x_0 + y - z|^{n - \rho}} \right. \right. \\
& \quad \left. \left. \times (b(z) - b_{B^*}) f_2(z) dz \right|^2 \frac{dy dt}{t^{n + 2\rho + 1}} \right)^{\frac{1}{2}} \\
& \quad + \left( \int_0^\infty \int_{|x_0 - y| < t} \left| \int_{|\omega - x_0 + y - z| < t} \left( \frac{\Omega(y - z)}{|y - z|^{n - \rho}} - \frac{\Omega(\omega - x_0 + y - z)}{|\omega - x_0 + y - z|^{n - \rho}} \right) \right. \right. \\
& \quad \left. \left. \times (b(z) - b_{B^*}) f_2(z) dz \right|^2 \frac{dy dt}{t^{n + 2\rho + 1}} \right)^{\frac{1}{2}} \\
& := J_{1.1} + J_{1.2} + J_{1.3}.
\end{aligned}$$

Take  $0 < \varepsilon < \min \left\{ \frac{1}{2}, \rho - \frac{n}{2}, \alpha - \frac{5}{2}, \frac{(\lambda - 2)}{2} n \right\}$  (we always restrict that  $\varepsilon$  satisfies this in the whole proof of this lemma). As for  $J_{1.1}$ , it follows from the Minkowski

inequality that

$$\begin{aligned}
J_{1.1} &\leq \int_{(8B^*)^c} |b(z) - b_{B^*}| |f(z)| \left[ \left( \iint_{\substack{y \in 2B^* \\ |y-z| < t \\ |x_0-y| < t \\ |\omega-x_0+y-z| \geq t}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dy dt}{t^{n+2\rho+1}} \right)^{\frac{1}{2}} \right. \\
&\quad \left. + \iint_{\substack{y \in (2B^*)^c \\ |y-z| < t \\ |x_0-y| < t \\ |\omega-x_0+y-z| \geq t}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dy dt}{t^{n+2\rho+1}} \right]^{\frac{1}{2}} dz \\
&\leq J_{1.1.1} + J_{1.1.2},
\end{aligned}$$

where

$$\begin{aligned}
J_{1.1.1} &= \int_{(8B^*)^c} |b(z) - b_{B^*}| |f(z)| \left( \iint_{\substack{y \in 2B^* \\ |y-z| < t \\ |x_0-y| < t \\ |\omega-x_0+y-z| \geq t}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dy dt}{t^{n+2\rho+1}} \right)^{\frac{1}{2}} dz, \\
J_{1.1.2} &= \int_{(8B^*)^c} |b(z) - b_{B^*}| |f(z)| \left( \iint_{\substack{y \in (2B^*)^c \\ |y-z| < t \\ |x_0-y| < t \\ |\omega-x_0+y-z| \geq t}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dy dt}{t^{n+2\rho+1}} \right)^{\frac{1}{2}} dz.
\end{aligned}$$

For  $J_{1.1.1}$ , since  $y \in 2B^*$ ,  $z \in (8B^*)^c$ , then  $|y-z| \sim |x_0-z| \sim |\omega-x_0+y-z|$ . We have

$$\begin{aligned}
(3.3) \quad J_{1.1.1} &\leq \int_{(8B^*)^c} |b(z) - b_{B^*}| |f(z)| \left( \int_{y \in 2B^*} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \right. \\
&\quad \left. \times \int_{|y-z| < t \leq |\omega-x_0+y-z|} \frac{dt}{t^{n+2\rho+1}} dy \right)^{\frac{1}{2}} dz \\
&\leq C \int_{(8B^*)^c} |b(z) - b_{B^*}| |f(z)| \left( \int_{y \in 2B^*} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{r}{|y-z|^{n+2\rho+1}} dy \right)^{\frac{1}{2}} dz \\
&\leq C \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}| |f(z)|}{|z-x_0|^{n+\varepsilon}} \left( \int_{|y-z| > 6r} \frac{r |\Omega(y-z)|^2}{|y-z|^{n+1-2\varepsilon}} dy \right)^{\frac{1}{2}} dz \\
&\leq Cr^\varepsilon \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}| |f(z)|}{|z-x_0|^{n+\varepsilon}} dz.
\end{aligned}$$

For  $J_{1.1.2}$ , we have

$$\begin{aligned}
J_{1.1.2} &\leq \int_{(8B^*)^c} |b(z) - b_{B^*}| |f(z)| \\
&\quad \times \left( \iint_{\substack{y \in (2B^*)^c, |y-z| < t \\ |x_0-y| < t, 2|y-z| \geq |z-x_0| \\ |\omega-x_0+y-z| \geq t}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dy dt}{t^{n+2\rho+1}} \right)^{\frac{1}{2}} dz \\
&\quad + \int_{(8B^*)^c} |b(z) - b_{B^*}| |f(z)|
\end{aligned}$$

$$\begin{aligned} & \times \left( \iint_{\substack{y \in (2B^*)^c, |y-z| < t \\ |x_0-y| < t, 2|y-z| < |z-x_0| \\ |\omega-x_0+y-z| \geq t}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dy dt}{t^{n+2\rho+1}} \right)^{\frac{1}{2}} dz \\ & := J_{1.1.2'} + J_{1.1.2''}. \end{aligned}$$

First we give the estimate of  $J_{1.1.2'}$ .

$$\begin{aligned} J_{1.1.2'} & \leq \int_{(8B^*)^c} |b(z) - b_{B^*}| |f(z)| \\ & \quad \times \left( \int_{\substack{y \in (2B^*)^c \\ 2|y-z| \geq |z-x_0|}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \int_{|y-z|}^{|\omega-x_0+y-z|} \frac{dt}{t^{n+2\rho+1}} dy \right)^{\frac{1}{2}} dz \\ & \leq C \int_{(8B^*)^c} |b(z) - b_{B^*}| |f(z)| \\ & \quad \times \left( \int_{\substack{y \in (2B^*)^c \\ 2|y-z| \geq |z-x_0|}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{r}{|y-z|^{n+2\rho+1}} dy \right)^{\frac{1}{2}} dz \\ & \leq C \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}| |f(z)|}{|z-x_0|^{n+\varepsilon}} \left( \int_{|y-z| > 3r} \frac{r |\Omega(y-z)|^2}{|y-z|^{n+1-2\varepsilon}} dy \right)^{\frac{1}{2}} dz. \end{aligned}$$

Similarly to the estimate of (3.3), we have

$$J_{1.1.2'} \leq Cr^\varepsilon \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}| |f(z)|}{|z-x_0|^{n+\varepsilon}} dz.$$

Then we give the estimate of  $J_{1.1.2''}$ .

$$\begin{aligned} J_{1.1.2''} & \leq \int_{(8B^*)^c} |b(z) - b_{B^*}| |f(z)| \\ & \quad \times \left( \iint_{\substack{y \in (2B^*)^c, |y-z| < t \\ |x_0-y| < t, 2|y-z| < |z-x_0| \\ |\omega-x_0+y-z| \geq t, |y-z| < 2r}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dy dt}{t^{n+2\rho+1}} \right)^{\frac{1}{2}} dz \\ & \quad + \int_{(8B^*)^c} |b(z) - b_{B^*}| |f(z)| \\ & \quad \times \left( \iint_{\substack{y \in (2B^*)^c, |y-z| < t \\ |x_0-y| < t, 2|y-z| < |z-x_0| \\ |\omega-x_0+y-z| \geq t, |y-z| \geq 2r}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dy dt}{t^{n+2\rho+1}} \right)^{\frac{1}{2}} dz \\ & := J_{1.1.2.1''} + J_{1.1.2.2''}. \end{aligned}$$

For  $J_{1.1.2.1''}$ , since  $|y-z| < \frac{|z-x_0|}{2}$ , then  $|y-x_0| > \frac{|z-x_0|}{2}$  and we get

$$\begin{aligned} J_{1.1.2.1''} & \leq \int_{(8B^*)^c} |b(z) - b_{B^*}| |f(z)| \\ & \quad \times \left( \int_{\substack{|y-z| < 2r \\ |y-x_0| > \frac{|z-x_0|}{2}}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \int_{|y-x_0|}^{\infty} \frac{dt}{t^{n+2\rho+1}} dy \right)^{\frac{1}{2}} dz \end{aligned}$$

$$\begin{aligned} &\leq C \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}| |f(z)|}{|z - x_0|^{\frac{n}{2} + \rho}} \left( \int_{|y-z| < 2r} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} dy \right)^{\frac{1}{2}} dz \\ &\leq Cr^{\rho - \frac{n}{2}} \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}| |f(z)|}{|z - x_0|^{\frac{n}{2} + \rho}} dz. \end{aligned}$$

