

## LOGARITHMIC BLOCH SPACE AND ITS PREDUAL

Miroslav Pavlović

ABSTRACT. We consider the space  $\mathfrak{B}_{\log^\alpha}^1$ , of analytic functions on the unit disk  $\mathbb{D}$ , defined by the requirement  $\int_{\mathbb{D}} |f'(z)| \phi(|z|) dA(z) < \infty$ , where  $\phi(r) = \log^\alpha(1/(1-r))$  and show that it is a predual of the “log $^\alpha$ -Bloch” space and the dual of the corresponding little Bloch space. We prove that a function  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  with  $a_n \downarrow 0$  is in  $\mathfrak{B}_{\log^\alpha}^1$  iff  $\sum_{n=0}^{\infty} \log^\alpha(n+2)/(n+1) < \infty$  and apply this to obtain a criterion for membership of the Libera transform of a function with positive coefficients in  $\mathfrak{B}_{\log^\alpha}^1$ . Some properties of the Cesàro and the Libera operator are considered as well.

### 1. Introduction and some results

Let  $H(\mathbb{D})$  denote the space of all functions analytic in the unit disk  $\mathbb{D}$  of the complex plane. Endowed with the topology of uniform convergence on compact subsets of  $\mathbb{D}$ , the class  $H(\mathbb{D})$  becomes a complete locally convex space. In this paper we are concerned with the predual of the space  $\mathfrak{B}_{\log^\alpha}$ ,  $\alpha \in \mathbb{R}$ ,

$$\mathfrak{B}_{\log^\alpha} = \left\{ f \in H(\mathbb{D}) : |f'(z)| = \mathcal{O}\left((1-|z|)^{-1} \log^\alpha \frac{2}{1-|z|}\right) \right\}.$$

The norm in  $\mathfrak{B}_{\log^\alpha}$  is defined by

$$\|f\|_{\mathfrak{B}_{\log^\alpha}} = |f(0)| + \sup_{z \in \mathbb{D}} |f'(z)| (1-|z|) \log^{-\alpha} \frac{1}{1-|z|}.$$

The subspace,  $\mathfrak{b}_{\log^\alpha}$ , of  $\mathfrak{B}_{\log^\alpha}$  is defined by replacing “ $\mathcal{O}$ ” with “ $o$ ”. It will be proved:

(A) The dual of  $\mathfrak{b}_{\log^\alpha}$  is isomorphic to  $\mathfrak{B}_{\log^\alpha}^1$ ,

$$(1.1) \quad \mathfrak{B}_{\log^\alpha}^1 = \left\{ f : \|f\|_{\mathfrak{B}_{\log^\alpha}^1} = |f(0)| + \int_{\mathbb{D}} |f'(z)| \log^\alpha \frac{2}{1-|z|} dA(z) < \infty \right\},$$

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and the dual of  $\mathfrak{B}_{\log^\alpha}^1$  is isomorphic to  $\mathfrak{B}_{\log^\alpha}$ , in both cases with respect to the bilinear form

$$(1.2) \quad \langle f, g \rangle = \lim_{r \uparrow 1} \sum_{n=0}^{\infty} \hat{f}(n) \hat{g}(n) r^{2n}.$$

(In (1.1)  $dA$  stands for the normalized Lebesgue measure on  $\mathbb{D}$ .) This extends the well-known result on the Bloch space and the little Bloch space  $\mathfrak{b} := \mathfrak{b}_{\log^0}$ .

These spaces are Banach spaces, and the space  $\mathfrak{b}_{\log^\alpha}$  coincides with the closure in  $\mathfrak{B}_{\log^\alpha}$  of the set of all polynomials. The space  $\mathfrak{B}_{\log} := \mathfrak{B}_{\log^1}$  occurs naturally in the study of pointwise multipliers on the usual Bloch space  $\mathfrak{B} := \mathfrak{B}_{\log^0}$  (see [3]).

One of interesting properties of  $\mathfrak{B}_{\log^\alpha}^1$  is described in the following theorem:

**THEOREM 1.1.** *Let  $f(z) = \sum_{n=0}^{\infty} a_n z^n$ , where  $\{a_n\}$  is a nonincreasing sequence, of real numbers, tending to zero. Let  $\alpha \geq -1$ . Then  $f$  belongs to  $\mathfrak{B}_{\log^\alpha}^1$  if and only if*

$$(1.3) \quad S_\alpha(f) := \sum_{n=0}^{\infty} \frac{a_n \log^\alpha(n+2)}{n+1} < \infty.$$

Moreover, there is a constant  $C$  independent of  $\{a_n\}$  such that  $S_\alpha(f)/C \leq \|f\|_{\mathfrak{B}_{\log^\alpha}^1} \leq CS_\alpha(f)$ .

**PROOF.** See Section 4. □

In the case  $\alpha = 0$ , this assertion is proved in [17]. We can take  $a_n$  to be the coefficients of the Libera transform of a function with positive coefficients. Namely, if  $g(z) = \sum_{n=0}^{\infty} \hat{g}(n) z^n$  and

$$(1.4) \quad \sum_{n=0}^{\infty} \frac{|\hat{g}(n)|}{n+1} < \infty,$$

then the Libera transform  $\mathcal{L}g$  of  $g$  is well defined as

$$(1.5) \quad \mathcal{L}g(z) = \frac{1}{1-z} \int_z^1 f(\zeta) d\zeta = \sum_{n=0}^{\infty} z^n \sum_{k=n}^{\infty} \frac{\hat{g}(k)}{k+1}$$

(see, e.g., [12]). If  $\hat{g} \geq 0$ , then condition (1.4) is also necessary for the existence of the integral in (1.8): take  $z = 0$  to conclude that (1.8) implies the convergence of the integral

$$\int_0^1 g(t) dt = \sum_{n=0}^{\infty} \frac{\hat{g}(n)}{n+1}.$$

Then, as an application of Theorem 1.1 we get

**THEOREM 1.2.** *Let  $\alpha > -1$ , let  $g \in H(\mathbb{D})$ , and  $\hat{g} \geq 0$ . Then  $\mathcal{L}g$  is in  $\mathfrak{B}_{\log^\alpha}^1$  if and only if*

$$K_\alpha(g) := \sum_{n=0}^{\infty} \frac{\hat{g}(n) \log^{\alpha+1}(n+2)}{n+1} < \infty.$$

We have  $K_\alpha(g)/C \leq \|\mathcal{L}g\|_{\mathfrak{B}_{\log^\alpha}^1} \leq CK_\alpha(g)$ .

PROOF. See Section 4.  $\square$

In the general case, the integral in (1.5) need not exist, but it certainly exists if  $g \in H(\overline{\mathbb{D}})$ , which means that  $g$  is analytic in a neighborhood of the closed disk. By using Theorem 1.1 we shall prove that  $\overline{\mathcal{L}}$  cannot be extended to a bounded operator from  $\mathfrak{B}_{\log^\alpha}^1$  to  $H(\mathbb{D})$ , if  $\alpha < 0$ . In the case  $\alpha \geq 0$ , every function  $g \in \mathfrak{B}_{\log^\alpha}^1$  satisfies (1.4), whence  $\mathcal{L}$  is well defined, and we will show that  $\mathcal{L}$  maps this space into  $\mathfrak{B}_{\log^{\alpha-1}}^1$ , when  $\alpha > 0$ . If  $\alpha = 0$  we need a sort of “iterated” logarithmic space.

