ON TAUBERIAN THEOREMS FOR THE RIEMANN-LIOUVILLE INTEGRAL

by

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1. *INTRODUCTION*

Let $\varphi(x)$ be a function of bounded variation in every finite interval of $x \geqslant 0$, assumed (for simplicity) to be such that $\varphi(0) = 0$. Then the Riemann-Liouville integral of $\varphi(x)$, of order $\alpha \geqslant 0$, is defined by

$$(1.1) \qquad \Phi_\alpha(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \varphi(t)\, dt, \quad \alpha > 0; \quad \Phi_0(x) = \varphi(x).$$

In special cases when we have to define $\Phi_\alpha(x)$ for $x < 0$, we suppose that $\varphi(x) = 0$ for $x < 0$, so that $\Phi_\alpha(x) = 0$ for $x \leqslant 0$. The object of this paper is to study a general "converse" or Tauberian theorem in which a conclusion regarding the order of $\Phi_{\alpha'}(x)$, $\alpha' \geqslant 0$, as $x \to \infty$, emerges from a standard order condition imposed on $\Phi_\alpha(x)$, $\alpha > \alpha'$, together with a condition on $\Phi_{\alpha'}(x)$ describable as an extension of the well-known condition of "slow increase". This theorem may be made the basis of a complete Tauberian theory for $\Phi_\alpha(x)$[1]; for, arising from it naturally, there is a host of other converse theorems which have been treated separately (e. g. [2], theorems in §§ 1.6, 1.7, 1.8; [4], Théorème 1′; [5], Theorem 2), among them a classical theorem of M. Riesz ([7], §§ 5,6).

The proof of our general theorem is by a (now familiar) method which depends on certain formulae for differences of integral and non-integral orders of the Riemann-Liouville integral, first introduced by Karamata ([4], Lemme 2) in the case of integral orders and then inde-

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[1] The theory will of course cover directly the case in which $\Phi_\alpha(x) = o(x^\alpha)$ or $\varphi(x)$ is summable$-(C, \alpha)$ to sum $s = 0$. The case of $\varphi(x)$ summable$-(C, \alpha)$ to sum $s \neq 0$, being simply that of $\varphi(x) - s$ summable$-(C, \alpha)$ to 0, will be indirectly covered.

pendently by Bosanquet ([1], Theorem 1) in the case of non-integral orders also[2]. Minakshisundaram and Rajagopal ([5], [6]) have used these difference formulae to prove some Tauberian theorems for the Riesz sum $A_\alpha(x)$, $\alpha \geqslant 0$, of a series $\sum_{n=0}^{\infty} a_n$ with respect to a sequence $\{\lambda_n\}$ such that

$$0 < \lambda_0 < \lambda_1 < \lambda_2 < \ldots, \quad \lim \lambda_n = \infty,$$

adopting a definition which (to suit our present purpose) may be modified as under:

$$(1.2) \qquad A_\alpha(x) \equiv \Phi_\alpha(x) \text{ when } \varphi(x) \equiv A_0(x) = \begin{cases} \sum\limits_{\lambda_n \leqslant x} a_n & \text{for } n = 0, 1, 2, \ldots, \\ 0 & \text{for } x < \lambda_0. \end{cases}$$

The principal theorem of this paper includes the said theorems of Minakshisundaram and Rajagopal besides the theorems referred to in the preceding paragraph.

2. *LEMMAS*

The "backward-difference-formula" of Karamata-Bosanquet, given in the following lemma, is easily verified by induction.

LEMMA 1*a*. *If* $h > 0$, $p = 0, 1, 2, \ldots,$ *then*

$$(2.1a) \qquad \Delta^p_{-h}\,\Phi_{\alpha+p}(x) \equiv \sum_{\nu=0}^{p} (-1)^\nu \binom{p}{\nu} \Phi_{\alpha+p}(x - \nu h) = \int_{x-h}^{x} dt_1 \int_{t_1-h}^{t_1} dt_2 \ldots \int_{t_{p-1}-h}^{t_{p-1}} \Phi_\alpha(t_p)\, dt_p .$$

Analogous to (2.1*a*) there is a backward-difference-formula of non-integral order δ, $0 < \delta < 1$, which Bosanquet has *defined* as follows replacing the p-th order integral in (2.1*a*) by a δ-th order integral:

$$(2.2a) \qquad \Delta^\delta_{-h}\,\Phi_{\alpha+\delta}(x) \equiv \delta \int_{x-h}^{x} (x-t)^{\delta-1}\,\Phi_\alpha(t)\,dt, \quad 0 < \delta < 1.$$

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[2]) Karamata's papers [4], [5] which introduce the difference-formulae in question are dated November 1939 although they remained unpublished, the first till 1948 and the second till 1950.