For  $J_{1.1.2.2''}$ , since  $t > |y - x_0| > \frac{|z - x_0|}{2}$ , and  $|y - z| \sim |\omega - x_0 + y - z|$ , we have

$$\begin{aligned} J_{1.1.2.2''} &\leq \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}| |f(z)|}{\left(\frac{|z - x_0|}{2}\right)^{n+\varepsilon}} \left( \int_{\substack{y \in (2B^*)^c \\ 2|y-z| < |z-x_0| \\ |y-z| \geq 2r}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \right. \\ &\quad \left. \times \int_{|y-z| < t \leq |\omega - x_0 + y - z|} \frac{dt}{t^{2\rho - n + 1 - 2\varepsilon}} dy \right)^{\frac{1}{2}} dz \\ &\leq C \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}| |f(z)|}{|z - x_0|^{n+\varepsilon}} \left( \int_{|y-z| \geq 2r} \frac{r |\Omega(y-z)|^2}{|y-z|^{n+1-2\varepsilon}} dy \right)^{\frac{1}{2}} dz. \end{aligned}$$

Similarly to the estimate of (3.3), we have

$$J_{1.1.2.2''} \leq Cr^\varepsilon \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}| |f(z)|}{|z - x_0|^{n+\varepsilon}} dz.$$

Combining the estimates of  $J_{1.1.1}$ ,  $J_{1.1.2'}$ ,  $J_{1.1.2.1''}$  and  $J_{1.1.2.2''}$ , we obtain

$$J_{1.1} \leq Cr^\varepsilon \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}| |f(z)|}{|z - x_0|^{n+\varepsilon}} dz + Cr^{\rho - \frac{n}{2}} \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}| |f(z)|}{|z - x_0|^{\frac{n}{2} + \rho}} dz.$$

Similarly as we deal with  $J_{1.1}$ , we can get

$$J_{1.2} \leq Cr^\varepsilon \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}| |f(z)|}{|z - x_0|^{n+\varepsilon}} dz + Cr^{\rho - \frac{n}{2}} \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}| |f(z)|}{|z - x_0|^{\frac{n}{2} + \rho}} dz.$$

Next we give the estimate of  $J_{1.3}$ . Apply the Minkowski inequality to  $J_{1.3}$  and divide the region by  $|y - z| \geq 6r$  and  $|y - z| < 6r$ . When  $|y - z| < 6r$ , we have  $y \in (2B^*)^c$ , so

$$\begin{aligned} J_{1.3} &\leq \int_{(8B^*)^c} |b(z) - b_{B^*}| |f(z)| \left( \iint_{\substack{y \in (2B^*)^c \\ |y-z| < t \\ |x_0-y| < t \\ |y-z| < 6r \\ |\omega-x_0+y-z| < t}} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} \right. \right. \\ &\quad \left. \left. - \frac{\Omega(\omega - x_0 + y - z)}{|\omega - x_0 + y - z|^{n-\rho}} \right|^2 \frac{dy dt}{t^{n+2\rho+1}} \right)^{\frac{1}{2}} dz \\ &\quad + \int_{(8B^*)^c} |b(z) - b_{B^*}| |f(z)| \left( \iint_{\substack{|y-z| \geq 6r \\ |y-z| < t \\ |x_0-y| < t \\ |\omega-x_0+y-z| < t}} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} \right. \right. \\ &\quad \left. \left. - \frac{\Omega(\omega - x_0 + y - z)}{|\omega - x_0 + y - z|^{n-\rho}} \right|^2 \frac{dy dt}{t^{n+2\rho+1}} \right)^{\frac{1}{2}} dz := J_{1.3.1} + J_{1.3.2}. \end{aligned}$$

When  $z \in (8B^*)^c$  and  $|y - z| < 6r$ , there are  $|y - x_0| \sim |z - x_0|$  and  $|\omega - x_0 + y - z| \leq |\omega - x_0| + |y - z| < 8r$ . Then

$$\begin{aligned}
J_{1.3.1} &\leq C \int_{(8B^*)^c} |b(z) - b_{B^*}| |f(z)| \left( \int_{\substack{y \in (2B^*)^c, |y-z| < 6r \\ |x_0-y| < t, |\omega-x_0+y-z| < 8r}} \left( \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \right. \right. \\
&\quad \left. \left. + \frac{|\Omega(\omega-x_0+y-z)|^2}{|\omega-x_0+y-z|^{2n-2\rho}} \right) \int_{|y-x_0|}^{\infty} \frac{dt}{t^{n+2\rho+1}} dy \right)^{\frac{1}{2}} dz \\
&\leq C \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}| |f(z)|}{|z-x_0|^{\frac{n}{2}+\rho}} \left( \int_{|y-z| < 6r} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} dy \right)^{\frac{1}{2}} dz \\
&\quad + C \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}| |f(z)|}{|z-x_0|^{\frac{n}{2}+\rho}} \\
&\quad \times \left( \int_{|\omega-x_0+y-z| < 8r} \frac{|\Omega(\omega-x_0+y-z)|^2}{|\omega-x_0+y-z|^{2n-2\rho}} dy \right)^{\frac{1}{2}} dz \\
&\leq Cr^{\rho-\frac{n}{2}} \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}| |f(z)|}{|z-x_0|^{\rho+\frac{n}{2}}} dz.
\end{aligned}$$

Next we estimate  $J_{1.3.2}$ . Note that  $|z-x_0| \leq |x_0-y| + |y-z| < 2t$ , so  $t > \frac{|z-x_0|}{2}$ .

$$\begin{aligned}
J_{1.3.2} &\leq \int_{(8B^*)^c} |b(z) - b_{B^*}| |f(z)| \left( \iint_{\substack{t > |z-x_0|/2, |y-z| < t \\ |y-x_0| < t, |y-z| \geq 6r \\ |\omega-x_0+y-z| < t}} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} \right. \right. \\
&\quad \left. \left. - \frac{\Omega(\omega-x_0+y-z)}{|\omega-x_0+y-z|^{n-\rho}} \right|^2 \frac{(\log \frac{t}{r})^{4+2\varepsilon} dy dt}{t^{2\rho-n+1} t^{2n} (\log \frac{t}{r})^{4+2\varepsilon}} \right)^{\frac{1}{2}} dz \\
&\leq C \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}| |f(z)|}{|z-x_0|^n (\log \frac{|z-x_0|}{2r})^{2+\varepsilon}} \left( \int_{|y-z| \geq 6r} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} \right. \right. \\
(3.4) \quad &\quad \left. \left. - \frac{\Omega(\omega-x_0+y-z)}{|\omega-x_0+y-z|^{n-\rho}} \right|^2 \int_{|y-z|}^{\infty} \frac{(\log \frac{t}{r})^{4+2\varepsilon}}{t^{2\rho-n+1}} dt dy \right)^{\frac{1}{2}} dz.
\end{aligned}$$

By Lemma 3.7 and Lemma 3.8, there is

$$\begin{aligned}
J_{1.3.2} &\leq C \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}| |f(z)|}{|z-x_0|^n (\log \frac{|z-x_0|}{2r})^{2+\varepsilon}} \\
(3.5) \quad &\quad \times \left( \int_{|y-z| \geq 6r} \frac{(1 + |\Omega(y-z)|)^2}{|y-z|^n (\log \frac{|y-z|}{r})^{2\alpha-4-2\varepsilon}} dy \right)^{\frac{1}{2}} dz \\
&\leq C \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}| |f(z)|}{|z-x_0|^n (\log \frac{|z-x_0|}{r})^{2+\varepsilon}} dz.
\end{aligned}$$

Combining the estimates of  $J_{1.3.1}$  and  $J_{1.3.2}$ , we obtain

$$J_{1.3} \leq Cr^{\rho-\frac{n}{2}} \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}| |f(z)|}{|z-x_0|^{\frac{n}{2}+\rho}} dz + C \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}| |f(z)|}{|z-x_0|^n (\log \frac{|z-x_0|}{r})^{2+\varepsilon}} dz.$$

As for  $J_2$ ,

$$\begin{aligned}
J_2 &\leq \left( \int_0^\infty \int_{|y| \geq 1} \left( \frac{1}{1+|y|} \right)^{\lambda n} \left| \int_{\substack{|\frac{x_0-z}{t}-y| < 1 \\ |\frac{w-z}{t}-y| \geq 1}} t^{-n} \right. \right. \\
&\quad \times \left. \left. \phi \left( \frac{x_0-z}{t} - y \right) (b(z) - b_{B^*}) f_2(z) dz \right|^2 \frac{dy dt}{t} \right)^{\frac{1}{2}} \\
&\quad + \left( \int_0^\infty \int_{|y| \geq 1} \left( \frac{1}{1+|y|} \right)^{\lambda n} \left| \int_{\substack{|\frac{x_0-z}{t}-y| \geq 1 \\ |\frac{w-z}{t}-y| < 1}} t^{-n} \right. \right. \\
&\quad \times \left. \left. \phi \left( \frac{w-z}{t} - y \right) (b(z) - b_{B^*}) f_2(z) dz \right|^2 \frac{dy dt}{t} \right)^{\frac{1}{2}} \\
&\quad + \left( \int_0^\infty \int_{|y| \geq 1} \left( \frac{1}{1+|y|} \right)^{\lambda n} \left| \int_{\substack{|\frac{x_0-z}{t}-y| < 1 \\ |\frac{w-z}{t}-y| < 1}} t^{-n} \right. \right. \\
&\quad \times \left. \left. \left( \phi \left( \frac{x_0-z}{t} - y \right) - \phi \left( \frac{w-z}{t} - y \right) \right) (b(z) - b_{B^*}) f_2(z) dz \right|^2 \frac{dy dt}{t} \right)^{\frac{1}{2}}.
\end{aligned}$$