**Cesàro operator.** The dual of  $H(\mathbb{D})$  is equal to  $H(\overline{\mathbb{D}})$ , where “ $g \in H(\overline{\mathbb{D}})$ ” means that  $g$  is holomorphic in a neighborhood of  $\overline{\mathbb{D}}$  (depending on  $g$ ). The duality pairing is given

$$(1.6) \quad \langle f, g \rangle = \sum_{n=0}^{\infty} \hat{f}(n) \hat{g}(n),$$

where  $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n \in H(\mathbb{D})$  and  $g(z) = \sum_{n=0}^{\infty} \hat{g}(n) z^n \in H(\overline{\mathbb{D}})$ , and the series is absolutely convergent (see, e.g., [8]). The Cesàro operator is defined on  $H(\mathbb{D})$  as

$$(1.7) \quad \mathcal{C}f(z) = \sum_{n=0}^{\infty} z^n \frac{1}{n+1} \sum_{k=0}^n a_k, \quad f \in H(\mathbb{D}).$$

It is easy to verify that the adjoint of  $\mathcal{C} : H(\mathbb{D}) \mapsto H(\mathbb{D})$  is equal to  $\overline{\mathcal{L}} : H(\overline{\mathbb{D}}) \mapsto H(\overline{\mathbb{D}})$ , under the pairing (1.6), and vice versa (see, e.g., [12]).

The operators  $\mathcal{C}$  and  $\overline{\mathcal{L}}$  acting on  $H^p$  spaces were first studied by Siskakis in 1987. In [21] he proved that  $\mathcal{C}$  is bounded on  $H^p$  for  $1 < p < \infty$ , and that  $\overline{\mathcal{L}}$  can be extended to a bounded operator on  $H^p$ ,  $1 < p < \infty$ , and obtained some results on their spectra and norms. A few years later he proved the boundedness of the Cesàro operator on  $H^1$  ([22]), while Miao proved its boundedness on  $H^p$  for  $0 < p < 1$  ([10]). A short proof of the boundedness of  $\mathcal{C}$  on  $H^p$ ,  $0 < p < \infty$ , as well as a stronger result, can be also found in Nowak [11]. However,  $H^\infty$  is not mapped into itself by  $\mathcal{C}$  (see [4]). If we write (1.7) as

$$z\mathcal{C}f(z) = \int_0^z \frac{f(\zeta)}{1-\zeta} d\zeta,$$

and hence

$$(z\mathcal{C}f(z))' = \frac{f(z)}{1-z},$$

we conclude that  $\mathcal{C}$  maps  $H^\infty$  into the Bloch space (see [4]).

On the other hand, by using the inequality

$$|f(z)| = \mathcal{O}\left(\log \frac{2}{1-|z|}\right), \quad f \in \mathfrak{B},$$

and the analogous inequality for  $f \in \mathfrak{b}$  (replace “ $\mathcal{O}$ ” with “ $o$ ”), we get:

(B) The operator  $\mathcal{C}$  maps the space  $\mathfrak{B}$  into  $\mathfrak{B}_{\log}$ , and  $\mathfrak{b}$  into  $\mathfrak{b}_{\log}$ .

One of our aims is to generalize this assertion to some other values of  $\alpha$  and then use assertion **(A)** together with the duality between  $\mathcal{C}$  and  $\overline{\mathcal{L}}$  to obtain an alternative proof of some results on the action of  $\mathcal{L}$  from  $\mathfrak{B}_{\alpha+1}^1$  to  $\mathfrak{B}_\alpha^1$ , where

$$(1.8) \quad \mathcal{L}f(z) = \int_0^1 f(t + (1-t)z) dt.$$

In particular we have:

(C) The operator  $\mathcal{L}$  is well defined on  $\mathfrak{B}_{\log}^1$  and maps it into  $\mathfrak{B}^1$ .

It should be noted that: (a)  $\mathfrak{B}^1 \subsetneq H^1$ ; (b)  $\mathcal{L}$  does not map  $\mathfrak{B}^1$  into  $H^1$  (see [17]); and (c)  $\mathcal{L}$  maps  $\mathfrak{B}$  into BMOA [12], which improves an earlier result, namely that  $\mathcal{L}$  maps  $\mathfrak{B}$  into  $\mathfrak{B}$  ([5, 24]).

The formula (1.8) is obtained from (1.5) by integrating over the straight line joining  $z$  and 1. A sufficient (not necessary [18]) condition for the possibility of such integration is (1.4) ( $g = f$ ).

In proving some of our results, in particular assertions (A) and (B), we use a sequence of polynomials constructed in [6] (see also [7] and [16]) to decompose the space into a sum which resembles a sum of finite-dimensional spaces (see Section 3).

## 2. Some more results

Some elementary facts concerning the cases when  $\mathcal{L}f$  is well defined are collected in the following theorem, where

$$\ell_{-1}^1 = \left\{ g \in H(\mathbb{D}) : \|g\|_{\ell_{-1}^1} = \sum_{n=0}^{\infty} \frac{|\hat{g}(n)|}{n+1} < \infty \right\}.$$

**THEOREM 2.1.** *Let  $\alpha \in \mathbb{R}$ . Then:*

- (a)  $\mathfrak{B}_{\log^\alpha}^1 \subset \ell_{-1}^1$  for all  $\alpha$ ;
- (b)  $\mathfrak{B}_{\log^\alpha}^1 \subset \ell_{-1}^1$  if and only if  $\alpha \geq 0$ ;
- (c) if  $\alpha < 0$ , then  $\overline{\mathcal{L}}$  cannot be extended to a continuous operator from  $\mathfrak{B}_{\log^\alpha}^1$  to  $H(\mathbb{D})$ .

**PROOF.** See Section 4. □

**REMARK 2.1.** The inclusions in (a) and (b) are continuous. Assertion (c) says much more than simply that  $\mathfrak{B}_{\log^\alpha}^1 \not\subset \ell_{-1}^1$ .

In the context of the action of  $\mathcal{C}$  and  $\mathcal{L}$  some new spaces occur: the space  $\mathfrak{B}_{\log\log}$  is defined by the requirement

$$|f'(z)| = \mathcal{O}\left(\log \log \frac{4}{1-|z|}\right),$$

the space  $\mathfrak{b}_{\log\log}$  defined by replacing “ $\mathcal{O}$ ” with “ $o$ ”, and the space  $\mathfrak{B}_{\log\log}^1$  defined by

$$\int_{\mathbb{D}} |f'(z)| \log \log \frac{4}{1-|z|} dA(z) < \infty.$$

Our next result is

THEOREM 2.2. (a) If  $\alpha > -1$ , then  $\mathcal{C}$  maps the space  $\mathfrak{B}_{\log^\alpha}$ , resp.  $\mathfrak{b}_{\log^\alpha}$ , into  $\mathfrak{B}_{\log^{\alpha+1}}$ , resp.  $\mathfrak{b}_{\log^{\alpha+1}}$ .