Combining (2.1a) with (2.2a), we obtain the next lemma.

LEMMA 2a. *If* $h > 0$, $p = 0,1,2\ldots$, $0 < \delta < 1$, *then*

$$(2.3a)\quad \Delta^{p+\delta}_{-h}\, \Phi_{\alpha+p+\delta}(x) = \delta \int_{x-h}^{x} (x-t)^{\delta-1}\, dt \int_{t-h}^{t} dt_1 \int_{t_1-h}^{t_1} dt_2 \ldots \int_{t_{p-1}-h}^{t_{p-1}} \Phi_\alpha(t_p)\, d\,t_p$$

where

$$\Delta^{p+\delta}_{-h}\, \Phi_{\alpha+p+\delta}(x) \equiv \Delta^{p}_{-h}\, \Delta^{\delta}_{-h}\, \Phi_{\alpha+p+\delta}(x).$$

Proof. Clearly (2.3a) is the same as the formula

$$\Delta^{p}_{-h}\, \delta \int_{x-h}^{x} (x-t)^{\delta-1}\, \Phi_{\alpha+p}(t)\, dt = \delta \int_{x-h}^{x} (x-t)^{\delta-1}\, \Delta^{p}_{-h}\, \Phi_{\alpha+p}(t)\, dt$$

which is an immediate deduction from the more general formula

$$\Delta^{p}_{-h}\, \delta \int_{x-h}^{x} f(x-t)\, g(t)\, dt = \delta \int_{x-h}^{x} f(x-t)\, \Delta^{p}_{-h}\, g(t)\, dt$$

true for all f and g integrable for the values of t in question.

The analogues of Lemmas 1a, 2a for "forward" differences, stated below, are proved just like these lemmas.

LEMMA 1b. *If* $h > 0$, $p = 0,1,2,\ldots$, *then*

$$(2.1b)\quad \Delta^{p}_{-h}\, \Phi_{\alpha+p}(x) \equiv \sum_{\nu=0}^{p} (-1)^\nu \binom{p}{\nu} \Phi_{\alpha+p}(x + \overline{p-\nu}\, h) = \int_{x}^{x+h} dt_1 \int_{t_1}^{t_1+h} dt_2 \ldots \int_{t_{p-1}}^{t_{p-1}+h} \Phi_\alpha(t_p)\, dt_p .$$

LEMMA 2b. *If* $h > 0$, $p = 0,1,2,\ldots$, $0 < \delta < 1$, *then*

$$(2.3b)\quad \Delta^{p+\delta}_{h}\, \Phi_{\alpha+p+\delta}(x) = \delta \int_{x}^{x+h} (x+h-t)^{\delta-1}\, dt \int_{t}^{t+h} dt_1 \int_{t_1}^{t_1+h} dt_2 \ldots \int_{t_{p-1}}^{t_{p-1}+h} \Phi_\alpha(t_p)\, dt_p,$$

where

$$(2.2b)\quad \Delta^{p+\delta}_{h}\, \Phi_{\alpha+p+\delta}(x) \equiv \Delta^{p}_{h}\, \Delta^{\delta}_{h}\, \Phi_{\alpha+p+\delta}(x) \equiv \Delta^{p}_{h}\, \delta \int_{x}^{x+h} (x+h-t)^{\delta-1}\, \Phi_{\alpha+p}(t)\, dt .$$

Besides the above lemmas, there is another auxiliary result required for our purpose:

LEMMA 3. *If* $\alpha \geqslant 0$, $0 < \delta < 1$, *and* $W(x)$ *is a positive monotonic increasing function of* $x > 0$, *then the hypothesis*

$$|\Phi_{\alpha+\delta}(x)| \leqslant m\, W(x), \quad m = \text{ a constant},$$

implies the conclusions:

$$\text{(i)} \qquad \delta \left| \int_0^{\xi} (x-t)^{\delta-1}\, \Phi_\alpha(t)\, dt \right| \leqslant cm\, W(x),$$

where $0 \leqslant \xi < x$ *and* $c =$ *a constant depending on* δ,

$$\left.\begin{array}{ll} \text{(ii}a\text{)} & |\Delta_{-h}^{p+\delta}\, \Phi_{\alpha+\delta}\,(x)| \leqslant 2^{p+1}\, cm\, W(x), \\ \text{(ii}b\text{)} & |\Delta_h^{p+\delta}\, \Phi_{\alpha+\delta}\,(x)| \leqslant 2^{p+1}\, cm\, W(x+\overline{p+1}\,h). \end{array}\right\} \; p = 0,1,2,\ldots$$

When m in the hypothesis can be made arbitrarily small for all large x, it can be made so in conclusions (i), (ii*a*), (ii*b*) *as well.*

Conclusion (i) of Lemma 3 is a consequence of well-known results ([2], Lemmas 1.41, 1.42; or [7], §§ 2, 8), while conclusions (iia), (iib), ar. obvious implications of conclusion (i).

3. *THE MAIN THEOREM*

The following theorem may be viewed as an extension of a result proved by Minakshisundaram and Rajagopal ([5], Theorem 2)‘ a version of the latter being the case $\alpha' = 0$ of the former.

THEOREM I. *Let* $\alpha, \alpha', \alpha > \alpha' \geqslant 0$, *and a function* $\theta(x) >$ *of* $x > 0$ *be given In the notation of* §§ 1, 2 *above, let*

$$\text{(i)} \qquad |\Phi_\alpha\,(x)| \leqslant m\, W(x).$$

Then

$$\text{(ii}a\text{)} \qquad \overline{\underset{0<t<\theta(x)}{\text{bound}}}\, [\Phi_{\alpha'}(x) - \Phi_{\alpha'}(x-t)] \leqslant \frac{m\, W(x)}{[\theta(x)]^{\alpha-\alpha'}}$$

implies

$$\Phi_{\alpha'}(x) \leqslant K \frac{m\, W(x)}{[\theta(x)]^{\alpha-\alpha'}}$$

where $K =$ *a constant generally involving* α, α'; *while*

(ii*b*) $$\overline{\text{bound}}_{0 < t < \theta(x)} [\Phi_{\alpha'}(x+t) - \Phi_{\alpha'}(x)] \leqslant \frac{m\, W(x)}{[\theta(x)]^{\alpha-\alpha'}}$$

implies

$$\Phi_{\alpha'}(x) \geqslant - K \frac{m\, W(x+\theta(x))}{[\theta(x)]^{\alpha-\alpha'}}$$

Proof. Let $p \geqslant 0$ be the greatest integral less than $\alpha - \alpha'$ so that $\alpha = \alpha' + p + \delta, \quad 0 < \delta \leqslant 1$.

To prove the conclusion associated with (ii*a*), we first note that we may assume (without loss of generality) $\theta(x) \leqslant x$. For, whenever $\theta(x) > x$, we can choose $t \geqslant x$ and obtain from (ii*a*) alone, without using (i) but remembering that $\Phi_{\alpha'}(x) = 0$ for $x \leqslant 0$, the desired conclusion in the form

$$\Phi_{\alpha'}(x) \leqslant \frac{m\, W(x)}{[\theta(x)]^{\alpha-\alpha'}}.$$

We can rewrite (2.3*a*) with α replaced by α', in the form

(3.1)
$$h^{\alpha-\alpha'} \Phi_{\alpha'}(x) = \\ = \Delta_{-h}^{p+\delta} \Phi_{\alpha'+p+\delta}(x) + \delta \int_{x-h}^{x} (x-t)^{\delta-1}\, dt \int_{t-h}^{t} dt_1 \cdots \int_{t_{p-1}-h}^{t_{p-1}} \{\Phi_{\alpha'}(x) - \Phi_{\alpha'}(t_p)\}\, dt_p,$$

where

$$x-(p+1)h < \cdots < t_{p-2} - 2h < t_{p-1} - h < t_p < t_{p-1} < t_{p-2} < \cdots < x$$

