A NOTE
ON A QUESTION SET BY P. ERDÖS AND L. K. HUA

by

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SUMMARY: If s_n are partial sums of a series $\sum a_\nu$ with positive terms, then from $\sum_{\nu=1}^{n} a_\nu s_{n-\nu} = {}^1/_2\, n^2 + O(n)$ follows $s_n = n + O\left(n^{\sqrt{3}-1+\varepsilon}\right)$ for every $\varepsilon > 0$.

1. In the "Supplementary Note" of his paper — "On a Tauberian theorem connected with the new proof of the prime number theorem" — (*Journ. Ind. Math. Soc.* **13** (1949), No. 4, pp. 131—147), referring to a question put by L. K. Hua, P. Erdös has shown that from

$$(a) \qquad \sum_{\nu=1}^{n} a_\nu s_{n-\nu} = {}^1/_2\, n^2 + O(n), \quad n \to \infty,$$

where

$$s_n = \sum_{\nu=1}^{n} a_\nu ,$$

and

$$(b) \qquad a_n \geqslant 0$$

one cannot infer that

$$s_n = n + o(\sqrt{n}), \quad n \to \infty,$$

and notes that from (a) resp. the somewhat weaker assumption

$$\sum_{\nu=1}^{n} a_\nu s_{n-\nu} \sim {}^1/_2\, n^2, \quad n \to \infty,$$

and (b) it follows the asymptotic formula

$$s_n \sim n, \quad n \to \infty.$$

I shall show now that

$$s_n = n + O(n^{\sqrt{3}-1+\varepsilon}), \quad n \to \infty \quad \text{for every } \varepsilon > 0.$$

If we put

$$a(x) = \sum_{\nu \leqslant x} a_\nu$$

our assertion becomes equivalent to the theorem:

THEOREM 1. *Let $a(x)$ be of bounded variation over every finite segment,*

(1) $$a(0) = 0,$$

(2) $$\int_0^x a(x-u)\, da(u) = {}^1/_2 x^2 + O(x), \quad x \to \infty,$$

and

(3) $$a(x) \text{ non-decreasing.}$$

Then we have

$$a(x) = x + O(x^{\sqrt{3}-1+\varepsilon}), \quad x \to \infty \quad \text{for every } \varepsilon > 0.$$

2. The proof of Theorem 1 is found on the following

THEOREM A. *Let k be fixed,*

$$1 < k < \frac{1+\sqrt{3}}{2},$$

and let $a(x)$ satisfy the assumptions of Theorem 1.
Then there exists

$$J(s) = s \int_0^\infty e^{-sx} \{a(x) - x\} \frac{dx}{x^{1/k}} \quad \text{for} \quad \Re\{s\} > 0,$$

and has the following properties: (i) *Let D_k denote the region*

$$D_k: \qquad |\Im(s)|^k \leqslant \Re\{s\}.$$

Hence

$$|J(s)| \leqslant M \quad \text{for} \quad s \in D_k.$$

(ii) *For all $\delta > 0$ there exists*

$$\int_{-\infty}^{+\infty} \frac{\sin y\sigma}{\sigma} J(\delta + |\sigma|^k + i\sigma)\, d\sigma$$

and we have

$$H(y) = \lim_{\delta=0} \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\sin y\sigma}{\sigma} J(\delta + |\sigma|^k + i\sigma)\, d\sigma = o(1), \quad y \to \infty.$$

Proof. From (1), (2) and (3) we have

$$a^2\left(\frac{x}{2}\right) = a\left(\frac{x}{2}\right) \int_0^{x/2} d\,a(u) \leqslant \int_0^{x/2} a(x-u)\, d\,a(u)$$

$$\leqslant \int_0^x a(x-u)\, d\,a(u) \leqslant C_1 \left(\frac{x}{2}\right)^2,$$

that is

(4) $$a(x) \leqslant C_1 x.$$

Hence $J(s)$ exists for $\Re\{s\} > 0$.