(b)  $\mathcal{C}$  maps the space  $\mathfrak{B}_{\log^{-1}}$ , resp.  $\mathfrak{b}_{\log^{-1}}$ , into  $\mathfrak{B}_{\log g}$ , resp.  $\mathfrak{b}_{\log g}$ .

PROOF. See Section 5. □

REMARK 2.2. If  $f \in \mathfrak{B}_{\log^\alpha}$  and  $\alpha < -1$ , then, as it can easily be shown,  $f \in A(\mathbb{D})$ , where  $A(\mathbb{D})$  is the disk-algebra, i.e., the subset of  $H^\infty$  consisting of those  $f$  which have a continuous extension to the closed disk. Moreover, the modulus of continuity of the boundary function  $f_*(\zeta)$ ,  $\zeta \in \partial\mathbb{D}$ , satisfies the condition

$$\omega(f_*, t) = \mathcal{O}\left(t\left(\log \frac{2}{t}\right)^{\alpha+1}\right), \quad t \downarrow 0.$$

This follows from the inequality

$$\omega(f_*, t) \leq C \int_{1-t}^1 M_\infty(r, f') dr,$$

see [15, Theorem 2.2]. It should be noted that the modulus of continuity of  $f_*$  is “proportional” to that of  $f(z)$ ,  $z \in \mathbb{D}$ , see [23, 19].

Concerning the Libera operator we shall prove, besides Theorem 2.1(c), the following facts.

THEOREM 2.3. (a) If  $\alpha > 0$ , then  $\mathcal{L}$  is well defined on  $\mathfrak{B}_{\log^\alpha}^1$  and maps this space to  $\mathfrak{B}_{\log^{\alpha-1}}^1$ .

(b)  $\mathcal{L}$  is well defined on  $\mathfrak{B}_{\log g}^1$  and maps this space into  $\mathfrak{B}_{\log^{-1}}^1$ .

(c)  $\mathcal{L}$  is well defined on  $\mathfrak{B}^1$  and maps it into  $\mathfrak{B}_\alpha^1$  for all  $\alpha < -1$ .

PROOF. See Section 5. □

THEOREM 2.4. Let  $\alpha \in \mathbb{R}$ . Then the dual of  $\mathfrak{b}_{\log^\alpha}$ , resp.  $\mathfrak{B}_{\log^\alpha}^1$ , is isomorphic to  $\mathfrak{B}_{\log^\alpha}^1$ , resp.  $\mathfrak{B}_{\log^\alpha}$  under the pairing (1.2). Similarly, the dual of  $\mathfrak{b}_{\log g}$ , resp.  $\mathfrak{B}_{\log g}^1$ , is isomorphic to  $\mathfrak{B}_{\log g}^1$ , resp.  $\mathfrak{B}_{\log g}$ , under the same pairing.

PROOF. See Section 6. □

REMARK 2.3. The phrase “the dual of  $X$  is isomorphic to  $Y$  under the pairing (1.2)” means that if  $f \in X$  and  $g \in Y$ , then the limit in (1.2) exists and the functional  $\Phi(f) = \langle f, g \rangle$  is bounded on  $X$ ; and on the other hand, if  $\Phi \in X^*$ , then there exists  $g \in Y$  such that  $\Phi(f) = \langle f, g \rangle$ , and moreover, there exists a constant  $C$  independent of  $g$  such that  $\|g\|_Y/C \leq \|\Phi\| \leq C\|g\|_Y$ .

As an application of Theorems 2.2, 2.3, and 2.4, one can prove the following fact.

THEOREM 2.5. Let  $\alpha > 0$ . Then the adjoint (with respect to (1.2)) of the operator  $\mathcal{L} : \mathfrak{B}_{\log^\alpha}^1 \mapsto \mathfrak{B}_{\log^{\alpha-1}}^1$  is equal to  $\mathcal{C} : \mathfrak{B}_{\log^{\alpha-1}} \mapsto \mathfrak{B}_{\log^\alpha}$ . The adjoint of the operator  $\mathcal{C} : \mathfrak{b}_{\log^{\alpha-1}} \mapsto \mathfrak{b}_{\log^\alpha}$  is equal to  $\mathcal{L} : \mathfrak{B}_{\log^\alpha}^1 \mapsto \mathfrak{B}_{\log^{\alpha-1}}^1$ . The analogous assertions hold in the case when  $\alpha = 0$ .

### 3. Decompositions

In [6], a sequence  $\{V_n\}_0^\infty$  was constructed in the following way. Let  $\omega$  be a  $C^\infty$ -function on  $\mathbb{R}$  such that

- (1)  $\omega(t) = 1$  for  $t \leq 1$ ,
- (2)  $\omega(t) = 0$  for  $t \geq 2$ ,
- (3)  $\omega$  is decreasing and positive on the interval  $(1, 2)$ .

Let  $\varphi(t) = \omega(t/2) - \omega(t)$ , and let  $V_0(z) = 1 + z$ , and, for  $n \geq 1$ ,

$$V_n(z) = \sum_{k=0}^{\infty} \varphi(k/2^{n-1})z^k = \sum_{k=2^{n-1}}^{2^{n+1}-1} \varphi(k/2^{n-1})z^k.$$

The polynomials  $V_n$  have the following properties:

$$(3.1) \quad g(z) = \sum_{n=0}^{\infty} V_n * g(z), \text{ for } g \in H(\mathbb{D});$$

$$(3.2) \quad \|V_n * g\|_p \leq C \|g\|_p, \text{ for } g \in H^p, p > 0;$$

$$(3.3) \quad \|V_n\|_p \asymp 2^{n(1-1/p)}, \text{ for all } p > 0,$$

where  $*$  denotes the Hadamard product. Here  $\|h\|_p$  denotes the norm in the  $p$ -Hardy space  $H^p$ ,

$$\|h\|_p = \sup_{0 < r < 1} \left( \frac{1}{2\pi} \int_0^{2\pi} |h(re^{i\theta})| d\theta \right)^{1/p} = \sup_{0 < r < 1} M_p(r, g).$$

We need additional properties.

LEMMA 3.1. *Let  $P(z) = \sum_{k=m}^j a_k z^k$ ,  $m < j$ . Then*

$$r^j \|P\|_p \leq M_p(r, P) \leq r^m \|P\|_p, \quad 0 < r < 1.$$

When applied to the polynomial  $P = V_n * g'$ , this gives:

$$(3.4) \quad r^{2^{n+1}-1} \|V_n * g'\|_p \leq M_p(r, V_n * g') \leq r^{2^{n-1}-1} \|V_n * g'\|_p \text{ for } n \geq 1.$$

Another inequality will be used (see [16, Exercise 7.3.5]):

$$(3.5) \quad 2^{n-1} \|V_n * g\|_p / C \leq \|V_n * g'\|_p \leq C 2^{n+1} \|V_n * g\|_p \text{ for } n \geq 1,$$

where  $C$  is a constant independent of  $n$  and  $g$ .