Using the transform  $y \rightarrow \frac{x_0-y'}{t}$  again (we still use  $y$  instead of  $y'$ ), we have

$$\begin{aligned}
J_2 &\leq \left( \int_0^\infty \int_{|x_0-y| \geq t} \left( \frac{t}{t+|x_0-y|} \right)^{\lambda n} \left| \int_{\substack{|y-z| < t \\ |w-x_0+y-z| \geq t}} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} \right. \right. \\
&\quad \times \left. \left. (b(z) - b_{B^*}) f_2(z) dz \right|^2 \frac{dy dt}{t^{n+2\rho+1}} \right)^{\frac{1}{2}} \\
&\quad + \left( \int_0^\infty \int_{|x_0-y| \geq t} \left( \frac{t}{t+|x_0-y|} \right)^{\lambda n} \left| \int_{\substack{|y-z| \geq t \\ |w-x_0+y-z| < t}} \frac{\Omega(w-x_0+y-z)}{|w-x_0+y-z|^{n-\rho}} \right. \right. \\
&\quad \times \left. \left. (b(z) - b_{B^*}) f_2(z) dz \right|^2 \frac{dy dt}{t^{n+2\rho+1}} \right)^{\frac{1}{2}} \\
&\quad + \left( \int_0^\infty \int_{|x_0-y| \geq t} \left( \frac{t}{t+|x_0-y|} \right)^{\lambda n} \left| \int_{\substack{|y-z| < t \\ |w-x_0+y-z| < t}} \left( \frac{\Omega(y-z)}{|y-z|^{n-\rho}} \right. \right. \right. \\
&\quad \left. \left. \left. - \frac{\Omega(w-x_0+y-z)}{|w-x_0+y-z|^{n-\rho}} \right) (b(z) - b_{B^*}) f_2(z) dz \right|^2 \frac{dy dt}{t^{n+2\rho+1}} \right)^{\frac{1}{2}} \\
&:= J_{2.1} + J_{2.2} + J_{2.3}.
\end{aligned}$$

For  $J_{2.1}$ , we claim that  $y \in (2B^*)^c$ . Otherwise if  $y \in 2B^*$ , then  $t \leq |x_0-y| < 4r$ . But  $z \in (8B^*)^c$  and  $t > |y-z| > 6r$ . Thus by the Minkowski inequality, we get

$$J_{2.1} \leq \int_{(8B^*)^c} |b(z) - b_{B^*}| |f(z)| \left( \int_0^\infty \int_{\substack{|x_0-y| \geq t \\ y \in (2B^*)^c \\ |y-z| < t \\ |w-x_0+y-z| \geq t}} \left( \frac{t}{t+|x_0-y|} \right)^{\lambda n} \right.$$

$$\times \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dy dt}{t^{n+2\rho+1}} \Big)^{\frac{1}{2}} dz \leq J_{2.1.1} + J_{2.1.2},$$

where

$$J_{2.1.1} = \int_{(8B^*)^c} |b(z) - b_{B^*}| |f(z)| \left( \int_0^\infty \int_{\substack{|y-z| < 8r, |x_0-y| \geq t \\ y \in (2B^*)^c, |y-z| < t \\ |w-x_0+y-z| \geq t}} \left( \frac{t}{t+|x_0-y|} \right)^{\lambda n} \right. \\ \left. \times \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dy dt}{t^{n+2\rho+1}} \right)^{\frac{1}{2}} dz,$$

$$J_{2.1.2} = \int_{(8B^*)^c} |b(z) - b_{B^*}| |f(z)| \left( \int_0^\infty \int_{\substack{|y-z| \geq 8r, |x_0-y| \geq t \\ y \in (2B^*)^c, |y-z| < t \\ |w-x_0+y-z| \geq t}} \left( \frac{t}{t+|x_0-y|} \right)^{\lambda n} \right. \\ \left. \times \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dy dt}{t^{n+2\rho+1}} \right)^{\frac{1}{2}} dz.$$

For  $J_{2.1.1}$ , since  $|y-z| < 8r$ ,  $z \in (8B^*)^c$  and  $y \in (2B^*)^c$ , then  $|y-x_0| \sim |z-x_0|$ . So

$$J_{2.1.1} \leq \int_{(8B^*)^c} |b(z) - b_{B^*}| |f(z)| \left( \int_0^\infty \int_{\substack{|y-z| < 8r, |x_0-y| \geq t \\ y \in (2B^*)^c, |y-z| < t}} \left( \frac{1}{t+|x_0-y|} \right)^{2n+2\varepsilon} \right. \\ \left. \times \left( \frac{t}{t+|x_0-y|} \right)^{\lambda n - 2n - 2\varepsilon} t^{2n+2\varepsilon} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dy dt}{t^{n+2\rho+1}} \right)^{\frac{1}{2}} dz \\ \leq C \int_{(8B^*)^c} |b(z) - b_{B^*}| |f(z)| \left( \int_0^\infty \int_{\substack{|y-z| < 8r \\ |x_0-y| \geq t \\ y \in (2B^*)^c \\ |y-z| < t}} \frac{1}{|z-x_0|^{2n+2\varepsilon}} \right. \\ \left. \times \frac{|\Omega(y-z)|^2}{|y-z|^{n-\varepsilon}} \frac{dy dt}{t^{1-\varepsilon}} \right)^{\frac{1}{2}} dz \\ \leq C \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}| |f(z)|}{|z-x_0|^{n+\varepsilon/2}} \left( \int_{|y-z| < 8r} \frac{|\Omega(y-z)|^2}{|z-x_0|^\varepsilon |y-z|^{n-\varepsilon}} \right. \\ \left. \times \left( \int_0^{|x_0-y|} \frac{1}{t^{1-\varepsilon}} dt \right) dy \right)^{\frac{1}{2}} dz \\ \leq Cr^{\varepsilon/2} \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}| |f(z)|}{|z-x_0|^{n+\varepsilon/2}} dz.$$

For  $J_{2.1.2}$ , there is

$$J_{2.1.2} \leq \int_{(8B^*)^c} |b(z) - b_{B^*}| |f(z)| \left( \int_0^\infty \int_{\substack{2|y-z| \geq |z-x_0| \\ |x_0-y| \geq t, y \in (2B^*)^c \\ |y-z| < t, |y-z| \geq 8r \\ |w-x_0+y-z| \geq t}} \left( \frac{t}{t+|x_0-y|} \right)^{\lambda n} \right. \\ \left. \times \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dy dt}{t^{n+2\rho+1}} \right)^{\frac{1}{2}} dz$$

$$\begin{aligned}
& + \int_{(8B^*)^c} |b(z) - b_{B^*}| |f(z)| \left( \int_0^\infty \int_{\substack{2|y-z| < |z-x_0| \\ |x_0-y| \geq t, y \in (2B^*)^c \\ |y-z| < t, |y-z| \geq 8r \\ |w-x_0+y-z| \geq t}} \left( \frac{t}{t+|x_0-y|} \right)^{\lambda n} \right. \\
& \quad \left. \times \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dy dt}{t^{n+2\rho+1}} \right)^{\frac{1}{2}} dz := J_{2.1.2'} + J_{2.1.2''}.
\end{aligned}$$

As for  $J_{2.1.2'}$ , we have

$$\begin{aligned}
J_{2.1.2'} & \leq \int_{(8B^*)^c} |b(z) - b_{B^*}| |f(z)| \left( \int_{\substack{y \in (2B^*)^c \\ |y-z| \geq 8r \\ 2|y-z| \geq |z-x_0|}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \right. \\
& \quad \left. \times \int_{|y-z|}^{|w-x_0+y-z|} \frac{dt}{t^{n+2\rho+1}} dy \right)^{\frac{1}{2}} dz \\
& \leq C \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}| |f(z)|}{|z-x_0|^{n+\varepsilon}} \left( \int_{|y-z| \geq 8r} \frac{r|\Omega(y-z)|^2}{|y-z|^{n+1-2\varepsilon}} dy \right)^{\frac{1}{2}} dz \\
& \leq Cr^\varepsilon \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}| |f(z)|}{|z-x_0|^{n+\varepsilon}} dz.
\end{aligned}$$