(ii). By (4) there exists

$$y(s) = \int_0^\infty e^{-sx}\, d\,a(x) \quad \text{for} \quad \Re\{s\} > 0.$$

Let

$$\gamma(x) = \int_0^x a(x-u)\, d\,a(u) = {}^1/_2\, x^2 + \theta(x) \cdot x$$

so that

(5) $$\gamma(0) = 0.$$

Thus we have

$$y^2(s) = \int_0^\infty e^{-sx}\, d\,\gamma(x)$$

and with respect to (5)

$$= s \int_0^\infty e^{-sx}\, \gamma(x)\, dx$$

$$= \frac{1}{s^2} + s \int_0^\infty e^{-sx}\, \theta(x)\, x\, dx$$

$$= \frac{1}{s^2} + I(s).$$

so that finally we have

$$y(s) - \frac{1}{s} = \frac{1}{s}\{[1 + s^2 I(s)]^{1/2} - 1\}. \tag{6}$$

Because of (2) there must be a $\theta > 1$ so that $|\theta(x)| \leqslant \theta$ and consequently

$$|I(s)| \leqslant \theta |s| \int_0^\infty e^{-\Re\{s\}x} x\,dx = \frac{\theta |s|}{[\Re\{s\}]^2}.$$

On account of (6) it follows tat: if

$$\frac{\theta |s|^3}{[\Re\{s\}]^2} < 1 \tag{7}$$

then

$$\left| y(s) - \frac{1}{s} \right| \leqslant C_2 \left[\frac{|s|}{\Re\{s\}}\right]^2 \tag{8}$$

for all s satisfying the inequality (7).

Let σ_0 be a fixed number,

$$\sigma_0 < \mathrm{Min}\left(\frac{1}{\{2^{3/2}\theta\}^{1/(3-2k)}},\ 1\right) \tag{9}$$

and let D_k^* denote the region

$$D_k^*: \qquad |\Im\{s\}|^k \leqslant \Re\{s\} \quad \text{with} \quad \Re\{s\} \leqslant \sigma_0{}^k.$$

I shall prove that D_k^* is contained within the region (7), i. e. that (8) is true for all s of D_k^*. It is clear that this assertion needs to be proved only for s not lying on the rightlined boundary of D_k^*.

If $s = |\sigma|^k + i\sigma$ is situated on the boundary of D_k^*, since $|\sigma| \leqslant \sigma_0$ and $1 < k < {}^3/_2$, we shall have

$$|s|^2 = |\sigma|^{2k} + \sigma^2 \leqslant 2\sigma^2 \leqslant 2\sigma_0^{2-4k/3} |\sigma|^{4k/3}.$$

Therefore, in consideration of (9)

$$|s|^2 \leqslant \frac{1}{\theta^{2/3}} |\sigma|^{4k/3} = \frac{1}{\theta^{2/3}} [\Re\{s\}]^{4/3}$$

so that our assertion is proved.

b) From (8) it follows that: if $|s| \neq 0$ and

$$-\frac{\pi}{2} < \arg s = \varphi < \frac{\pi}{2}$$

there exists

$$\frac{s}{\Gamma(1/k)} \int_0^{\infty \cdot e^{i\varphi}} \left\{ y(s) - \frac{1}{s} \right\} (\tau - s)^{1/k-1} \frac{d\tau}{\tau}.$$

On account of (4), the expression $\{a(x) - x\} e^{-\tau x} (\tau - s)^{1/k-1}$ is absolutely integrable both with respect to x (from 0 to ∞) and to τ (from s to $\infty \cdot e^{i\varphi}$). Consequently, we may, in the last integral, express $y(\tau) - 1/\tau$ by

$$\tau \int_0^{\infty} e^{-\tau x} \{a(x) - x\} \, dx$$

and then in the integral thus obtained invert the order of integration. And so we see that

$$\frac{s}{\Gamma(1/k)} \int_0^{\infty \cdot e^{i\varphi}} \left\{ y(\tau) - \frac{1}{\tau} \right\} (\tau - s)^{1/k-1} \frac{d\tau}{\tau} =$$

$$= s \int_0^{\infty} e^{-sx} \{a(x) - x\} \frac{dx}{\Gamma(1/k)} \int_0^{\infty \cdot e^{i\varphi}} e^{-zx} z^{1/k-1} \, dz$$

$$= s \int_0^{\infty} e^{-sx} \{a(x) - x\} \frac{dx}{x^{1/k}}$$

$$= J(s).$$

Now I shall prove: *when*

$$1 < k < {}^{8}/_{2},$$

then we have

$$|J(s)| \leqslant C_3 \frac{[\Re\{s\}]^{\varkappa}}{1+[\Re\{s\}]^{\varkappa+2-2/k}} \quad \textit{for} \quad s \in D_k \tag{10}$$

with

$$\varkappa = \frac{1}{k^2} + \frac{2}{k} - 2.$$

First of all, let us take $|s| < 1$ and $s = \delta + |\sigma|^k + i\sigma$. Then it is $|\sigma| < 1$ and $0 < \delta < 1$. Hence

$$(\delta + |\sigma|)^k \leqslant 2^k (\delta^k + |\sigma|^k) \leqslant 2^k (\delta + |\sigma|^k) \leqslant 2^k \Re\{s\}$$

and also

$$|s|^2 = (\delta + |\sigma|^k)^2 + \sigma^2 \leqslant (\delta + |\sigma|)^2 + (\delta + |\sigma|)^2 = 2^3 [\Re\{s\}]^{2/k}. \tag{11}$$