THEOREM 3.1. *Let  $\alpha \in \mathbb{R}$ , and  $f \in H(\mathbb{D})$ . Then:*

- (i)  $f \in \mathfrak{B}_{\log^\alpha}$  if and only if  $\sup_{n \geq 0} (n+1)^{-\alpha} \|V_n * f\|_\infty < \infty$ .
- (ii)  $f \in \mathfrak{b}_{\log^\alpha}$  if and only if  $\lim_{n \rightarrow \infty} (n+1)^{-\alpha} \|V_n * f\|_\infty = 0$ .
- (iii)  $f \in \mathfrak{B}_{\log^\alpha}^1$  if and only if  $\sum_{n=0}^{\infty} (n+1)^\alpha \|V_n * f\|_1 < \infty$ .

Moreover, the inequality

$$C^{-1} \|f\|_{\mathfrak{B}_{\log^\alpha}} \leq \sup_{n \geq 0} (n+1)^{-\alpha} \|V_n * g\|_\infty \leq C^{-1} \|f\|_{\mathfrak{B}_{\log^\alpha}}$$

holds, where  $C$  is independent of  $f$ . The analogous inequality holds in the case of (iii) as well.

For the proof we need the following reformulation of [9, Proposition 4.1].

LEMMA 3.2. *Let  $\varphi$  be a continuous function on the interval  $(0, 1]$  such that  $\varphi(x)/x^\gamma$  ( $0 < x < 1$ ) is nonincreasing, and  $\varphi(x)/x^\beta$  ( $0 < x < 1$ ) is nondecreasing, where  $\beta$  and  $\gamma$  are positive constants independent of  $x$ .<sup>1</sup> Let*

$$F_1(r) = (1-r)^{-1/q} \varphi(1-r) \sup_{n \geq 1} \lambda_n r^{2^{n+1}-1},$$

$$F_2(r) = (1-r)^{1/q} \varphi(1-r) \sum_{n=0}^{\infty} \lambda_n r^{2^{n-1}-1},$$

where  $\lambda_n \geq 0$ ,  $0 < q \leq \infty$ . If  $F = F_1$  or  $F = F_2$ , then

$$C^{-1} \|F\|_{L^q(0,1)} \leq \|\{\varphi(2^{-n})\lambda_n\}\|_{\ell^q} \leq C \|F\|_{L^q(0,1)}.$$

PROOF OF THEOREM 3.1. CASE (i). Let  $\varphi(x) = x \log^{-\alpha}(2/x)$ , and  $q = \infty$ . That  $\varphi$  is normal follows from the relation

$$\lim_{x \downarrow 0} \frac{x\varphi'_\alpha(x)}{\varphi_\alpha(x)} = 1.$$

Let  $\lambda_n = 2^n \|V_n * f\|_\infty$ . By (3.1), (3.2), (3.4), and (3.5), we have

$$C^{-1} |\hat{f}(1)| + C^{-1} \sup_{n \geq 1} \lambda_n r^{2^{n+1}-1} \leq M_\infty(r, f') \leq C |\hat{f}(1)| + C \sum_{n=1}^{\infty} \lambda_n r^{2^{n-1}-1}.$$

Hence, by Lemma 3.2, we obtain the desired result.

CASE (ii). In this case we can proceed in two ways:

1° Modify the proof of Lemma 3.2 to get the inequalities

$$C^{-1} \|F\|_{C_0[0,1]} \leq \|\{\varphi(2^{-n})\lambda_n\}\|_{\mathfrak{c}_0} \leq C \|F\|_{C_0[0,1]},$$

where  $C_0[0, 1] = \{u \in C[0, 1] : u(1) = 0\}$  and  $\mathfrak{c}_0$  is the set of the sequences tending to zero.

2° Consider the spaces  $\mathfrak{b}_{\log^\alpha} \subset \mathfrak{B}_{\log^\alpha}$  and  $X = \{f : \|V_n * f\| = o((n+1)^\alpha)\}$ , which is, by assertion (i) and its proof, a subspace of a space  $Y$  isomorphic to  $\mathfrak{B}_{\log^\alpha}$ . It is not hard to show that the polynomials are dense in both  $\mathfrak{b}_{\log^\alpha}$  and  $X$ . This proves (ii).

CASE (iii). In this case we use the function  $\varphi(x) = x \log^\alpha(2/x)$ , let  $q = 1$ , and then proceed as in the proof of (i). The details are omitted. This concludes the proof of the theorem.  $\square$

REMARK 3.1. By choosing  $\phi(x) = x \log \log(\frac{4}{x})$ , then we can conclude that Theorem 3.1 remains true if  $\log^\alpha$ , resp.  $(n+1)^\alpha$ , are replaced with  $\log \log$ , resp.  $\log(n+2)$ .

<sup>1</sup>Following Shields and Williams [20], we call such a function **normal**.

#### 4. Functions with decreasing coefficients

PROOF OF THEOREM 1.1. Assuming that (1.3) holds, we want to prove that

$$\|f\|_{\mathfrak{B}_{\log^\alpha}^1} \leq C a_0 + C \sum_{n=1}^{\infty} a_{2^{n-1}} (n+1)^\alpha.$$

According to Theorem 3.1 and its proof, we have

$$C^{-1} \|f\|_{\mathfrak{B}_{\log^\alpha}^1} \leq a_0 + \sum_{n=1}^{\infty} (n+1)^\alpha \|V_n * f\|_1 \leq C \|f\|_{\mathfrak{B}_{\log^\alpha}^1}.$$

Let  $n \geq 1$ ,  $m = 2^{n-1}$ , and  $Q_k = \sum_{j=m}^k \varphi(j/m) e_j$ . Since  $Q_{4m-1} = V_n$ , we have

$$\begin{aligned} V_n * f &= \sum_{k=m}^{4m-1} \varphi(k/m) a_k e_k = \sum_{k=m}^{4m-1} (a_k - a_{k+1}) Q_k + a_{4m} Q_{4m-1} \\ &= \sum_{k=m}^{4m-1} (a_k - a_{k+1}) Q_k + a_{4m} V_n. \end{aligned}$$