For  $J_{2.1.2''}$ , by  $|z-x_0| > 2|y-z|$ , we have  $|y-x_0| > |z-x_0|/2$  and  $|y-z| \sim |w-x_0+y-z|$ . Then

$$\begin{aligned}
J_{2.1.2''} & \leq \int_{(8B^*)^c} |b(z) - b_{B^*}| |f(z)| \left( \iint_{\substack{y \in (2B^*)^c, |y-z| \geq 8r \\ |y-x_0| \geq |z-x_0|/2 \\ |y-z| < t, |w-x_0+y-z| \geq t}} \frac{t^{2n+2\varepsilon}}{|x_0-y|^{2n+2\varepsilon}} \right. \\
& \quad \left. \times \left( \frac{t}{t+|x_0-y|} \right)^{\lambda n - 2n - 2\varepsilon} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dy dt}{t^{n+2\rho+1}} \right)^{\frac{1}{2}} dz \\
& \leq C \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}| |f(z)|}{|z-x_0|^{n+\varepsilon}} \left( \int_{\substack{y \in (2B^*)^c, |y-z| \geq 8r \\ |y-x_0| \geq |z-x_0|/2}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \right. \\
& \quad \left. \times \left( \int_{|y-z|}^{|w-x_0+y-z|} \frac{1}{t^{2\rho-n-2\varepsilon+1}} dt \right) dy \right)^{\frac{1}{2}} dz \\
& \leq C \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}| |f(z)|}{|z-x_0|^{n+\varepsilon}} \left( \int_{|y-z| \geq 8r} \frac{r|\Omega(y-z)|^2}{|y-z|^{n+1-2\varepsilon}} dy \right)^{\frac{1}{2}} dz.
\end{aligned}$$

Similarly to the estimate of (3.3), we have  $J_{2.1.2''} \leq Cr^\varepsilon \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}| |f(z)|}{|z-x_0|^{n+\varepsilon}} dz$ .

Combining the estimates of  $J_{2.1.1}$ ,  $J_{2.1.2'}$  and  $J_{2.1.2''}$ , we obtain

$$J_{2.1} \leq Cr^{\varepsilon/2} \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}| |f(z)|}{|z-x_0|^{n+\varepsilon/2}} dz + Cr^\varepsilon \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}| |f(z)|}{|z-x_0|^{n+\varepsilon}} dz.$$

Similarly as  $J_{2.1}$ , we can obtain

$$J_{2.2} \leq Cr^{\varepsilon/2} \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}| |f(z)|}{|z-x_0|^{n+\varepsilon/2}} dz + Cr^\varepsilon \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}| |f(z)|}{|z-x_0|^{n+\varepsilon}} dz.$$



Finally, we deal with the last part  $J_{2.3}$ . By the Minkowski inequality,

$$\begin{aligned} J_{2.3} &\leq \int_{(8B^*)^c} |b(z) - b_{B^*}| |f(z)| \left( \iint_{\substack{|y-z| < 6r, |x_0-y| \geq t \\ |y-z| < t, |w-x_0+y-z| < t}} \left( \frac{t}{t+|x_0-y|} \right)^{\lambda n} \right. \\ &\quad \times \left. \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(w-x_0+y-z)}{|w-x_0+y-z|^{n-\rho}} \right|^2 \frac{dy dt}{t^{n+2\rho+1}} \right)^{\frac{1}{2}} dz \\ &+ \int_{(8B^*)^c} |b(z) - b_{B^*}| |f(z)| \left( \iint_{\substack{|y-z| \geq 6r, |x_0-y| \geq t \\ |y-z| < t, |w-x_0+y-z| < t}} \left( \frac{t}{t+|x_0-y|} \right)^{\lambda n} \right. \\ &\quad \times \left. \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(w-x_0+y-z)}{|w-x_0+y-z|^{n-\rho}} \right|^2 \frac{dy dt}{t^{n+2\rho+1}} \right)^{\frac{1}{2}} dz := J_{2.3.1} + J_{2.3.2}. \end{aligned}$$

For  $J_{2.3.1}$ , it follows from  $|y-z| < 6r$  and  $z \in (8B^*)^c$  that  $|y-\bar{x}| \geq |z-\bar{x}| - |y-z| > 2r$ . We can get  $y \in (2B^*)^c$  and  $|w-x_0+y-z| \leq |w-x_0| + |y-z| < 8r$ .

$$\begin{aligned} J_{2.3.1} &\leq \int_{(8B^*)^c} |b(z) - b_{B^*}| |f(z)| \left( \iint_{\substack{y \in (2B^*)^c, |x_0-y| \geq t \\ |y-z| < 6r, |y-z| < t}} \left( \frac{t}{t+|x_0-y|} \right)^{\lambda n} \right. \\ &\quad \times \left. \left| \frac{\Omega(y-z)}{|y-z|^{2n-2\rho}} \right|^2 \frac{dy dt}{t^{n+2\rho+1}} \right)^{\frac{1}{2}} dz \\ &+ \int_{(8B^*)^c} |b(z) - b_{B^*}| |f(z)| \left( \iint_{\substack{y \in (2B^*)^c, |x_0-y| \geq t \\ |w-x_0+y-z| < 8r \\ |w-x_0+y-z| < t}} \left( \frac{t}{t+|x_0-y|} \right)^{\lambda n} \right. \\ &\quad \times \left. \left| \frac{\Omega(w-x_0+y-z)}{|w-x_0+y-z|^{2n-2\rho}} \right|^2 \frac{dy dt}{t^{n+2\rho+1}} \right)^{\frac{1}{2}} dz := J_{2.3.1'} + J_{2.3.1''}. \end{aligned}$$

Estimate  $J_{2.3.1'}$  and  $J_{2.3.1''}$  by the similar method as we deal with  $J_{2.1.1}$ . There is

$$J_{2.3.1} \leq Cr^{\varepsilon/2} \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}| |f(z)|}{|z-x_0|^{n+\varepsilon/2}} dz.$$

As for  $J_{2.3.2}$ , we have

$$\begin{aligned} J_{2.3.2} &\leq \int_{(8B^*)^c} |b(z) - b_{B^*}| |f(z)| \left( \iint_{\substack{2|y-z| \geq |z-x_0| \\ |y-z| \geq 6r, |x_0-y| \geq t \\ |y-z| < t, |w-x_0+y-z| < t}} \left( \frac{t}{t+|x_0-y|} \right)^{\lambda n} \right. \\ &\quad \times \left. \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(w-x_0+y-z)}{|w-x_0+y-z|^{n-\rho}} \right|^2 \frac{dy dt}{t^{n+2\rho+1}} \right)^{\frac{1}{2}} dz \\ &+ \int_{(8B^*)^c} |b(z) - b_{B^*}| |f(z)| \left( \iint_{\substack{2|y-z| < |z-x_0| \\ |y-z| \geq 6r, |x_0-y| \geq t \\ |y-z| < t, |w-x_0+y-z| < t}} \left( \frac{t}{t+|x_0-y|} \right)^{\lambda n} \right. \\ &\quad \times \left. \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(w-x_0+y-z)}{|w-x_0+y-z|^{n-\rho}} \right|^2 \frac{dy dt}{t^{n+2\rho+1}} \right)^{\frac{1}{2}} dz \\ &:= J_{2.3.2'} + J_{2.3.2''}. \end{aligned}$$

For  $J_{2.3.2'}$ , there is

$$\begin{aligned}
J_{2.3.2'} &\leq C \int_{(8B^*)^c} |b(z) - b_{B^*}| |f(z)| \\
&\quad \times \left( \int_{|y-z| \geq 6r} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(w-x_0+y-z)}{|w-x_0+y-z|^{n-\rho}} \right|^2 \right. \\
&\quad \times \left. \left( \int_{\max\{|y-z|, |z-x_0|/2\}}^{\infty} \frac{(\log \frac{t}{r})^{4+2\varepsilon} dt}{t^{2\rho-n+1} |z-x_0|^{2n} (\log \frac{|z-x_0|}{2r})^{4+2\varepsilon}} dy \right)^{\frac{1}{2}} dz \\
&\leq C \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}| |f(z)|}{|z-x_0|^n (\log \frac{|z-x_0|}{2r})^{2+\varepsilon}} \\
&\quad \times \left( \int_{|y-z| \geq 6r} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(w-x_0+y-z)}{|w-x_0+y-z|^{n-\rho}} \right|^2 \right. \\
&\quad \times \left. \left( \int_{|y-z|}^{\infty} \frac{(\log \frac{t}{r})^{4+2\varepsilon} dt}{t^{2\rho-n+1}} dy \right)^{\frac{1}{2}} dz.
\end{aligned}$$

By the estimate of (3.4), we get  $J_{2.3.2'} \leq C \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}| |f(z)|}{|z-x_0|^n (\log \frac{|z-x_0|}{2r})^{2+\varepsilon}} dz$ .

