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SOME CHARACTERIZATION BASED EXPONENTIALITY TESTS AND THEIR BAHADUR EFFICIENCIES

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Abstract. We propose new exponentiality tests based on a recent characterization. We construct integral and Kolmogorov-type statistics, derive their asymptotics and calculate the Bahadur efficiency against some common alternatives. We also obtain a class of locally optimal alternatives for each test. In case of small samples tests are compared with some common exponentiality tests.

1. Introduction

Exponential distribution is one of the most exploited distributions thanks to its numerous applications in queueing theory, reliability theory, survival analysis etc.

Due to its importance on one hand, and to its numerous suitable properties on the other, the exponential distribution probably has the largest number of characterizing theorems. Many books and chapters are devoted to this topic, e.g., [**2**, **3**, **7**, **8**].

One of the main directions in goodness-of-fit testing in recent times have become tests based on characterizations. Such tests for exponential distribution are studied in papers [**1**, **4**, **12**, **16**, **17**], among others. In particular, the Bahadur efficiency of such tests has been considered in, e.g., [**14**, **21**, **26**, **28**].

The characterization we present here is the special case of the characterization from [**19**].

Let X_0, X_1, X_2 be independent and identically distributed non-negative random variables from the distribution whose density $f(x)$ has the Maclaurin expansion for $x > 0$. If

$$X_0 + \min\{X_1, X_2\} \overset{d}{=} \max\{X_1, X_2\} \tag{1.1}$$

then $f(x) = \lambda e^{-\lambda x}$ for some $\lambda > 0$.

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Let $X_1, X_2, \dots, X_n$ be a random sample from non-negative continuous distribution F. We test the composite hypothesis that F belongs to a family of exponential distributions $\mathcal{E}(\lambda)$, where $\lambda > 0$ is an unknown parameter.

We consider two test statistics, namely integral-type and Kolmogorov-type. Both of our statistics are invariant with respect to the scale parameter λ (see [**15**]). Following (1.1) we define two so-called V-empirical distribution functions:

$$G_n(t) = \frac{1}{n^3}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \mathrm{I}\{X_i + \min(X_j, X_k) < t\},$$
$$H_n(t) = \frac{1}{n^2}\sum_{j=1}^{n}\sum_{k=1}^{n} \mathrm{I}\{\max(X_j, X_k) < t\}.$$

Our test statistics can now be defined as

$$I_n = \int_0^\infty (H_n(t) - G_n(t))\, dF_n(t),$$
$$K_n = \sup_{t \geqslant 0} \big|H_n(t) - G_n(t)\big|.$$

We consider large values of our statistics to be significant.

The rest of the paper is organized as follows. In Section 2 we examine the asymptotics of integral-type statistic and find Bahadur efficiencies for a choice of common alternatives. In Section 3 we do the analogous study for the Kolomogorov-type statistic. Some classes of locally optimal alternatives are determined in Section 4 and a power study is conducted in Section 5.

2. Integral-type statistic I_n

The statistic I_n is asymptotically equivalent to U-statistic with symmetric kernel [**15**]

$$\Psi(X_1, X_2, X_3, X_4) = \frac{1}{4!}\sum_{\pi(4)} \big(\mathrm{I}\{\max(X_{i_2}, X_{i_3}) < X_{i_4}\} \\ - \mathrm{I}\{X_{i_1} + \min(X_{i_2}, X_{i_3}) < X_{i_4}\}\big),$$

where $\pi(m)$ is the set of all permutations $\{i_1, i_2, \dots, i_m\}$ of set $\{1, 2, \dots, m\}$.

Its projection on X_1 under null hypothesis is

$$\begin{aligned}\psi(s) &= E(\Psi(X_1, X_2, X_3, X_4)|X_1 = s)\\ &= \frac{1}{4}\big(P\{\max(X_2, X_3) < X_4\} - P\{s + \min(X_2, X_3) < X_4\}\big)\\ &\quad + \frac{1}{2}\big(P\{\max(s, X_3) < X_4\} - P\{X_2 + \min(s, X_3) < X_4\}\big)\\ &\quad + \frac{1}{4}\big(P\{\max(X_3, X_4) < s\} - P\{X_2 + \min(X_3, X_4) < s\}\big).\end{aligned}$$

After some calculations we get

$$\psi(s) = \frac{1}{12} - \frac{1}{8}e^{-2s} - \frac{1}{6}e^{-s}.$$

The expected value of this projection is equal to zero, while its variance is

$$\sigma_I^2 = E(\psi^2(X_1)) = \frac{1}{1080}.$$

Hence this kernel is non-degenerate. Applying Hoeffding's theorem (see [**11**]) we get that the asymptotic distribution of $\sqrt{n}I_n$ is normal $\mathcal{N}(0, \frac{2}{135})$.