On the other hand,  $Q_k = V_n * \Delta_{n,k}$ , where

$$\Delta_{n,k} = \sum_{j=2^{n-1}}^k z^j, \quad 2^{n-1} \leq k \leq 2^{n+1}.$$

By (3.2), with  $g = \Delta_{n,k}$ , we have

$$\|Q_k\|_1 \leq C \|\Delta_{n,k}\|_1 \leq C \log(k+1 - 2^{n-1}) \leq C(n+1).$$

Combining these inequalities we get

$$\begin{aligned} \|V_n * f\|_1 (n+1)^\alpha &\leq C \sum_{k=m}^{4m-1} (a_k - a_{k+1}) (n+1)^{\alpha+1} + C a_{4m} \|V_n\|_1 (n+1)^\alpha \\ &\leq C(n+1)^{\alpha+1} (a_m - a_{4m}) + C a_{4m} (n+1)^\alpha \\ &= C(n+1)^{\alpha+1} (a_{2^{n-1}} - a_{2^{n+1}}) + C(n+1)^\alpha a_{2^{n+1}}. \end{aligned}$$

Here we have used the relation  $\|V_n\|_1 \leq C$  (see (3.3))! Thus

$$\begin{aligned} (n+1)^\alpha \|V_n * f\|_1 &\leq C(n+1)^{\alpha+1} (a_{2^{n-1}} - a_{2^n}) \\ &\quad + C(n+1)^{\alpha+1} (a_{2^n} - a_{2^{n+1}}) + C(n+1)^\alpha a_{2^{n+1}}, \end{aligned}$$

and therefore it remains to estimate the sums

$$S_1 = \sum_{n=1}^{\infty} (n+1)^{\alpha+1} (a_{2^{n-1}} - a_{2^n}) \quad \text{and} \quad S_2 = \sum_{n=1}^{\infty} (n+1)^{\alpha+1} (a_{2^n} - a_{2^{n+1}}).$$

If  $\alpha > -1$ , then

$$(n+1)^{\alpha+1} \leq C \sum_{k=1}^n (k+1)^\alpha,$$

and hence

$$\begin{aligned} S_1 &\leq C \sum_{n=1}^{\infty} (a_{2^{n-1}} - a_{2^n}) \sum_{k=1}^n (k+1)^\alpha = C \sum_{k=1}^{\infty} (k+1)^\alpha \sum_{n=k}^{\infty} (a_{2^{n-1}} - a_{2^n}) \\ &= C \sum_{k=1}^{\infty} (k+1)_{2^{k-1}}^\alpha. \end{aligned}$$

In the case of  $S_2$  we get

$$S_2 \leq C \sum_{k=1}^{\infty} (k+1)^\alpha a_{2^k},$$

which completes the proof of “if” part of the theorem in the case  $\alpha > -1$ . If  $\alpha = -1$ , then

$$\|V_n * f\|_1 (n+1)^{-1} \leq C(a_{2^{n-1}} - a_{2^{n+1}}) + C(n+1)^{-1} a_{2^{n+1}},$$

from which we get the desired result in the case  $\alpha = -1$ .

To prove “only if” part we use Hardy’s inequality in the form

$$\pi M_1(r, g) \geq \sum_{n=0}^{\infty} \frac{|\hat{g}(n)|}{n+1} r^n.$$

It follows that

$$\begin{aligned} \int_{\mathbb{D}} |f'(z)| \log^\alpha \frac{2}{1-|z|} dA(z) &= 2 \int_0^1 M_1(r, f') \log^\alpha \frac{2}{1-r} r dr \\ &\geq \frac{2}{\pi} \sum_{n=1}^{\infty} a_n \frac{n}{n+1} \int_0^1 \log^\alpha \frac{2}{1-r} r^n dr \end{aligned}$$

Now the desired result follows from the inequality

$$\int_0^1 \frac{\varphi(1-r)}{1-r} r^n dr \geq c\varphi\left(\frac{1}{n+1}\right) \quad (c = \text{const.} > 0),$$

valid for any normal function  $\varphi$  (see [9, Lemma 4.1]).  $\square$

Before proving Theorem 1.2, some remarks are in order. Let  $f(z) = \sum_{n=0}^{\infty} a_n z^n$ ,  $a_n \geq 0$ . In order that  $\mathcal{L}f$  be well defined by (1.8) it is necessary and sufficient that

$$(4.1) \quad \sum_{n=0}^{\infty} \frac{a_n}{n+1} < \infty.$$

We already have mentioned in Introduction that this condition implies the existence of the integral in (1.8). In fact, this integral converges uniformly on compact subsets of  $\mathbb{D}$ , which means that the limit

$$\lim_{x \uparrow 1} \int_0^x f(t + (1-t)z) dt$$

exists and is uniform in  $|z| < \rho$ , for every  $\rho < 1$ . This guarantees that  $\mathcal{L}f$  is analytic. On the other hand, if the integral in (1.8) exists, then we take  $z = 0$  to conclude that (4.1) holds.

PROOF OF THEOREM 1.2. The Taylor coefficients of  $\mathcal{L}f$  are

$$b_n = \sum_{k=n}^{\infty} \frac{a_k}{k+1}.$$

The sequence  $\{b_n\}$  is nonincreasing so we can apply Theorem 1.1 to conclude that  $\mathcal{L}f \in \mathfrak{B}_{\log^\alpha}^1$  if and only if

$$\sum_{n=0}^{\infty} \frac{\log^\alpha(n+2)}{n+1} \sum_{k=n}^{\infty} \frac{a_k}{k+1} = \sum_{k=0}^{\infty} \frac{a_k}{k+1} \sum_{n=0}^k \frac{\log^\alpha(n+2)}{n+1} < \infty.$$

Now the desired result follows from the estimate

$$C^{-1} \log^{\alpha+1}(k+2) \leq \sum_{n=0}^k \frac{\log^\alpha(n+2)}{n+1} \leq C \log^{\alpha+1}(k+2),$$

which holds because  $\alpha > -1$ .  $\square$

REMARK 4.1. The above proof shows that  $\mathcal{L}f$  belongs to  $\mathfrak{B}_{\log^{-1}}^1$  if and only if

$$\sum_{n=0}^{\infty} \frac{a_n}{\log \log(n+4)} < \infty.$$

Now we pass to the proof of Theorem 2.1.

PROOF OF THEOREM 1.1(c). Since  $\mathfrak{B}_{\log^\alpha} \subset \mathfrak{B}_{\log^\beta}$  for  $\beta < \alpha$ , we may assume that  $-1 < \alpha < 0$ . Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad a_n = \log^{-\varepsilon-\alpha}(n+2),$$

where  $\varepsilon > 1$ . Condition (4.1) holds because  $\varepsilon > 1$ . For every  $r \in (0, 1)$  the function  $f_r(z) = f(rz)$  belongs to  $H(\overline{\mathbb{D}})$  and, by Theorem 1.1 and its proof, the set  $\{f_r : 0 < r < 1\}$  is bounded in  $\mathfrak{B}_{\log^\alpha}^1$ . On the other hand,

$$\overline{\mathcal{L}}(f_r)(0) = \sum_{k=0}^{\infty} \frac{r^k}{(k+1) \log^{\alpha+\varepsilon}(k+2)}.$$

Now choose  $\varepsilon = 1 - \alpha > 1$  (because  $\alpha < 0$ ) to get

$$\overline{\mathcal{L}}(f_r)(0) = \sum_{k=0}^{\infty} \frac{r^k}{(k+1) \log(k+2)} \longrightarrow \infty \quad (r \uparrow 1).$$