For  $J_{2.3.2''}$ , denote  $C(\varepsilon) = e^{(4+2\varepsilon)/\varepsilon}$ . Since  $2|y-z| < |z-x_0|$ , then  $|x_0-y| \geq |z-x_0| - |y-z| > \frac{|z-x_0|}{2}$ , thus

$$\begin{aligned}
J_{2.3.2''} &\leq \int_{(8B^*)^c} |b(z) - b_{B^*}| |f(z)| \left( \iint_{\substack{|y-z| \geq 6r, |x_0-y| \geq t \\ |x_0-y| > |z-x_0|/2 \\ |y-z| < t}} \frac{t^{\lambda n}}{(t+|x_0-y|)^{\lambda n-2n+2n}} \frac{(\log \frac{t+|y-x_0|+C(\varepsilon)r}{r})^{4+2\varepsilon}}{(\log \frac{t+|y-x_0|+C(\varepsilon)r}{r})^{4+2\varepsilon}} \right. \\
&\quad \times \left. \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(w-x_0+y-z)}{|w-x_0+y-z|^{n-\rho}} \right|^2 \frac{dy dt}{t^{n+2\rho+1}} \right)^{\frac{1}{2}} dz \\
&\leq C \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}| |f(z)|}{|z-x_0|^n (\log \frac{|z-x_0|}{2r})^{2+\varepsilon}} \\
&\quad \times \left( \int_{|y-z| \geq 6r} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(w-x_0+y-z)}{|w-x_0+y-z|^{n-\rho}} \right|^2 \right. \\
&\quad \times \left. \left( \int_{|y-z|}^{|x_0-y|} \frac{t^{\lambda n} (\log \frac{t+|y-x_0|+C(\varepsilon)r}{r})^{4+2\varepsilon}}{(t+|x_0-y|)^{\lambda n-2n}} \frac{dt}{t^{n+2\rho+1}} dy \right)^{\frac{1}{2}} dz.
\end{aligned}$$

Notice that the function  $G(s) = \frac{(\log s)^{4+2\varepsilon}}{s^\varepsilon}$  is decreasing when  $s \geq e^{(4+2\varepsilon)/\varepsilon}$  and

$$\frac{t+|y-x_0|+C(\varepsilon)r}{r} \geq \frac{|y-z|+C(\varepsilon)r}{r} \geq C(\varepsilon) = e^{(4+2\varepsilon)/\varepsilon},$$

then

$$\frac{[\log(\frac{t+|y-x_0|+C(\varepsilon)r}{r})]^{4+2\varepsilon}}{(\frac{t+|y-x_0|+C(\varepsilon)r}{r})^\varepsilon} \leq \frac{[\log(\frac{|y-z|+C(\varepsilon)r}{r})]^{4+2\varepsilon}}{(\frac{|y-z|+C(\varepsilon)r}{r})^\varepsilon}.$$

Since  $t + |y - x_0| \sim t + |y - x_0| + C(\varepsilon)r$  and  $0 < \varepsilon < \min\{\frac{1}{2}, \frac{(\lambda-2)n}{2}, \rho - \frac{n}{2}, \alpha - \frac{5}{2}\}$ , then

$$\begin{aligned} & \int_{|y-z|}^{|x_0-y|} \frac{t^{\lambda n} (\log \frac{t+|y-x_0|+C(\varepsilon)r}{r})^{4+2\varepsilon}}{(t+|x_0-y|)^{\lambda n-2n}} \frac{dt}{t^{n+2\rho+1}} \\ & \leq C \int_{|y-z|}^{\infty} \frac{[\log(\frac{|y-z|+C(\varepsilon)r}{r})]^{4+2\varepsilon}}{(|y-z|+C(\varepsilon)r)^\varepsilon} \frac{dt}{t^{2\rho-n+1-\varepsilon}} \\ & \leq C \frac{[\log(\frac{|y-z|+C(\varepsilon)r}{r})]^{4+2\varepsilon}}{|y-z|^{2\rho-n}}. \end{aligned}$$

Since  $|y-z| \geq 6r$ , there exists a constant  $l \geq 1$  such that  $|y-z|+C(\varepsilon)r \leq 2^l|y-z|$ . Hence

$$\begin{aligned} J_{2.3.2''} & \leq C \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}| |f(z)|}{|z-x_0|^n (\log \frac{|z-x_0|}{2r})^{2+\varepsilon}} \left( \int_{|y-z| \geq 6r} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} \right. \right. \\ & \quad \left. \left. - \frac{\Omega(w-x_0+y-z)}{|w-x_0+y-z|^{n-\rho}} \right|^2 \frac{(\log \frac{2^l|y-z|}{r})^{4+2\varepsilon}}{|y-z|^{2\rho-n}} dy \right)^{\frac{1}{2}} dz \\ & \leq C \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}| |f(z)|}{|z-x_0|^n (\log \frac{|z-x_0|}{2r})^{2+\varepsilon}} \\ & \quad \times \left( \int_{|y-z| \geq 6r} \frac{(1+|\Omega(y-z)|)^2}{|y-z|^n (\log \frac{|y-z|}{r})^{2\alpha-4-2\varepsilon}} dy \right)^{\frac{1}{2}} dz. \end{aligned}$$

By the estimate of (3.5), we get  $J_{2.3.2''} \leq C \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}| |f(z)|}{|z-x_0|^n (\log \frac{|z-x_0|}{r})^{2+\varepsilon}} dz$ .

Combining the estimates of  $J_{2.3.1}$ ,  $J_{2.3.2'}$  and  $J_{2.3.2''}$ , we obtain

$$J_{2.3} \leq Cr^{\varepsilon/2} \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}| |f(z)|}{|z-x_0|^{n+\varepsilon/2}} dz + C \int_{(8B^*)^c} \frac{|b(z) - b_{B^*}| |f(z)|}{|z-x_0|^n (\log \frac{|z-x_0|}{r})^{2+\varepsilon}} dz.$$

Then we complete the proof of (3.2). Next we will show that  $T_i \leq C \|b\|_* M^2(f)(x)$ , for  $i = 1, 2, 3, 4$ .

Denote  $B_j = \{z : |z - \bar{x}| < 2^j r\}$ , then  $|b_{B_{j+1}} - b_{B^*}| \leq j \|b\|_*$ . Since  $M^2(f) \approx M_{L \log L}(f)$ , by Lemma 3.3 we get

$$\begin{aligned} T_1 & \leq Cr^\varepsilon \sum_{j=3}^{\infty} \int_{2^j r \leq |z-\bar{x}| < 2^{j+1} r} \frac{|b(z) - b_{B^*}| |f(z)|}{|z-\bar{x}|^{n+\varepsilon}} dz \\ & \leq C \sum_{j=3}^{\infty} \frac{1}{2^{j\varepsilon}} \left( \int_{B_{j+1}} \frac{|b(z) - b_{B_{j+1}}| |f(z)|}{(2^j r)^n} dz + |b_{B_{j+1}} - b_{B^*}| M(f)(x) \right) \\ & \leq C \sum_{j=3}^{\infty} \frac{1}{2^{j\varepsilon}} (\|b - b_{B_{j+1}}\|_{\text{exp}L, B_{j+1}} \|f\|_{L \log L, B_{j+1}} + j \|b\|_* M(f)(x)) \\ & \leq C \sum_{j=3}^{\infty} \frac{1}{2^{j\varepsilon}} (\|b\|_* M_{L \log L}(f)(x) + j \|b\|_* M(f)(x)) \leq C \|b\|_* M^2(f)(x). \end{aligned}$$

Taking  $\varepsilon/2$  and  $\rho - \frac{n}{2}$  instead of  $\varepsilon$  in the above inequality respectively, we get  $T_2 \leq C\|b\|_* M^2(f)(x)$  and  $T_3 \leq C\|b\|_* M^2(f)(x)$ .