Taking $(p+1)h = \theta(x)$ in (3.1), we find that

(3.2)
$$h^{\alpha-\alpha'} \Phi_{\alpha'}(x) \leqslant \\ \leqslant |\Delta_{-h}^{p+\delta} \Phi_{\alpha'+p+\delta}(x)| + \delta \int_{x-h}^{x} (x-t)^{\delta-1}\, dt \int_{t-h}^{t} dt_1 \cdots \int_{t_{p-1}-h}^{t_{p-1}} \{\Phi_{\alpha'}(x) - \Phi_{\alpha'}(t_p)\}\, dt_p,$$

where $0 < x - t_p < (p+1)h = \theta(x)$ and consequently, by hypothesis (ii*a*),

(3.3) $$\Phi_{\alpha'}(x) - \Phi_{\alpha'}(t_p) \leqslant \frac{m\, W(x)}{[\theta(x)]^{\alpha-\alpha'}};$$

while, by Lemma 3 applied to hypothesis (i),

$$(3.4)\qquad |\Delta_{-h}^{p+\delta}\,\Phi_{\alpha'+p+\delta}(x)| \leqslant 2^{p+1}\,cm\,W(x).$$ [8]

Using (3.3) and (3.4) in (3.2), and then dividing by $h^{\alpha-\alpha'}$ both sides of the inequality resulting from (3.2), we conclude that

$$(3.5a)\qquad \Phi_{\alpha'}(x) \leqslant K\,\frac{m\,W(x)}{[\theta(x)]^{\alpha-\alpha'}},\quad K = 2^{p+1}\,c\,(p+1)^{\alpha-\alpha'} + 1.$$

This is the conclusion sought, K being a constant which depends on α, α' (the constant δ involved in c and the constant p being such).

We can prove in the same way the conclusion associated with (ii*b*), namely

$$(3.5b)\qquad \Phi_{\alpha'}(x) \geqslant -K\,\frac{m\,W(x+\theta(x))}{[\theta(x)]^{\alpha-\alpha'}},\quad K = \text{ the constant in } (3.5a),$$

by changing α to α' in (2.3*b*) and then rewriting (2.3*b*) in the form

$$h^{\alpha-\alpha'}\,\Phi_{\alpha'}(x) =$$

$$= \Delta_h^p\,\Phi_{\alpha'+p+\delta}(x) - \delta\int_x^{x+h}(x+h-t)^{\delta-1}\,dt\int_t^{t+h}dt_1\ldots\int_{t_{p-1}}^{t_{p-1}+h}\{\Phi_{\alpha'}(t_p) - \Phi_{\alpha'}(x)\}\,dt_p\,.$$

Note. In (3.5*b*) we can replace $W(x+\theta(x))$ by $H\,W(x)$ when there is a constant H such that

$$(3.6)\qquad W(x+\theta(x)) \leqslant H\,W(x).$$

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[8] In appealing to Lemma 3 to establish (3.4) we tacitly assume that $\delta < 1$. In the case $\delta = 1$, we get (3.4) without appealing to Lemma 3, in the form

$$|\Delta_{-h}^{p+1}\,\Phi_\alpha(x)| \leqslant 2^{p+1}cm\,W(x),\quad c = 1.$$

It may be remarked that, in the above inequality, m can be made arbitrarily small for all large x provided that it can be made so in hypothesis (i) and provided also that $x - \theta(x) \to \infty$ as $x \to \infty$.

The two cases (a) and (b) of Theorem I can be combined to produce the next theorem which, in the case $\alpha' = 0$, reduces to a theorem of Chandrasekharan and Minakshisundaram ([2], Theorem 1.83).

THEOREM I′. *Suppose that, for* $x > 0$, $W(x)$ *is positive monotonic increasing and subject to* (3.6), *and* $V(x)$ *is positive. Suppose that* α, α', γ *are given numbers such that* $\alpha > \alpha' \geqslant 0$, $\gamma > 0$,

$$\text{(i)} \qquad |\Phi_\alpha(x)| \leqslant m\, W(x),$$

$$\left.\begin{array}{ll}\text{(ii}a) & \Phi_{\alpha'}(x) - \Phi_{\alpha'}(x-t) \leqslant t^\gamma\, l\, V(x) \\ \text{(ii}b) & \Phi_{\alpha'}(x+t) - \Phi_{\alpha'}(x) \leqslant t^\gamma\, l\, V(x)\end{array}\right\} \text{for } 0 < t < \left[\frac{m\, W}{l\, V}\right]^{1/(\alpha-\alpha'+\gamma)} \equiv \theta(x),$$

where l, m *are (positive) constants. Then*

$$|\Phi_{\alpha'}(x)| \leqslant H K [l\, V]^{(\alpha-\alpha')/(\alpha-\alpha'+\gamma)} [m\, W]^{\gamma/(\alpha-\alpha'+\gamma)} \equiv H K \frac{m\, W(x)}{[\theta(x)]^{\alpha-\alpha'}}$$

where K *is a constant independent of* l *and* m.