This contradicts the fact that if a set  $X \subset \mathfrak{B}_{\log^\alpha}^1$  is bounded and  $\overline{\mathcal{L}}$  is bounded on  $\mathfrak{B}_{\log^\alpha}^1$ , then the set  $\{\overline{\mathcal{L}}f(0) : f \in X\}$  is bounded because the functional  $h \mapsto h(0)$  is continuous on  $H(\overline{\mathbb{D}})$ . This completes the proof.  $\square$

PROOF OF THEOREM 2.1(a). Let  $g \in \mathfrak{B}_{\log^\alpha}$ . Then

$$M_2(r, g') \leq C(1-r)^{-1} \log \frac{2}{1-r}.$$

It follows that

$$2^n \left( \sum_{k=2^n}^{2^{n+1}-1} |\hat{g}(k)|^2 \right)^{1/2} r^{2^{n+1}} \leq C(1-r)^{-1} \log^\alpha \frac{2}{1-r}.$$

Taking  $r = 1 - 2^{-n}$ ,  $n \geq 1$ , we get

$$\left( \sum_{k=2^n}^{2^{n+1}-1} |\hat{g}(k)|^2 \right)^{1/2} \leq C \log^\alpha(n+1).$$

Hence,

$$2^{-n} \sum_{k=2^n}^{2^{n+1}-1} |\hat{g}(k)| \leq \left( 2^{-n} \sum_{k=2^n}^{2^{n+1}-1} |\hat{g}(k)|^2 \right)^{1/2} \leq 2^{-n/2} \log^\alpha(n+1).$$

This gives the result.  $\square$

PROOF OF THEOREM 2.1(b). In this case we use Hardy's inequality as in the proof of Theorem 1.1 to get

$$\|g\|_{\mathfrak{B}_{\log^\alpha}^1} \geq c \sum_{n=0}^{\infty} \frac{|\hat{g}(n)| \log^\alpha(n+2)}{n+1}.$$

This proves the result because  $\alpha \geq 0$ .  $\square$

### 5. Proofs of Theorem 2.2 and 2.3

Define the operator  $\mathcal{R} : H(\mathbb{D}) \mapsto H(\mathbb{D})$  by

$$\mathcal{R}f(z) = \sum_{n=0}^{\infty} (n+1) \hat{f}(n) z^n = \frac{d}{dz} (zf(z)).$$

By using Theorem 3.1 and the relation

$$(5.1) \quad C^{-1} 2^n \|V_n * f\|_p \leq \|V_n * \mathcal{R}f\|_p \leq C 2^n \|V_n * f\|_p \quad (n \geq 0),$$

one proves that the norm in  $\mathfrak{B}_{\log^\alpha}$  is equivalent to

$$\sup_{z \in \mathbb{D}} (1 - |z|) \log^{-\alpha} \frac{2}{1 - |z|} |\mathcal{R}f(z)|.$$

PROOF OF THEOREM 2.2(a). Let  $\alpha > -1$  and  $f \in \mathfrak{B}_{\log^\alpha}$ . Then, by integration,

$$|f(z)| \leq \log^{\alpha+1} \frac{1}{1 - |z|}.$$

Since

$$\mathcal{R}f(z) = \frac{f(z)}{1 - z},$$

we see that

$$|\mathcal{R}f(z)| \leq C(1 - |z|)^{-1} \log^{\alpha+1} \frac{1}{1 - |z|}.$$

The result follows.  $\square$

PROOF OF THEOREM 2.2(b). The function  $\varphi(x) = x \log \log(4/x)$  is normal because  $\lim_{x \rightarrow 0} x\varphi'(x)/\varphi(x) = 1$ . Hence, arguing as in the proof of Theorem 3.1 we conclude that  $f \in \mathfrak{B}_{\log \log}$  if and only if  $\sup_{n \geq 0} \|V_n * f\|_\infty / \log(n+2) < \infty$ . Then using (5.1) we find that  $g \in \mathfrak{B}_{\log \log}$  if and only if

$$|\mathcal{R}g(z)| \leq C(1 - |z|)^{-1} \log \log \frac{4}{1 - |z|}.$$

The rest of the proof is the same as in the case of (a).  $\square$

REMARK 5.1. In the case of the small spaces the proofs are similar and therefore omitted.

For the proof of Theorem 2.3 we need the following lemma [18]:

LEMMA 5.1. *If  $f \in \ell_{-1}^1$ , then  $\mathcal{L}f$  is well defined by (1.8) and the inequality*

$$(5.2) \quad rM_1(r, (\mathcal{L}f)') \leq 2(1 - r)^{-1} \int_r^1 M_1(s, f') ds, \quad 0 < r < 1,$$

*holds.*

Before passing to the proof observe that  $\mathfrak{B}_{\log^\alpha}^1 \subset \mathfrak{B}^1$  and  $\mathfrak{B}_{\log \log}^1 \subset \mathfrak{B}^1$ , and, since  $\mathfrak{B}_1 \subset H^1$ , we see that in all cases of Theorem 2.3 the operator  $\mathcal{L}$  is well defined.

PROOF OF THEOREM 2.3(a). We have, by (5.2),

$$\begin{aligned} \int_{\mathbb{D}} |(\mathcal{L}f)'(z)| \log^{\alpha-1} \frac{2}{1 - |z|} dA(z) &= 2 \int_0^1 M_1(r, (\mathcal{L}f)') \log^{\alpha-1} \frac{2}{1 - r} r dr \\ &\leq 4 \int_0^1 (1 - r)^{-1} \log^{\alpha-1} \frac{2}{1 - r} dr \int_r^1 M_1(s, f') ds \\ &= 4 \int_0^1 M_1(s, f') ds \int_0^s (1 - r)^{-1} \log^{\alpha-1} \frac{2}{1 - r} dr \\ &\leq C \int_0^1 M_1(s, f') \log^\alpha \frac{2}{1 - s} ds. \end{aligned}$$

A standard application of the maximum modulus principle shows that the inequality remains valid if we replace  $ds$  with  $s ds$ . This gives the result.  $\square$

The proofs of Theorem 2.3, (b) and (c), are similar and we omit them.

## 6. Proof of Theorem 2.4

We consider a more general situation. Let  $X \subset H(\mathbb{D})$  (with continuous inclusion) be a Banach space such that the functions  $f_w(z) = f(wz)$ ,  $|w| \leq 1$ , belong to  $X$  whenever  $f \in X$ , and  $\sup_{|w| \leq 1} \|f_w\|_X \leq \|f\|_X$ . Such a space is said to be *homogeneous* (see [2]). A homogeneous space satisfies the condition

$$(6.1) \quad \|V_n * f\|_X \leq C \|f\|_X, \quad f \in X,$$

where  $C$  is independent of  $n$  and  $f$ .