2.1. Local Bahadur efficiency. The asymptotic efficiency is an established way of measuring the quality of the tests. The Bahadur efficiency has an advantage that it can also be applied to non-normal test statistics, unlike e.g. its Pitman counterpart. For asymptotically normal test statistics these efficiencies coincide (see [**6**]). The Bahadur efficiency can be expressed as the ratio of the Bahadur exact slope, function describing the rate of exponential decrease for the attained level under the alternative, and double Kullback–Leibler distance between null and alternative distribution. More details on the Bahadur theory can be found in [**5**,**20**].

The Bahadur exact slopes are defined as follows. Suppose that the sequence $\{T_n\}$ of test statistics under alternative converges in probability to some finite function $b(\theta)$. Suppose also that the following large deviations limit

$$\lim_{n\to\infty} n^{-1} \ln P_{H_0}(T_n \geqslant t) = -f(t) \tag{2.1}$$

exists for any t in an open interval I, on which f is continuous and $\{b(\theta), \theta > 0\} \subset I$. Then the Bahadur exact slope is

$$c_T(\theta) = 2f(b(\theta)). \tag{2.2}$$

The exact slopes always satisfy the inequality

$$c_T(\theta) \leqslant 2K(\theta), \quad \theta > 0, \tag{2.3}$$

where $K(\theta)$ is the Kullback–Leibler "distance" between the alternative H_1 and the null hypothesis H_0.

In view of (2.3), the local Bahadur efficiency of the sequence of statistics T_n is naturally defined as

$$e^B(T) = \lim_{\theta\to 0} \frac{c_T(\theta)}{2K(\theta)}. \tag{2.4}$$

The local Bahadur efficiency is measured for alternative distributions that are "close" to the null. Therefore we define the following class of alternatives that are close to exponential.

Let $G(\cdot,\theta)$, $\theta \geqslant 0$, be a family of distributions with densities $g(\cdot,\theta)$, such that $G(\cdot,0)$ is exponential, and the regularity conditions from [**20**, Chapter 6], and [**23**, assumptions ND] hold.

Denote $h(x) = g'_\theta(x,0)$. It is obvious that $\int_0^\infty h(x)\,dx = 0$.

We now calculate the Bahadur exact slope for the test statistic I_n. The functions necessary for its calculations are obtained from the following lemmas.

LEMMA 2.1. *For statistic I_n the function f_I from* (2.1) *is analytic for sufficiently small $\varepsilon > 0$ and it holds*

$$f_I(\varepsilon) = \frac{135}{2}\varepsilon^2 + o(\varepsilon^2), \quad \varepsilon \to 0.$$

PROOF. The kernel Ψ is bounded, centered and non-degenerate. Therefore we can apply the theorem of large deviations for non-degenerate U-statistics (see [**24**]) and get the statement of the lemma. $\square$

LEMMA 2.2. *For a given alternative density $g(x;\theta)$ whose distribution belongs to $\mathcal{G}$, it holds*

$$b(\theta) = 4\theta \int_0^\infty \psi(x)h(x)\,dx + o(\theta), \quad \theta \to 0.$$

PROOF. The proof follows from the general result from [**23**]. $\square$

The double Kullback–Leibler distance from the densities of the class $\mathcal{G}$ to the class of exponential distributions can be for small θ expressed as (see [**25**]):

$$2K(\theta) = \Big(\int_0^\infty h^2(x)e^x dx - \Big(\int_0^\infty xh(x)\,dx\Big)^2\Big) \cdot \theta^2 + o(\theta^2). \tag{2.5}$$

We are going to calculate the local Bahadur efficiency of our test for some common close alternatives. They are:

- a Makeham distribution with the density

$$g(x;\theta) = (1+\theta(1-e^{-x}))\exp(-x-\theta(e^{-x}-1+x)), \quad \theta \in (0,1), \quad x \geqslant 0; \tag{2.6}$$

- a Weibull distribution with the density

$$g(x;\theta) = e^{-x^{1+\theta}}(1+\theta)x^\theta, \quad \theta \in (0,1), \quad x \geqslant 0; \tag{2.7}$$