Similarly to the way in estimating  $T_1$ , we obtain

$$\begin{aligned} T_4 &\leq C \sum_{j=3}^{\infty} \int_{2^j r \leq |z-\bar{x}| < 2^{j+1} r} \frac{|b(z) - b_{B^*}| |f(z)|}{|z-x_0|^n (\log \frac{|z-\bar{x}|}{r})^{2+\varepsilon}} dz \\ &\leq C \sum_{j=3}^{\infty} \frac{1}{j^{2+\varepsilon}} (\|b\|_* M_{L \log L}(f)(x) + j\|b\|_* M(f)(x)) \\ &\leq C\|b\|_* M^2(f)(x). \end{aligned}$$

So we have  $J \leq C\|b\|_* M^2(f)(x)$  and  $I_3 \leq C\|b\|_* M^2(f)(x)$ . Combining the estimates of  $I_1$ ,  $I_2$  and  $I_3$ , we have

$$\begin{aligned} M_{\delta}^{\sharp, \Delta}(\mu_{\lambda, b}^{*, \rho}(f))(x) &= [M^{\sharp, \Delta}(|\mu_{\lambda, b}^{*, \rho}(f)|^{\delta})(x)]^{\frac{1}{\delta}} \\ &\leq \sup_{Q \ni x} \left( \frac{1}{|Q|} \int_Q \left| |\mu_{\lambda, b}^{*, \rho}(f)|^{\delta} - |C_Q|^{\delta} \right| du \right)^{\frac{1}{\delta}} \\ &\leq C\|b\|_* (M_{\lambda}^{\Delta}(\mu_{\lambda}^{*, \rho}(f))(x) + M^2(f)(x)), \end{aligned}$$

where the supremum is taken over all dyadic cubes  $Q$  with  $x \in Q$ .  $\square$

LEMMA 3.10. *Suppose that  $0 < \delta < 1$ ,  $\rho > n/2$ ,  $\lambda > 2$ , and  $\Omega \in L^2(S^{n-1})$  is a homogeneous function of degree zero satisfying (1.1) and (1.2) with  $\alpha > \frac{5}{2}$ . Then for any smooth function with compact support  $f$ , there exists a positive constant  $0 < C = C_{\delta}$  such that*

$$M_{\delta}^{\sharp, \Delta}(\mu_{\lambda}^{*, \rho}(f))(x) \leq CM(f)(x).$$

PROOF. Let  $f_1, f_2, Q$  and  $B^*$  be the same as in the proof of Lemma 3.9. Then applying Lemma 3.4 with  $\frac{1}{s} = \frac{1}{\delta} - 1$  and Theorem 1.1, similarly to get (3.1), we have

$$\left( \frac{1}{|Q|} \int_Q |\mu_{\lambda}^{*, \rho}(f_1)(y)|^{\delta} dy \right)^{\frac{1}{\delta}} \leq \frac{C}{|8B^*|} \int_{8B^*} |f(y)| dy \leq CM(f)(x).$$

Since  $f \in L^p$  for  $1 < p < \infty$ , and  $\mu_{\lambda}^{*, \rho}$  is  $L^p$  bounded, then  $\mu_{\lambda}^{*, \rho}(f_2)(u) < \infty$  a.e. on  $Q$ , so except a subset  $E$  with measure zero, for all  $u \in Q \setminus E$ ,  $\mu_{\lambda}^{*, \rho}(f_2)(u) < \infty$ . Next we will prove the following fact. For any  $x_0, w \in Q \setminus E$ ,

$$J = |\mu_{\lambda}^{*, \rho}(f_2)(x_0) - \mu_{\lambda}^{*, \rho}(f_2)(w)| \leq CM(f)(x).$$

In fact, similarly to (3.2), we know that

$$\begin{aligned} J &\leq Cr^{\varepsilon} \int_{(8B^*)^c} \frac{|f(z)|}{|z-x_0|^{n+\varepsilon}} dz + Cr^{\varepsilon/2} \int_{(8B^*)^c} \frac{|f(z)|}{|z-x_0|^{n+\varepsilon/2}} dz \\ &\quad + Cr^{\rho - \frac{n}{2}} \int_{(8B^*)^c} \frac{|f(z)|}{|z-x_0|^{\frac{n}{2} + \rho}} dz + C \int_{(8B^*)^c} \frac{|f(z)|}{|z-x_0|^n (\log \frac{|z-x_0|}{r})^{2+\varepsilon}} dz \end{aligned}$$

$$\begin{aligned}
&\leq Cr^\varepsilon \sum_{j=3}^{\infty} \int_{2^j r \leq |z-\bar{x}| < 2^{j+1} r} \frac{|f(z)|}{|z-\bar{x}|^{n+\varepsilon}} dz \\
&\quad + Cr^{\varepsilon/2} \sum_{j=3}^{\infty} \int_{2^j r \leq |z-\bar{x}| < 2^{j+1} r} \frac{|f(z)|}{|z-\bar{x}|^{n+\varepsilon/2}} dz \\
&\quad + Cr^{\rho-\frac{n}{2}} \sum_{j=3}^{\infty} \int_{2^j r \leq |z-\bar{x}| < 2^{j+1} r} \frac{|f(z)|}{|z-\bar{x}|^{\frac{n}{2}+\rho}} dz \\
&\quad + C \sum_{j=3}^{\infty} \int_{2^j r \leq |z-\bar{x}| < 2^{j+1} r} \frac{|f(z)|}{|z-\bar{x}|^n (\log \frac{|z-\bar{x}|}{r})^{2+\varepsilon}} dz \\
&\leq C \sum_{j=3}^{\infty} \frac{1}{2^{j\varepsilon}} \frac{1}{(2^{j+1}r)^n} \int_{|z-\bar{x}| < 2^{j+1} r} |f(z)| dz \\
&\quad + C \sum_{j=3}^{\infty} \frac{1}{2^{j\varepsilon/2}} \frac{1}{(2^{j+1}r)^n} \int_{|z-\bar{x}| < 2^{j+1} r} |f(z)| dz \\
&\quad + C \sum_{j=3}^{\infty} \frac{1}{2^{j(\rho-\frac{n}{2})}} \frac{1}{(2^{j+1}r)^n} \int_{|z-\bar{x}| < 2^{j+1} r} |f(z)| dz \\
&\quad + C \sum_{j=3}^{\infty} \frac{1}{j^{2+\varepsilon}} \frac{1}{(2^{j+1}r)^n} \int_{|z-\bar{x}| < 2^{j+1} r} |f(z)| dz \leq CM(f)(x).
\end{aligned}$$

Let  $C_Q = (\mu_{\lambda}^{*,\rho}(f_2))_Q$ . Since  $|\mu_{\lambda}^{*,\rho}(f)(u) - C_Q| \leq |\mu_{\lambda}^{*,\rho}(f_1)(u)| + |\mu_{\lambda}^{*,\rho}(f_2)(u) - C_Q|$ , then

$$\begin{aligned}
&M_{\delta}^{\sharp, \Delta} \left( \mu_{\lambda}^{*,\rho}(f) \right) (x) \\
&\leq C_{\delta} \sup_{Q \ni x} \left( \frac{1}{|Q|} \int_Q |\mu_{\lambda}^{*,\rho}(f_1)(u)|^{\delta} du \right)^{\frac{1}{\delta}} \\
&\quad + C_{\delta} \sup_{Q \ni x} \left( \frac{1}{|Q|} \int_Q |\mu_{\lambda}^{*,\rho}(f_2) - C_Q|^{\delta} du \right)^{\frac{1}{\delta}} \\
&\leq CM(f)(x) + C \sup_{Q \ni x} \frac{1}{|Q|^2} \int_{Q \setminus E} \int_{Q \setminus E} |\mu_{\lambda}^{*,\rho}(f_2)(u) - \mu_{\lambda}^{*,\rho}(f_2)(v)| dv du \\
&\leq CM(f)(x).
\end{aligned}$$

□

For  $b \in BMO$ , let  $b_k(x) = b(x)$  if  $|b(x)| \leq k$ ,  $b_k(x) = k$  if  $b(x) > k$  and  $b_k(x) = -k$  if  $b(x) < -k$  for  $k = 1, 2, 3, \dots$ . Then  $b_k \in L^{\infty}$  and  $\|b_k\|_* \leq \|b\|_*$ . The following lemma shows that  $\mu_{\lambda, b_k}^{*,\rho}(f)$  can be controlled by the maximal operator.

LEMMA 3.11. [4] Suppose that  $\Omega \in L^2(S^{n-1})$  is a homogeneous function of degree zero. If  $\rho > n/2$ ,  $\lambda > 2$ ,  $\text{supp} f \subset B(0, R)$  and  $|x| \geq 2R$ , then for any  $k$ , there exists a constant  $C$  independent of  $k, f, R$  and  $x$  such that  $\mu_{\lambda, b_k}^{*,\rho}(f)(x) \leq CkMf(x)$ .

REMARK 3.2. By checking the proof of Lemma 3.7 in [4], we find out that the condition “ $\Omega \in L^2(S^{n-1})$  and  $\Omega$  is a homogeneous function of degree zero” is sufficient to get the desired result.

#### 4. Proof of theorems

First, we give the proof of Theorem 2.1 as follows.

PROOF. It is easy to check that  $\mu_{\Omega,S}^{\rho,b} \leq 2^{\lambda n/2} \mu_{\lambda,b}^{*,\rho}$ , so we only give the proof of Theorem 2.1 for  $\mu_{\lambda,b}^{*,\rho}$ .