If in addition

$$(6.2) \quad \lim_{r \uparrow 1} \|f - f_r\|_X = 0, \quad f \in X,$$

then the dual of  $X$  can be identified with the space,  $X'$ , of those  $g \in H(\mathbb{D})$  for which limit (1.2) exists for all  $f \in X$  (see [1, 2]). Also, the dual of a homogeneous space  $X$  satisfying (6.2) can be realized as the space of coefficient multipliers,  $(X, A(\mathbb{D}))$ , from  $X$  to  $A(\mathbb{D})$ ; in this case we have  $(X, A(\mathbb{D})) = (X, H^\infty) =: X^*$  (see [2]). The norm in  $X^*$  is introduced as

$$\|g\|_{X^*} = \sup \|f * g\|_\infty : f \in X, \|f\|_X \leq 1,$$

and, if  $X$  is homogeneous and satisfies (6.2), it is equal to

$$\|g\|_{X'} = \sup \{ |\langle f, g \rangle| : f \in X, \|f\|_X \leq 1 \}.$$

There is another way to express  $\langle f, r \rangle$ , when  $f \in X$ ,  $X$  satisfies (6.2), and  $g \in X'$ ; namely, in this case, the function  $f * g$  belongs to  $A(\mathbb{D})$ , and we have  $\langle f, g \rangle = (f * g)(1)$  (see [13, 2]).

We fix a sequence  $\lambda = \{\lambda_n\}_0^\infty$  of positive real numbers such that

$$(6.3) \quad 0 < \inf_{n \geq 0} \frac{\lambda_{n+1}}{\lambda_n}, \quad \sup_{n \geq 0} \frac{\lambda_{n+1}}{\lambda_n} < \infty.$$

It is clear that the spaces  $H^p$  ( $0 < p \leq \infty$ ),  $A(\mathbb{D})$ ,  $\mathfrak{B}_{\log^\alpha}$ ,  $\mathfrak{b}_{\log^\alpha}$ , and  $\mathfrak{B}_{\log^\alpha}^1$  are homogeneous. Among them only  $H^\infty$  and  $\mathfrak{B}_{\log^\alpha}$  do not satisfy condition (6.2).

Consider the following three spaces of sequences  $\{f_n\}_0^\infty$ ,  $f_n \in H(\mathbb{D})$ :

- (a)  $\mathfrak{c}_0(\lambda, X) = \{\{f_n\} : \lim_{n \rightarrow \infty} \|f_n\|_X / \lambda_n = 0\}$ ;
- (b)  $\ell^\infty(\lambda, X) = \{\{f_n\} : \sup_{n \geq 0} \lambda_n \|V_n * f\|_X < \infty\}$ ;
- (c)  $\ell^1(\lambda, X) = \{\{f_n\} : \sum_{n=0}^\infty \|f_n\|_X / \lambda_n < \infty\}$ .

We also define the spaces  $v_0(\lambda, X)$ ,  $V^\infty(\lambda, X)$ , and  $V^1(\lambda, X)$  (as subsets of  $H(\mathbb{D})$ ) by replacing  $f_n$  with  $V_n * f$  in (a), (b), and (c), respectively. The proof of the following lemma is rather easy, and is therefore left to the reader.

**LEMMA 6.1.** *If  $X$  is a homogeneous space, then so are  $v(\lambda, X)$ ,  $V^\infty(\lambda, X)$ , and  $V^1(\lambda, X)$ . The spaces  $v_0(\lambda, X)$  and  $V^1(\lambda, X)$  satisfy (6.2). The space  $v_0(\lambda, X)$  is equal to the closure in  $V^\infty(\lambda, X)$  of the sets of all polynomials.*

Theorem 2.4 will be deduced from Theorem 3.1 and the following.

**PROPOSITION 6.1.** *If  $X$  is a homogeneous space satisfying (6.2), then the dual of  $v_0(\lambda, X)$ , resp.  $V^1(\lambda, X)$ , is isomorphic to  $V^1(\lambda, X')$ , resp.  $V^\infty(\lambda, X')$ , with respect to (1.2).*

In proving we use ideas from [13, 14, 7]. For the proof we need the following lemma.

**LEMMA 6.2.** *The operator  $T(\{f_n\}) = \sum_{n=0}^\infty V_n * f_n$  acts as a bounded operator from  $Y$  to  $Z$ , where  $Y$  is one of the spaces  $\mathfrak{c}_0(\lambda, X)$ ,  $\ell^\infty(\lambda, X)$ , and  $\ell^1(\lambda, X)$ , while  $Z$  is  $v_0(\lambda, X)$ ,  $V^\infty(\lambda, X)$ , and  $V^1(\lambda, X)$ , respectively.*

PROOF. We have  $V_n * V_j = 0$  for  $|j - n| \geq 2$  and hence

$$V_n * T(\{f_j\}) = \sum_{j=n-1}^{n+1} V_n * V_j * f_j, \quad n \geq 0,$$

where, by definition,  $w_j = f_j = 0$  for  $j < 0$ . It follows that

$$\|V_n * T(\{f_j\})\|_X \leq C \sum_{j=n-1}^{n+1} \|f_j\|_X,$$

where we have used (6.1). Now the proof is easily completed by using (6.3).  $\square$

LEMMA 6.3. *Let  $g \in (v_0(\lambda, X))'$ , resp.  $g \in (V^1(\lambda, X))'$ , and define the operator  $S$  on  $\mathfrak{c}_0(\lambda, X)$ , resp.  $\ell^1(\lambda, X)$ , by*

$$S(\{f_n\}) = T(\{f_n\}) * g = \sum_{k=0}^{\infty} f_k * V_k * g.$$

*Then  $S$  maps  $\mathfrak{c}_0(\lambda, X)$ , resp.  $\ell^1(\lambda, X)$ , into  $H^\infty$  and  $\|S\| \leq C \|g\|_{(v_0(\lambda, X))'}$ , resp.  $\|S\| \leq C \|g\|_{(V^1(\lambda, X))'}$ .*

PROOF. By the preceding lemma, we have

$$\|S(\{f_n\})\|_\infty \leq \|T(\{f_n\})\|_{v_0(\lambda, X)} \|g\|_{(v_0(\lambda, X))^*} \leq C \|\{f_n\}\|_{\mathfrak{c}_0(\lambda, X)} \|g\|_{(v_0(\lambda, X))^*}.$$

This proves the result in one case. In the other case the proof is the same.  $\square$

PROOF OF PROPOSITION 6.1. Define the polynomials  $P_n$  ( $n \geq 0$ ) by

$$P_n = V_{n-1} + V_n + V_{n+1}.$$

Hence

$$V_n = \sum_{j=0}^{\infty} V_j * V_n = (V_{n-1} + V_n + V_{n+1}) * V_n = P_n * V_n.$$

Let  $f \in v_0(\lambda, X)$  and  $g \in V^1(\lambda, X')$ . It is easily verified that, when  $0 < r < 1$ ,

$$(f * g)(z) = \sum_{n=0}^{\infty} (f * V_n * g)(z) = \sum_{n=0}^{\infty} (P_n * f * V_n * g)(z), \quad z \in \mathbb{D}.$$

the series being absolutely convergent. Since

$$\|P_n * f_r * V_n * g\|_\infty \leq \|P_n * f\|_X \|V_n * g\|_{X^*},$$

we have

$$\begin{aligned} \|f * g\|_\infty &\leq \sum_{n=0}^{\infty} \|P_n * f\|_X \|V_n * g\|_{X^*} \leq C \sum_{n=0}^{\infty} \|P_n * f\|_X \|V_n * g\|_{X^*} \\ &= C \sum_{n=0}^{\infty} (\|P_n * f\|_X / \lambda_n) (\lambda_n \|V_n * g\|_{X^*}) \leq C \|f\|_{v_0(\lambda, X)} \|g\|_{V^1(\lambda, X^*)} \end{aligned}$$

This proves the inclusion  $V^1(\lambda, X') \subset (v_0(\lambda, X))'$ .