The deduction of Theorem I′ from Theorem I is obvious.

Remarks. (1) By taking $-\varphi(x)$ instead of $\varphi(x)$, we can rewrite Theorem I′ with the inequalities in hypotheses (ii$a$), (ii$b$) reversed and V in these hypotheses changed to $-V$, but without any other change.

(2) We may restate Theorem I′, with its *two* one-sided hypotheses (ii$a$), (ii$b$) changed to the *single* two-sided hypothesis:

$$\text{(3.7)} \quad \left.\begin{array}{ll}\textit{either} & |\Phi_{\alpha'}(x) - \Phi_{\alpha'}(x-t)| \\ \textit{or} & |\Phi_{\alpha'}(x+t) - \Phi_{\alpha'}(x)|\end{array}\right\} \leqslant t^\gamma\, l\, V(x) \text{ for } 0 < t < \left[\frac{m\, W}{l\, V}\right]^{1/(\alpha-\alpha'+\gamma)},$$

but without any change in the essential character of the conclusion, even dispensing with (3.6) in the case of the first alternative of (3.7). In such a restatement of Theorem I we may further suppose that either the numbers l, m are both finite or one of them can be made arbitrarily small for all large x.[4] The final result will then be as follows in the case of the first alternative of (3.7).

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[4] We justify taking m arbitrarily small for all large x in the conclusion of Theorem I′, when it is so in the hypothesis of the theorem, by appealing to Lemma 3 in case $\alpha-\alpha'$ is not an integer and to the remark in footnote [3] in case $\alpha-\alpha$ is an integer.

COROLLARY I′. *Suppose that, for* $x > 0$, $W(x) > 0$, $V(x) > 0$, *and the first function is monotonic increasing. Suppose that* $\alpha > \alpha' \geqslant 0$, $\gamma > 0$, *and that, when* $x \to \infty$,

(i) $$\Phi_\alpha(x) = O(W(x)),$$

(ii) $$\Phi_{\alpha'}(x) - \Phi_{\alpha'}(x-t) = O(t^\gamma V(x)) \text{ for } 0 < t = O\left(\left(\frac{W}{V}\right)^{1/(\alpha-\alpha'+\gamma)}\right)$$

where the constant in the last O-restriction depends on the constants in the remaining two O-restrictions. Then, as $x \to \infty$,

$$\Phi_{\alpha'}(x) = O(V^{(\alpha-\alpha')/(\alpha-\alpha'+\gamma)} W^{\gamma/(\alpha-\alpha'+\gamma)}).$$

Further, when O is replaced by o in either (i) or (ii), *O in the conclusion will be replaced by o.*

4. *A THEOREM OF M. RIESZ*

The following theorem of M. Riesz, referred to at the outset, is a deduction from Corollary I′.

THEOREM II. *Suppose that* $W(x)$, $V(x)$ *are positive monotonic increasing functions of* $x > 0$. *Suppose that, when* $x \to \infty$,

(i) $$\Phi_\alpha(x) = O(W(x)) \text{ for a given } \alpha > 0,$$

(ii) $$\varphi(x) = O(V(x)).$$

Then, as $x \to \infty$,

$$\Phi_{\alpha'}(x) = O(V^{1-\alpha'/\alpha} W^{\alpha'/\alpha}) \text{ for all } \alpha' \text{ such that } 0 < \alpha' < \alpha.$$

Further, when O is replaced by o in either (i) or (ii), *O in the conclusion will be replaced by o.*

Proof. Case 1: $0 < \alpha' \leqslant 1$.