Let us put now

$$\tau_0 = \frac{\sigma_0}{|s| \cos\varphi}$$

and

$$J(s) = \frac{s^{1/k}}{\Gamma(1/k)} \int\limits_1^{\infty} \left\{ y(s\tau) - \frac{1}{s\tau} \right\} (\tau - 1)^{1/k-1} \frac{d\tau}{\tau}$$

$$= \int\limits_0^{\tau_0} + \int\limits_{\tau_0}^{\infty} = J_1(s) + J_2(s).$$

If $1 \leqslant \tau \leqslant \tau_0$ then $s\tau$ is contained in D_k^*. But, as it was proved that the inequality (8) is correct for all s of D_k^*, we have

$$|J_1(s)| \leqslant \frac{C_2}{\Gamma(1/k)} \frac{|s|^{1/k+2}}{[\Re\{s\}]^2} \int\limits_1^{\tau_0} (\tau - 1)^{1/k-1} \frac{d\tau}{\tau} \leqslant C_4 \frac{|s|^{1/k+2}}{[\Re\{s\}]^2}.$$

On account of (11) we have

$$|J_1(s)| \leqslant C_4 2^{\frac{3+2k}{2k}} [\Re\{s\}]^{\varkappa}.$$

By the inequality (4) we obtain

$$(12)\quad \left| y(s) - \frac{1}{s} \right| \leqslant |s| \int_0^\infty e^{-\Re\{s\}x} |a(x) - x|\, dx \leqslant (1 + C_1) \frac{|s|}{[\Re\{s\}]^2}.$$

Hence

$$|J_2(s)| \leqslant \frac{C_5}{\Gamma(1/k)} \frac{|s|^{1/k+1}}{[\Re\{s\}]^2} \int_{\tau_0}^\infty (\tau - 1)^{1/k-1} \frac{d\tau}{\tau^2} \leqslant C_6 \frac{|s|^{1/k+1}}{[\Re\{s\}]^2} \tau_0^{1/k-2} \leqslant$$

$$\leqslant (\cos\varphi)^{2-1/k} \sigma_0^{1/k-2} C_6 \frac{|s|^3}{[\Re\{s\}]^2}.$$

Further, by (11) we have

$$|J_2(s)| \leqslant C_7 [\Re\{s\}]^{3/k-2} \leqslant C_7 [\Re\{s\}]^{\varkappa}$$

as is obvious that

$$\frac{3}{k} - 2 > \frac{1}{k^2} + \frac{2}{k} - 2 = \varkappa.$$

Summing up it follows that

$$(13)\quad |J(s)| \leqslant C_8 [\Re\{s\}]^{\varkappa} \quad \text{for} \quad s \in D_k \quad \text{and} \quad |s| < 1.$$

Let us take $|s| \geqslant 1$. From (4) we have

$$|J(s)| \leqslant (1 + C_1)|s| \int_0^\infty e^{-\Re\{s\}x} x^{1-1/k}\, dx$$

$$\leqslant (1 + C_1)\, \Gamma(2 - 1/k) \frac{|s|}{[\Re\{s\}]^{2-1/k}}$$

and also, on account of (11),

$$(14)\quad |J(s)| \leqslant C_9 [\Re\{s\}]^{2/k-2} \quad \text{for} \quad s \in D_k \quad \text{and} \quad |s| \geqslant 1.$$

For $k > 1$ we see that $2/k - 2 < 0$ and that (10) is a consequence of (13) and (14).

If $1 < k < (1+\sqrt{3})/2$ then $\varkappa > 0$; therefore and by (10)

$$|J(s)| \leqslant M \quad \text{for } s \in D_k \quad \text{if} \quad 1 < k < (1+\sqrt{3})/2 \tag{15}$$

which proves the first part of theorem A.

(ii) On account of (15) we have

$$\frac{\sin y\sigma}{\sigma} J(\delta + |\sigma|^k + i\sigma) = O(1), \quad |\sigma| \to 0, \tag{16}$$

and by (14)

$$\frac{1}{\sigma} J(\delta + |\sigma|^k + i\sigma) = O(|\sigma|^{1-2k}), \quad |\sigma| \to \infty. \tag{17}$$

Therefore

$$\int_{-\infty}^{+\infty} \frac{\sin y\sigma}{\sigma} J(\delta + |\sigma|^k + i\sigma)\, d\sigma$$

exists for all $\delta > 0$.