- a gamma distribution with the density

$$g(x;\theta) = \frac{x^\theta}{\Gamma(\theta+1)}e^{-x}, \quad \theta \in (0,1), \quad x \geqslant 0; \tag{2.8}$$

- an exponential mixture with negative weights (EMNW(β)) [**13**] with density

$$g(x;\theta) = (1+\theta)e^{-x} - \beta\theta e^{\beta x}, \quad \theta \in \Big(0, \frac{1}{\beta-1}\Big], \quad x \geqslant 0; \tag{2.9}$$

- an exponential distribution with resilience parameter (see [**18**]) with density

$$g(x;\theta) = e^{-x}(1-e^{-x})^\theta(1+\theta), \quad \theta \in (0,1), \quad x \geqslant 0. \tag{2.10}$$

In the following example we present calculation of the local Bahadur efficiency.

EXAMPLE 2.1. Let the alternative hypothesis be a Makeham distribution with density function (2.6). The first derivative along θ of its density at $\theta = 0$ is

$$h(x) = -2e^{-2x} + 2e^{-x} - e^{-x}x.$$

Using (2.5) we get that the Kullback–Leibler distance is $K(\theta) = \frac{1}{12}\theta^2 + o(\theta^2)$, $\theta \to 0$. Applying Lemma 2.2 we have

$$\begin{aligned} b_I(\theta) &= 4\theta \int_0^\infty \psi(x)(-2e^{-2x} + 2e^{-x} - e^{-x}x)\,dx + o(\theta) \\ &= \frac{1}{36}\theta + o(\theta) \approx 0.0278\theta + o(\theta), \quad \theta \to 0. \end{aligned}$$

According to Lemma 2.1 and (2.4) we get that local Bahadur efficiency $e_B(I) = 0.625$.

The calculation procedure for alternatives (2.7)–(2.10) is similar. Therefore we omit it here and present the efficiencies in Table 1.

3. Kolmogorov-type statistic K_n

For a fixed $t > 0$ the expression $H_n(t) - G_n(t)$ is a V-statistic with the following kernel:

$$\begin{aligned}\Xi(X_1, X_2, X_3, t) = \frac{1}{2!} \sum_{\pi(3)} \big(&\mathrm{I}\{\max(X_{i_2}, X_{i_3}) < t\} \\ &- \mathrm{I}\{X_{i_1} + \min(X_{i_2}, X_{i_3}) < t\}\big).\end{aligned}$$

The projection of this family of kernels on X_1 under H_0 is

$$\begin{aligned}\xi(s,t) &= E(\Xi(X_1, X_2, X_3, t) | X_1 = s) \\ &= \frac{1}{3}\big(P\{\max(X_2, X_3) < t\} - P\{s + \min(X_2, X_3) < t\}\big) \\ &\quad + \frac{2}{3}\big(P\{\max(s, X_3) < t\} - P\{X_2 + \min(s, X_3) < t\}\big).\end{aligned}$$

After some calculations we get

$$\xi(s,t) = \frac{1}{3}e^{-2t} - \frac{1}{3} + \frac{2}{3}te^{-t} + \frac{1}{3}\mathrm{I}\{s < t\}(e^{-2t+2s} + 1 - 2e^{-t}(1 - s + t)).$$

The variances of these projections $\sigma_K^2(t)$ under H_0 are

$$\sigma_K^2(t) = \frac{4}{27}e^{-4t}(-1 + e^t)^3.$$

The plot of this function is shown in Figure 1.

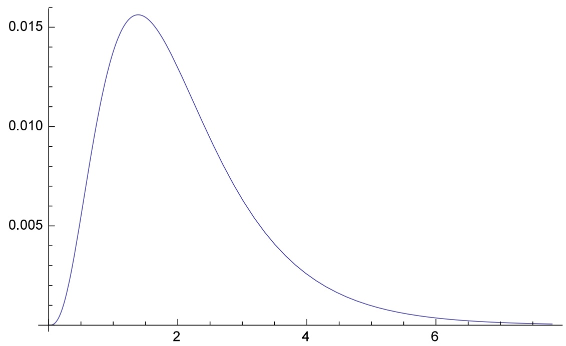

FIGURE 1. Plot of the function $\sigma_K^2(t)$,

The supremum is reached for $t_0 = 1.387$, hence

$$\sigma_K^2 = \sup_{t \geqslant 0} \sigma_K^2(t) = 0.0156.$$

Therefore, our family of kernels $\Xi(X_1, X_2, X_3, t)$ is non-degenerate as defined in [**22**]. Using argumentation of Silverman (see [**27**]) one can show that U-empirical process $\rho_n(t) = \sqrt{n}(H_n(t) - G_n(t))$, $t \geqslant 0$, weakly converges in $D(0,\infty)$ to a centered Gaussian process $\rho(t)$ with complicated, but calculable covariance. Thus, the sequence of our test statistics K_n converges in distribution to the random variable $\sup_{t\geqslant 0}|\rho(t)|$. However, we are unable to find its distribution and the critical values should be determined by using Monte Carlo simulations.