Since  $\omega \in A_p$ , there is an  $\varepsilon > 0$  such that  $\omega^{1+\varepsilon} \in A_p$ , then by Theorem 1.2, we have

$$(4.1) \quad \|\mu_{\lambda}^{*,\rho}(\phi)\|_{p,\omega^{1+\varepsilon}} \leq \bar{C}_1 \|\phi\|_{p,\omega^{1+\varepsilon}}, \text{ for } \phi \in L^p(\omega^{1+\varepsilon}).$$

Take  $\lambda' = p(1 + \varepsilon)/\varepsilon$  and  $\eta = \min\{C_2/\lambda', C_2(p - 1)/\lambda'\}$ , where  $C_2$  is the constant in Lemma 3.1. Without loss of generality we may assume that  $\|b\|_* < \eta$ . Otherwise we take  $0 < \delta < \eta$  and set  $b_0(x) = \delta b(x)/\|b\|_*$ , then  $\|b_0\|_* = \delta < \eta$  and  $\mu_{\lambda,b}^{*,\rho}(f)(x) = (\|b\|_*/\delta) \mu_{\lambda,b_0}^{*,\rho}(f)(x)$ . Therefore, it suffices to consider  $\mu_{\lambda,b_0}^{*,\rho}(f)(x)$ . By Lemma 3.2, we have  $e^{p(1+\varepsilon)b(x)/\varepsilon} \in A_p$  for  $b(x) \in BMO$  with  $\|b\|_* < \eta$ . Since  $b(x) \in BMO$  implies that  $tb(x) \in BMO$  with  $\|tb\|_* \leq \|b\|_*$  for  $|t| \leq 1$ , by Remark 3.1 we have

$$(4.2) \quad e^{p(1+\varepsilon)tb(x)/\varepsilon} \in A_p, \text{ for } b(x) \in BMO \text{ with } \|b\|_* < \eta \text{ and } |t| \leq 1,$$

where  $[e^{p(1+\varepsilon)tb(x)/\varepsilon}]_{A_p}$  can be dominated by a constant independent of  $t$ .

By (4.2) and Theorem 1.2, we know that for any  $\phi \in L^p(e^{p(1+\varepsilon)b(x)\cos\theta/\varepsilon})$  and  $\theta \in [0, 2\pi]$ ,

$$(4.3) \quad \|\mu_{\lambda}^{*,\rho}(\phi)\|_{p,e^{p(1+\varepsilon)b(x)\cos\theta/\varepsilon}} \leq \bar{C}_2 \|\phi\|_{p,e^{p(1+\varepsilon)b(x)\cos\theta/\varepsilon}},$$

where  $\bar{C}_2$  depends on  $n, p, b, \Omega$ , but not on  $\theta$  and  $\phi$ .

Applying the Stein-Weiss interpolation theorem with change of measure in [13] between (4.1) and (4.3), we have for any  $\theta \in [0, 2\pi]$  and  $\phi \in L^p(\omega e^{pb(x)\cos\theta})$ ,

$$(4.4) \quad \|\mu_{\lambda}^{*,\rho}(\phi)\|_{p,\omega e^{pb(x)\cos\theta}} \leq \bar{C} \|\phi\|_{p,\omega e^{pb(x)\cos\theta}},$$

where  $\bar{C} = \max\{\bar{C}_1, \bar{C}_2\}$  depending only on  $n, p, b, \omega, \Omega$ , but not on  $\theta$  and  $\phi$ .

Denote  $F(y) = e^{y[b(x)-b(z)]}$ ,  $y \in \mathbb{C}$ , then by the analyticity of  $F(y)$  on  $\mathbb{C}$  and the Cauchy integration formula, we have

$$(4.5) \quad b(x) - b(z) = F'(0) = \frac{1}{2\pi i} \int_{|y|=1} \frac{F(y)}{y^2} dy = \frac{1}{2\pi} \int_0^{2\pi} e^{e^{i\theta}(b(x)-b(z))} e^{-i\theta} d\theta.$$

By (4.5) and the Minkowski inequality we get

$$\begin{aligned} \mu_{\lambda,b}^{*,\rho}(f)(x) &= \left( \iint_{\mathbb{R}_+^{n+1}} \left( \frac{t}{t + |x - y|} \right)^{\lambda n} \left| \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{t^\rho} \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} \right. \right. \\ &\quad \left. \left. \times e^{e^{i\theta}(b(x)-b(z))} e^{-i\theta} f(z) dz d\theta \right|^2 \frac{dy dt}{t^{n+1}} \right)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{2\pi} \int_0^{2\pi} \left( \iint_{\mathbb{R}_+^{n+1}} \left( \frac{t}{t+|x-y|} \right)^{\lambda n} \left| \frac{1}{t^\rho} \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} \right. \right. \\ &\quad \left. \left. \times e^{-e^{i\theta}b(z)} f(z) dz \right|^2 \frac{dy dt}{t^{n+1}} \right)^{\frac{1}{2}} |e^{e^{i\theta}b(x)}| |e^{-i\theta}| d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \mu_{\lambda}^{*,\rho}(f^\theta)(x) e^{b(x)\cos\theta} d\theta, \end{aligned}$$

where  $f^\theta(z) = f(z)e^{-e^{i\theta}b(z)}$  for  $\theta \in [0, 2\pi]$ . Then by the Minkowski inequality and (4.4) we get

$$\begin{aligned} \|\mu_{\lambda,b}^{*,\rho}(f)\|_{p,\omega} &\leq \left( \int_{\mathbb{R}^n} \left| \frac{1}{2\pi} \int_0^{2\pi} \mu_{\lambda}^{*,\rho}(f^\theta)(x) e^{b(x)\cos\theta} d\theta \right|^p \omega(x) dx \right)^{\frac{1}{p}} \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \left( \int_{\mathbb{R}^n} |\mu_{\lambda}^{*,\rho}(f^\theta)(x)|^p \omega(x) e^{pb(x)\cos\theta} dx \right)^{\frac{1}{p}} d\theta \\ &\leq C \int_0^{2\pi} \left( \int_{\mathbb{R}^n} |f^\theta(x)|^p \omega(x) e^{pb(x)\cos\theta} dx \right)^{\frac{1}{p}} d\theta = C\|f\|_{p,\omega}. \end{aligned}$$

Thus we complete the proof of Theorem 2.1. □

Now, we turn to the proof of Theorem 2.2.

PROOF. Since  $\mu_{\Omega,S}^{\rho,b} \leq 2^{\lambda n/2} \mu_{\lambda,b}^{*,\rho}$ , we only give the proof of Theorem 2.2 for  $\mu_{\lambda,b}^{*,\rho}$ . Let  $\Phi(t) = t(1 + \log^+ t)$ . We first prove the following inequality

$$\begin{aligned} &\sup_{t>0} \frac{1}{\Phi(1/t)} \omega(\{x \in \mathbb{R}^n : \mu_{\lambda,b}^{*,\rho}(f)(x) > t\}) \\ (4.6) \quad &\leq C_{\|b\|_*} \sup_{t>0} \frac{1}{\Phi(1/t)} \omega(\{x \in \mathbb{R}^n : M^2(f)(x) > t\}). \end{aligned}$$

If  $\|b\|_* = 0$ , (4.6) holds obviously. As below we may assume  $\|b\|_* > 0$ . Denote

$$L_{\delta,b}(f) = \sup_{t>0} \frac{1}{\Phi(1/t)} \omega(\{x \in \mathbb{R}^n : M_\delta^\Delta(\mu_{\lambda,b}^{*,\rho}(f))(x) > t\}),$$

then it is easy to see that

$$(4.7) \quad \sup_{t>0} \frac{1}{\Phi(1/t)} \omega(\{x \in \mathbb{R}^n : \mu_{\lambda,b}^{*,\rho}(f)(x) > t\}) \leq L_{\delta,b}(f).$$

First we will prove that for any  $0 < \delta < 1$  and  $r > 0$ , there is

$$(4.8) \quad L_{\delta,b}(f) \leq C_\delta r L_{\delta,b}(f) + C\Phi\left(\frac{\|b\|_*}{r^{1/\delta}}\right) \sup_{t>0} \frac{1}{\Phi(1/t)} \omega(\{x \in \mathbb{R}^n : M^2(f)(x) > t\}).$$

By Lemma 3.5(1), we get for any  $t > 0$ ,

$$\begin{aligned} &\omega(\{x \in \mathbb{R}^n : M_\delta^\Delta(\mu_{\lambda,b}^{*,\rho}(f))(x) > t\}) \\ &\leq \omega(\{x \in \mathbb{R}^n : M^\Delta([\mu_{\lambda,b}^{*,\rho}(f)]^\delta)(x) > t^\delta, M^{\sharp,\Delta}([\mu_{\lambda,b}^{*,\rho}(f)]^\delta)(x) \leq rt^\delta\}) \\ &\quad + \omega(\{x \in \mathbb{R}^n : M^{\sharp,\Delta}([\mu_{\lambda,b}^{*,\rho}(f)]^\delta)(x) > rt^\delta\}) \end{aligned}$$

$$\begin{aligned} &\leq Cr\omega(\{x \in \mathbb{R}^n : M_\delta^\Delta(\mu_{\lambda,b}^{*,\rho}(f))(x) > t/2^{\frac{1}{\delta}}\}) \\ &\quad + \omega(\{x \in \mathbb{R}^n : M_\delta^{\sharp,\Delta}(\mu_{\lambda,b}^{*,\rho}(f))(x) > r^{\frac{1}{\delta}}t\}). \end{aligned}$$