To prove the converse, let  $g \in (v_0(\lambda, X))^*$ . Let  $S$  denote the operator defined in Lemma 6.3. By Lemma 6.3,  $S$  acts as a bounded operator from  $\mathfrak{c}_0(\lambda, X)$  into  $H^\infty$  and we have  $\|S\| \leq C\|g\|_{v_0(\lambda, X)^*}$ . Now it suffices to prove that

$$\|S\| \geq (1/2)\|\{g_n\}\|_{\ell^1(\lambda, X^*)} = (1/2)\|g\|_{V^1(\lambda, X)}.$$

For each  $n \geq 0$  choose  $f_n \in X$  so that  $\|f_n\|_X = 1$  and  $\langle f_n, g_n \rangle$  is a real number such that  $\langle f_n, g_n \rangle \geq (1/2)\|g_n\|_{X^*}$ . If  $\{a_n\}$  is a finite sequence of nonnegative real numbers, then

$$S(\{a_n f_n\}) = \sum_{n=0}^{\infty} a_n \langle f_n, g_n \rangle \geq (1/2) \sum_{n=0}^{\infty} a_n \|g_n\|_{X^*} = (1/2) \sum_{n=0}^{\infty} (a_n/\lambda_n) \lambda_n \|g_n\|_{X^*}.$$

Hence, by taking the supremum over all  $\{a_n\}$  such that  $0 \leq a_n \leq \lambda_n$ , we get  $S\{\lambda_n f_n\} \geq (1/2) \sum_{n=0}^{\infty} \lambda_n \|g_n\|_{X^*}$ . Since  $\|\{a_n f_n\}\|_{\mathfrak{c}_0(\lambda, X)} \leq 1$ , where  $a_n = \lambda_n$  for  $0 \leq n \leq N$  ( $N \in \mathbb{N}$ ) and  $a_n = 0$  for  $n > N$  we see that  $\|S\| \geq (1/2)\|g\|_{V^1(\lambda, X^*)}$ , as desired. This completes the proof that  $v_0(\lambda, X)' = V^1(\lambda, X')$ . In a similar way one proves that  $V^1(\lambda, X)' = V^\infty(\lambda, X')$ , which is all what has to be proved.  $\square$

PROOF OF THEOREM 2.4. First we prove that  $(\mathfrak{b}_{\log^\alpha})' = \mathfrak{B}_{\log^\alpha}^1$ . By Theorem 3.1, we have  $\mathfrak{b}_{\log^\alpha} = v_0(\lambda, A(\mathbb{D}))$ , where  $\lambda_n = (n+1)^\alpha$ . Hence, by Proposition 6.1, the dual of  $\mathfrak{b}_{\log^\alpha}$  is isomorphic to  $V^1(\lambda, A(\mathbb{D})')$ . In order to estimate  $\|V_n * g\|_{A(\mathbb{D})'}$  first observe that  $H^1 \subset A(\mathbb{D})'$  and moreover  $\|V_n * g\|_{A(\mathbb{D})'} \leq \|V_n * g\|_1$ . On the other hand, let  $\Phi$  be a bounded linear functional on  $A(\mathbb{D})$ , let  $\Phi_0$  be the Hahn/Banach extension of  $\Phi$  to  $hC(\mathbb{D})$ , and choose  $g \in A(\mathbb{D})^a$  so that  $\Phi(f) = \langle f, g \rangle$  for all  $f \in A(\mathbb{D})$ . By the Riesz representation theorem, we have

$$\begin{aligned} \Phi_0(f) &= \frac{1}{2\pi} \int_0^{2\pi} f(e^{-i\theta}) d\mu(e^{i\theta}) \\ &= \lim_{r \rightarrow 1^-} \frac{1}{2\pi} \int_0^{2\pi} f(re^{-i\theta}) g(re^{i\theta}) d\theta = \lim_{r \rightarrow 1^-} \sum_{n=0}^{\infty} \hat{f}(n) \hat{g}(n) r^{2n}, \end{aligned}$$

and  $\|\mu\| = \|\Phi\| = \|\Phi_0\|$ . In particular, taking  $f(w) = (1-zw)^{-1}$ , where  $z \in \mathbb{D}$  is fixed, we get  $\frac{1}{2\pi} \int_0^{2\pi} (1-e^{-i\theta}z)^{-1} d\mu(e^{i\theta}) = g(z)$ . Hence

$$\mathcal{R}^1 g(z) = \frac{1}{2\pi} \int_0^{2\pi} (1-e^{-i\theta}z)^{-2} d\mu(e^{i\theta}),$$

and hence, by integration,  $M_1(r, \mathcal{R}^1 f) \leq \|\mu\|(1-r^2)^{-1} = \|g\|_{A(\mathbb{D})'}(1-r^2)^{-1}$ . Now we proceed as in the proof of Theorem 3.1 to conclude that  $\|V_n * g\|_1 \leq C\|V_n * g\|_{A(\mathbb{D})'}$ . It follows that  $g \in (\mathfrak{b}_{\log^\alpha})'$  if and only if  $g \in V^1(\lambda, H^1)$ , i.e., by Theorem 3.1,  $g \in \mathfrak{B}_{\log^\alpha}^1$ .

In proving that  $(\mathfrak{B}_{\log^\alpha}^1)'$  is isomorphic to  $\mathfrak{B}_{\log^\alpha}$ , we use the inclusions  $H^\infty \subset (H^1)' \subset \mathfrak{B}$ , and then proceed as above.  $\square$

REMARK 6.1. The above proof of Theorem 2.4 certainly is not the simplest one. However, it can be applied to prove some general duality and multipliers theorems (see [13, 14, 7]). For instance, the dual of  $\mathfrak{b}_{\log}$  is isomorphic to  $\mathfrak{B}_{\log}^1$ , and the dual of  $\mathfrak{B}_{\log}^1$  is isomorphic to  $\mathfrak{B}_{\log}$ .

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Faculty of Mathematics, University of Belgrade  
 Belgrade, Serbia  
 pavlovic@matf.bg.ac.rs

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