The inequalities (16) and (17) are uniformly valid in $\delta > 0$.

Hence, there exists a $f(\sigma) = f(\sigma; y)$ not depending on δ and integrable from 0 to ∞, so that

$$\left| \frac{\sin y\sigma}{\sigma} J(\delta + |\sigma|^k + i\sigma) \right| \leqslant f(\sigma).$$

As, what is more, for all σ except eventually for $\sigma = 0$,

$$\lim_{\delta=0} J(\delta + |\sigma|^k + i\sigma) = Q(\sigma)$$

exists, so, on the basis of the well-known theorem of Lebesgue, we have

$$H(y) = \lim_{\delta=0} \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\sin y\sigma}{\sigma} J(\delta + |\sigma|^k + i\sigma)\, d\sigma$$

$$= \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{Q(\sigma)}{\sigma} \sin y\sigma\, d\sigma.$$

Because of $Q(\sigma) = \lim\limits_{\delta=0} J(\delta + |\sigma|^k + i\sigma)$ and (10) we have

$$\left| \frac{Q(\sigma)}{\sigma} \right| \leqslant C_3 \frac{|\sigma|^{k\varkappa - 1}}{1 + |\sigma|^{(\varkappa+2)k-2}}$$

hence $\sigma^{-1} Q(\sigma)$ is absolutely integrable from 0 to ∞ if $1 < k < (1+\sqrt{3})/2$ as in that case both $\varkappa > 0$ and $2k - 1 > 1$.

Therefore

$$H(y) = o(1), \quad y \to \infty \quad \text{if} \quad 1 < k < (1+\sqrt{3})/2$$

which proves the second part of theorem A.

3. *PROOF OF THEOREM* 1. Now I shall use the following

THEOREM B. *Let $J(s)$ be defined for $\Re\{s\} > 0$ by*

$$J(s) = s \int_0^\infty e^{-sx} A(x)\, dx$$

and let it satisfy the following conditions: (i) *$J(s)$ is uniformly bounded in a convex region, the boundary of which has for the point $s = 0$ with the imaginary axis a contact of the order $k-1$, $k > 1$. Therefore, if this region is denoted by D_k then*

$$|J(s)| \leqslant M \quad \text{if} \quad s \in D_k.$$

$$\text{(ii)} \quad H(y) = \lim_{\delta = 0} \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\sin y\sigma}{\sigma} J(\delta + |\sigma|^k + ia\sigma)\, d\sigma = o(1), \quad y \to \infty,$$

where $|\sigma|^k + ia\sigma$ with a fixed a is contained within D_k.

Moreover, let $A(x)$ be of bounded variation over every finite segment, and

$$(K) \qquad A(x') - A(x) > -m \quad \text{for} \quad x \leqslant x' \leqslant x + x^{1/k}.$$

Then

$$A(x) = O(1), \quad x \to \infty.$$

I have proved this Theorem in a note "Über das Verhalten Dirichletscher Reihen am Rande des Konvergenzgebietes" (*Math. Zeit.* **46** (1940), pp. 650—664, Satz B).

$J(s)$ of the Theorem A fulfils the assumptions (i) and (ii) of the Theorem B (with $a = 1$). Now, I have not but to prove that

$$A(x) = \frac{a(x) - x}{x^{1/k}}$$

satisfies the assumption (K).

As

$$A(x') - A(x) = \frac{1}{x^{1/k}}\{a(x') - a(x)\} - \frac{1}{x^{1/k}}(x' - x) - a(x')\left[\frac{1}{x^{1/k}} - \frac{1}{x'^{1/k}}\right]$$

hence, on account of (4)

$$A(x') - A(x) \geqslant -\frac{x' - x}{x^{1/k}} - C_1 x'^{\,1-1/k} \cdot \frac{x'^{1/k} - x^{1/k}}{x^{1/k}}.$$

If $x \leqslant x' \leqslant x + x^{1/k}$ we have

$$x'^{1/k} - x^{1/k} = O(x^{2/k-1}), \quad x \to \infty,$$

and hence

$$A(x') - A(x) > -m \quad \text{for} \quad x \leqslant x' \leqslant x + x^{1/k}.$$

In consequence we can apply the Theorem B so that we obtain

$$\frac{a(x) - x}{x^{k/1}} = O(1), \quad x \to \infty,$$

wherefrom, because of $1 < k < (1 + \sqrt{3})/2$ and $2/(1 + \sqrt{3}) = \sqrt{3} - 1$ the theorem is proved.

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