3.1. Local Bahadur efficiency. In this subsection we calculate the local Bahadur efficiency for statistic K_n. Similarly to the case of integral-type statistic we determine the large deviation function f and the limit in probability $b_K(\theta)$ in the following lemmas.

LEMMA 3.1. *For statistic K_n the function f_K from* (2.1) *is analytic for sufficiently small $\varepsilon > 0$ and it holds*

$$f_K(\varepsilon) = \frac{1}{18\sigma_K^2}\varepsilon^2 + o(\varepsilon^2) \approx 3.56\varepsilon^2 + o(\varepsilon^2), \quad \varepsilon \to 0.$$

PROOF. The family of kernels $\{\Xi_4(X_1, X_2, X_3, t),\ t \geqslant 0\}$ is centered and bounded in the sense described in [**22**]. Applying the large deviation theorem for the supremum of the family of non-degenerate U-and V-statistics (see [**22**]), we get the statement of the lemma. $\square$

LEMMA 3.2. *For a given alternative density $g(x;\theta)$ whose distribution belongs to $\mathcal{G}$ holds*

$$b_K(\theta) = 3\theta \sup_{t\geqslant 0}\left|\int_0^\infty \xi(x;t)h(x)\,dx\right| + o(\theta), \quad \theta \to 0.$$

PROOF. Using Glivenko–Cantelli theorem for U-statistics (see [**9**]) we get that the statistic K_n uniformly converges to

$$\begin{aligned} b_K(\theta) &= \sup_{t>0}\left|P_\theta\{\max\{X_1, X_2\} < t\} - P_\theta\{X_1 + \min\{X_2, X_3\} < t\}\right| \\ &= \sup_{t>0}\left|(1-e^{-t})^2 - \int_0^t\int_0^{t-x} g(x;\theta)2(1-G(y;\theta))g(y;\theta)\,dy\,dx\right|. \end{aligned}$$

Denote

$$a(t;\theta) = (1-e^{-t})^2 - \int_0^t\int_0^{t-x} g(x;\theta)2(1-G(y;\theta))g(y;\theta)\,dy\,dx.$$

After some calculations we get that its derivative along θ at $\theta = 0$ is

$$a'_\theta(t;0) = 3\int_0^\infty \xi(x;t)h(x)\,dx.$$

Applying Maclaurin's expansion to the function $a(t;\theta)$ we get the statement of the theorem. $\square$

In the next example we calculate the local Bahadur efficiency in the same manner as we did for integral-type statistic. As before the case of Makeham alternative is presented while for the others the values of efficiencies are given in Table 1.

EXAMPLE 3.1. Let the alternative hypothesis be Makeham distribution with density function (2.6). Using Lemma 3.2 we have

$$b_K(\theta) = \sup_{t \geqslant 0} |a(t,\theta)| = 3\theta \int_0^\infty \xi(x,t)(-2e^{-2x} + 2e^{-x} - e^{-x}x)\,dx + o(\theta), \quad \theta \to 0.$$

The plot of the function function $a'_\theta(t,0)$, is shown in Figure 2.

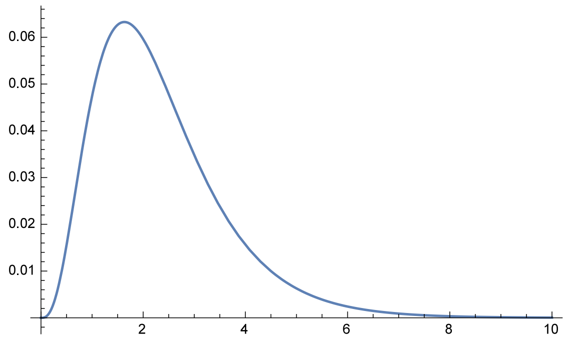

FIGURE 2. Plot of the function $a'_\theta(t,0)$

Supremum of $a(t,\theta)$ is reached at $t_1 = 1.632$, thus $b_K(\theta) = 0.063\theta + o(\theta)$, $\theta \to 0$. Using Lemma 3.1 and equations (2.2) and (2.4), we get that the local Bahadur efficiency in the case of statistic K_n is 0.342.