Since  $0 < \delta < 1$ , we can choose an  $l$  satisfying  $0 < \delta < l < 1$ . By Lemma 3.9,

$$\begin{aligned} &\omega(\{x \in \mathbb{R}^n : M_\delta^{\sharp,\Delta}(\mu_{\lambda,b}^{*,\rho}(f))(x) > r^{\frac{1}{\delta}}t\}) \\ &\leq \omega(\{x \in \mathbb{R}^n : M_l^\Delta(\mu_\lambda^{*,\rho}(f))(x) > r^{\frac{1}{\delta}}t/(2C\|b\|_*)\}) \\ &\quad + \omega(\{x \in \mathbb{R}^n : M^2(f)(x) > r^{\frac{1}{\delta}}t/(2C\|b\|_*)\}). \end{aligned}$$

Note that  $\Phi(ab) \leq \Phi(a)\Phi(b)$  for  $a, b \geq 0$ , and  $\Phi$  is increasing and doubling, so we have

$$\begin{aligned} &\frac{1}{\Phi(1/t)}\omega(\{x \in \mathbb{R}^n : M_\delta^\Delta(\mu_{\lambda,b}^{*,\rho}(f))(x) > t\}) \\ &\leq \frac{Cr}{\Phi(1/t)}\omega(\{x \in \mathbb{R}^n : M_\delta^\Delta(\mu_{\lambda,b}^{*,\rho}(f))(x) > t/2^{\frac{1}{\delta}}\}) \\ &\quad + \frac{1}{\Phi(1/t)}\omega(\{x \in \mathbb{R}^n : M_l^\Delta(\mu_\lambda^{*,\rho}(f))(x) > r^{\frac{1}{\delta}}t/(2C\|b\|_*)\}) \\ &\quad + \frac{1}{\Phi(1/t)}\omega(\{x \in \mathbb{R}^n : M^2(f)(x) > r^{\frac{1}{\delta}}t/(2C\|b\|_*)\}) \\ &\leq C_\delta r L_{\delta,b}(f) + \Phi\left(\frac{2C\|b\|_*}{r^{1/\delta}}\right) \sup_{t>0} \frac{1}{\Phi(1/t)}\omega(\{x \in \mathbb{R}^n : M_l^\Delta(\mu_\lambda^{*,\rho}(f))(x) > t\}) \\ &\quad + \Phi\left(\frac{2C\|b\|_*}{r^{1/\delta}}\right) \sup_{t>0} \frac{1}{\Phi(1/t)}\omega(\{x \in \mathbb{R}^n : M^2(f)(x) > t\}). \end{aligned}$$

Applying Lemma 3.5(2) and Lemma 3.10, we have

$$\begin{aligned} L_{\delta,b}(f) &\leq C_\delta r L_{\delta,b}(f) + C\Phi\left(\frac{\|b\|_*}{r^{1/\delta}}\right) \sup_{t>0} \frac{1}{\Phi(1/t)}\omega(\{x \in \mathbb{R}^n : M^2(f)(x) > t\}) \\ &\quad + C\Phi\left(\frac{\|b\|_*}{r^{1/\delta}}\right) \sup_{t>0} \frac{1}{\Phi(1/t)}\omega(\{x \in \mathbb{R}^n : M_l^{\sharp,\Delta}(\mu_\lambda^{*,\rho}(f))(x) > t\}) \\ &\leq C_\delta r L_{\delta,b}(f) + C\Phi\left(\frac{\|b\|_*}{r^{1/\delta}}\right) \sup_{t>0} \frac{1}{\Phi(1/t)}\omega(\{x \in \mathbb{R}^n : M^2(f)(x) > t\}). \end{aligned}$$

Thus we obtain the result of (4.8). Next we will show that

$$(4.9) \quad L_{\delta,b}(f) \leq C\|b\|_* \sup_{t>0} \frac{1}{\Phi(1/t)}\omega(\{x \in \mathbb{R}^n : M^2(f)(x) > t\}).$$

For  $b \in BMO$ , let  $b_k$  be the same as in Lemma 3.11. Then  $b_k \in L^\infty$  and  $\|b_k\|_* \leq \|b\|_*$ . Since  $f$  is smooth with compact support, we may assume  $\text{supp } f \subset B(0, R)$ . Then by Lemma 3.11, Theorem 2.1, Lemma 3.6 and  $t\Phi(1/t) \geq 1$ , we get

$$\begin{aligned} &\frac{1}{\Phi(1/t)}\omega(\{x \in \mathbb{R}^n : M_\delta^\Delta(\mu_{\lambda,b_k}^{*,\rho}(f))(x) > t\}) \\ &\leq \frac{1}{\Phi(1/t)}\omega(\{x \in \mathbb{R}^n : M(\chi_{B(0,2R)}\mu_{\lambda,b_k}^{*,\rho}(f))(x) > t/2\}) \end{aligned}$$



$$\begin{aligned}
& + \frac{1}{\Phi(1/t)} \omega(\{x \in \mathbb{R}^n : M(\chi_{B^c(0,2R)} \mu_{\lambda, b_k}^{*,\rho}(f))(x) > t/2\}) \\
\leq & \frac{C}{t\Phi(1/t)} \int_{B(0,2R)} |\mu_{\lambda, b_k}^{*,\rho}(f)(x)| \omega(x) dx \\
& + \frac{1}{\Phi(1/t)} \omega(\{x \in \mathbb{R}^n : M^2(f)(x) > t/(Ck)\}) \\
\leq & C\omega(B(0,2R))^{\frac{1}{2}} \left( \int_{B(0,2R)} |\mu_{\lambda, b_k}^{*,\rho}(f)(x)|^2 \omega(x) dx \right)^{\frac{1}{2}} \\
& + \frac{C}{\Phi(1/t)} \int_{\mathbb{R}^n} \Phi\left(\frac{Ck|f(x)|}{t}\right) \omega(x) dx \\
\leq & C\omega(B(0,2R))^{\frac{1}{2}} \left( \int_{B(0,R)} |f(x)|^2 \omega(x) dx \right)^{\frac{1}{2}} + C_k \int_{B(0,R)} \Phi(|f(x)|) \omega(x) dx.
\end{aligned}$$

So  $L_{\delta, b_k}(f) < \infty$ . Then choose an  $r > 0$  with  $r < 1/C_\delta$ , applying (4.8) for  $b_k$ , we have

$$(1 - C_\delta r)L_{\delta, b_k}(f) \leq C_{\delta, r, \|b\|_*} \sup_{t>0} \frac{1}{\Phi(1/t)} \omega(\{x \in \mathbb{R}^n : M^2(f)(x) > t\}).$$

That is

$$(4.10) \quad L_{\delta, b_k}(f) \leq C_{\|b\|_*} \sup_{t>0} \frac{1}{\Phi(1/t)} \omega(\{x \in \mathbb{R}^n : M^2(f)(x) > t\}),$$

where  $C$  is independent of  $k$ . Thus we get (4.9) by letting  $k \rightarrow \infty$  in (4.10). By (4.7) and (4.9), we prove that (4.6) holds.

For  $\beta = 1$ , applying (4.6) and Lemma 3.6, we obtain

$$\begin{aligned}
& \omega(\{x \in \mathbb{R}^n : \mu_{\lambda, b}^{*,\rho}(f)(x) > 1\}) \\
\leq & \sup_{t>0} \frac{1}{\Phi(1/t)} \omega(\{x \in \mathbb{R}^n : \mu_{\lambda, b}^{*,\rho}(f)(x) > t\}) \\
\leq & C_{\|b\|_*} \sup_{t>0} \frac{1}{\Phi(1/t)} \omega(\{x \in \mathbb{R}^n : M^2(f)(x) > t\}) \\
\leq & C_{\|b\|_*} \sup_{t>0} \frac{1}{\Phi(1/t)} \int_{\mathbb{R}^n} \frac{|f(x)|}{t} \left(1 + \log^+ \frac{|f(x)|}{t}\right) \omega(x) dx \\
\leq & C_{\|b\|_*} \int_{\mathbb{R}^n} \Phi(|f(x)|) \omega(x) dx \\
= & C_{\|b\|_*} \int_{\mathbb{R}^n} |f(x)|(1 + \log^+ |f(x)|) \omega(x) dx.
\end{aligned}$$

Then by homogeneity, we complete the proof of Theorem 2.2.  $\square$

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