4. Locally optimal alternatives

The problem of locally optimal alternatives, i.e., the alternatives for which the tests are locally asymptotically optimal in the Bahadur sense, and its importance is described in [**20**]. In the following theorem we give some classes of such alternatives for our two test statistics.

THEOREM 4.1. *Let $g(x;\theta)$ be the density from $\mathcal{G}$ that satisfies condition*

$$\int_0^\infty e^x h^2(x)\,dx < \infty.$$

Alternative densities

$$g(x;\theta) = e^{-x} + e^{-x}\theta(C\psi(x) + D(x-1)), \quad x \geqslant 0, \quad C > 0, \quad D \in \mathbb{R},$$

are for small θ locally asymptotically optimal for the test based on I_n.

Also, alternative densities

$$g(x;\theta) = e^{-x} + e^{-x}\theta(C\xi(x,t_0) + D(x-1)), \quad x \geqslant 0, \quad C > 0, \quad D \in \mathbb{R},$$

where $t_0 = 1.387$, are for small θ locally asymptotically optimal for the test based on K_n.

PROOF. Denote

$$h_0(x) = h(x) - (x-1)e^{-x}\int_0^\infty h(s)s\,ds. \tag{4.1}$$

It is easy to show that this function satisfies the following equalities.

$$\int_0^\infty h_0^2(x)e^x dx = \int_0^\infty e^x h^2(x)\,dx - \Big(\int_0^\infty h(x)x\,dx\Big)^2;$$
$$\int_0^\infty \psi(x)h_0(x)\,dx = \int_0^\infty \psi(x)h(x)\,dx;$$
$$\int_0^\infty \xi(x)h_0(x)\,dx = \int_0^\infty \xi(x)h(x)\,dx.$$

The local asymptotic efficiency for the test based on statistic I_n is

$$\begin{aligned}
e_I^B &= \lim_{\theta\to 0}\frac{c_I(\theta)}{2K(\theta)} = \lim_{\theta\to 0}\frac{2f(b_I(\theta))}{2K(\theta)} = \lim_{\theta\to 0}\frac{2\cdot\frac{135}{2}b_I^2(\theta)}{2K(\theta)} = \lim_{\theta\to 0}\frac{b_I^2(\theta)}{16\sigma_I^2 2K(\theta)}\\
&= \lim_{\theta\to 0}\frac{16\theta^2\big(\int_0^\infty \psi(x)h(x)\,dx\big)^2 + o(\theta^2)}{16\int_0^\infty \psi^2(x)e^{-x}dx\big(\theta^2\big(\int_0^\infty e^x h^2(x)\,dx - \big(\int_0^\infty h(x)x\,dx\big)^2\big) + o(\theta^2)\big)}\\
&= \frac{\big(\int_0^\infty \psi(x)h(x)\,dx\big)^2}{\int_0^\infty \psi^2(x)e^{-x}dx\big(\int_0^\infty e^x h^2(x)\,dx - \big(\int_0^\infty h(x)x\,dx\big)^2\big)}\\
&= \frac{\big(\int_0^\infty \psi(x)h_0(x)\,dx\big)^2}{\int_0^\infty \psi^2(x)e^{-x}dx\int_0^\infty h_0^2(x)e^x\,dx}.
\end{aligned}$$

The alternative is locally optimal if $e_I^B = 1$. From the Cauchy–Schwarz inequality we get that this holds if and only if $h_0(x) = C\psi(x)e^{-x}$. Inserting that in (4.1) we obtain $h(x)$. The densities from the statement of the theorem have the same $h(x)$, hence the proof of the first part of the theorem is completed.

The local asymptotic efficiency of the test based on statistic K_n is

$$\begin{aligned}
e_K &= \lim_{\theta\to 0}\frac{c_K(\theta)}{2K(\theta)} = \lim_{\theta\to 0}\frac{2f(b_K(\theta))}{2K(\theta)} = \lim_{\theta\to 0}\frac{b_K^2(\theta)}{9\sigma_K(t_0)^2 2K(\theta)}\\
&= \lim_{\theta\to 0}\frac{9\theta^2\sup_{t\geqslant 0}\big(\int_0^\infty \xi(x,t)h(x)\,dx\big)^2 + o(\theta^2)}{9\sup_{t\geqslant 0}\int_0^\infty \xi^2(x,t)e^{-x}dx\big(\theta^2\big(\int_0^\infty e^x h^2(x)\,dx - \big(\int_0^\infty h(x)x\,dx\big)^2\big) + o(\theta^2)\big)}\\
&= \frac{\sup_{t\geqslant 0}\big(\int_0^\infty \xi(x,t)h(x)\,dx\big)^2}{\sup_{t\geqslant 0}\int_0^\infty \xi^2(x,t)e^{-x}dx\big(\int_0^\infty e^x h^2(x)\,dx - \big(\int_0^\infty h(x)x\,dx\big)^2\big)}\\
&= \frac{\sup_{t\geqslant 0}\big(\int_0^\infty \xi(x,t)h_0(x)\,dx\big)^2}{\int_0^\infty \xi^2(x,t)e^{-x}dx\int_0^\infty h_0^2(x)e^x\,dx}.
\end{aligned}$$

Once again, using the Cauchy-Schwarz inequality we have that $e_K = 1$ if and only if $h_0(x) = C\xi(x,t_0)e^{-x}$. Inserting that in (4.1) we obtain $h(x)$. The densities from the statement of the theorem have the same $h(x)$, hence the proof is completed. □

5. Discussion

The local Bahadur efficiencies for our test statistics are presented in Table 1. We may notice that the integral type statistic is more efficient than the Kolmogorov type statistic. Comparing with other exponentiality tests based on characterizations (see e.g. [**14**,**28**]) we conclude that the I_n statistic is very efficient and K_n is reasonably efficient (compared to some other Kolmogorov-type tests based on characterizations).

Table 1. Local Bahadur efficiency for the statistic I_n and K_n

Alternative	Efficiency I_n	Efficiency K_n
Weibull	0.750	0.277
Makeham	0.625	0.342
Gamma	0.796	0.161
EMNW(3)	0.844	0.400
LM	0.749	0.272
GED	0.420	0.247
Resilience	0.803	0.256

6. Power comparison

In this section we present empirical powers of our tests for sample sizes $n = 20$ and $n = 50$ for some common distributions and compare results with other tests for exponentiality which can be found in [**10**]. The powers are shown in Tables 2 and 3. The labels used are identical to the ones in [**10**]. It can be noticed that for sample size $n = 20$ in majority of the cases the statistic I_n is the most powerful, and reasonably competitive for $n = 50$. The powers of statistic K_n are satisfactory. However there are few cases where the powers of both our tests are not suitable.

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TABLE 2. Percentage of significant samples for different exponentiality tests $n = 20$, $\alpha = 0.05$

Alternative	EP	$\overline{KS}$	$\overline{CM}$	ω^2	KS	KL	S	CO	I	K
W(1.4)	36	35	35	34	28	29	35	37	**45**	35
Γ(2)	48	46	47	47	40	44	46	54	**61**	47
HN	21	24	22	21	18	16	21	19	**24**	23
U	66	72	70	66	52	61	70	50	66	**81**
CH(0.5)	63	47	61	61	56	77	63	80	17	18
CH(1.0)	15	18	16	14	13	11	15	13	**28**	17
CH(1.5)	84	79	83	79	67	76	84	81	17	18
LF(2.0)	28	32	30	28	24	23	29	25	**32**	30
LF(4.0)	42	44	43	41	34	34	42	37	**47**	42
EW(0.5)	15	18	16	14	13	11	15	13	**18**	17
EW(1.5)	45	48	47	43	35	37	46	37	**47**	46

TABLE 3. Percentage of significant samples for different exponentiality tests $n = 50$, $\alpha = 0.05$

Alternative	EP	$\overline{KS}$	$\overline{CM}$	ω^2	KS	KL	S	CO	I	K
W(1.4)	80	71	77	75	64	72	79	82	**83**	67
Γ(2)	91	86	90	90	83	93	90	96	**96**	83
HN	54	50	53	48	39	37	54	45	49	44
U	98	99	99	98	93	97	99	91	**96**	**100**
CH(0.5)	94	90	94	95	92	99	94	99	33	32
CH(1.0)	38	36	37	32	26	23	38	30	33	31
CH(1.5)	100	100	100	100	98	100	100	100	32	31
LF(2.0)	69	65	69	64	53	54	69	60	66	58
LF(4.0)	87	82	87	83	72	75	87	80	84	76
EW(0.5)	38	36	37	32	26	23	38	30	33	32
EW(1.5)	90	88	90	86	75	79	90	78	82	82

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