For $0 < t < x$, we have

$$\Gamma(\alpha')[\Phi_{\alpha'}(x) - \Phi_{\alpha'}(x-t)] = \int_0^{x-t} [(x-u)^{\alpha'-1} - (x-t-u)^{\alpha'-1}]\varphi(u)\,du +$$

$$+ \int_{x-t}^{x} (x-u)^{\alpha'-1}\varphi(u)\,du.$$

Hence, by hypothesis (ii),

$$|\Phi_{\alpha'}(x) - \Phi_{\alpha'}(x-t)| <$$

$$< O(V(x)) \left[\int_0^{x-t} \{(x-t-u)^{\alpha'-1} - (x-u)^{\alpha'-1}\} \, du + \int_{x-t}^{x} (x-u)^{\alpha'-1} du \right]$$

i. e.

$$|\Phi_{\alpha'}(x) - \Phi_{\alpha'}(x-t)| <$$

$$< O(V(x)) \left[\frac{(x-t)^{\alpha'}}{\alpha'} - \frac{x^{\alpha'}}{\alpha'} + \frac{t^{\alpha'}}{\alpha'} + \frac{t^{\alpha'}}{\alpha'} \right] < O(V(x)) \frac{2t^{\alpha'}}{\alpha'}.$$

Thus, in the case $0 < \alpha' \leqslant 1$, comparing the above inequality with hypothesis (ii) of Corollary I' we find that Theorem II is simply the case $\gamma = \alpha$ of Corollary I'.

Case 2: $\alpha' > 1$. First, supposing the theorem to be true for $\alpha' =$ a positive integer m, we can prove it for $m < \alpha' \leqslant m+1$ where of course $m + 1 < \alpha$. For, this supposition is, when (for the sake of definiteness) we choose O in both hypotheses (i) and (ii),

$$\Phi_m(x) = O(V^{1-m/\alpha} W^{m/\alpha}) \text{ along with } \Phi_\alpha(x) = O(W),$$

or

$$\Phi_0^*(x) = O(V^*) \text{ along with } \Phi_{\alpha-m}^*(x) = O(W),$$

where V^* is positive monotonic increasing. Since $0 < \alpha' - m \leqslant 1$, Case 1 shows that our supposition carries with it the implication

$$\Phi_{\alpha'-m}^*(x) = O\left(V^{*\left(1-\frac{\alpha'-m}{\alpha-m}\right)} W^{\frac{\alpha'-m}{\alpha-m}}\right),$$

or

$$\Phi_{\alpha'}(x) = O\left(V^{1-\frac{\alpha'}{\alpha}} W^{\frac{\alpha'}{\alpha}}\right), \quad m < \alpha' \leqslant m+1.$$

Next let r be the greatest integer less than α. Then $r \geqslant 1$ since $\alpha > \alpha' > 1$. Also, the theorem being true for $\alpha' = 1$ (by Case 1), we can take $m = 1, 2, \ldots r$ successively in the argument of the preceding paragraph and establish the theorem for $1 < \alpha' < \alpha \leqslant r+1$. This completes the proof.

Note. Chandrasekharan and Minakshisundaram have proved ([2], Theorem 1.84) that Theorem II is true in the case $\alpha > 1$, for any α' such that $1 \leqslant \alpha' < \alpha$, with O in hypothesis (ii) replaced by O_R (or O_L) provided that (3.6) holds for W and also for V in place of W. This result is readily obtained by combining Theorem II with the case $\gamma = \alpha' = 1$ of Theorem I'.[5)]

5. *A LIMITATION THEOREM FOR SERIES SUMMABLE*—$R(\lambda_n, \alpha)$.

The next theorem, due to Chandrasekharan, Minakshisundaram and Rajagopal ([6] – A, § 2.4, p. 157; [2], Theorem 1.61, Cases (i), (ii)), is an easy deduction from Theorem I with hypothesis (ii*b*). In the case $W(x) \equiv x^\alpha$, $\alpha > \alpha' = 0$, it brings out a necessary restriction or limitation on series summable – $R(\lambda_n, \alpha)$ to 0 and thence a restriction on series so summable to $s \neq 0$.[6)]

THEOREM III. *Let $W(x)$ be a positive monotonic increasing function of $x > 0$. Let $A_\alpha(x)$, $\alpha \geqslant 0$, be defined according to* (1.2). *Then the hypothesis*

$$A_\alpha(x) = o(W(x)), \quad x \to \infty, \tag{5.1}$$

for a given $\alpha > 0$ implies the conclusion

$$A_{\alpha'}(x) = o\left(\frac{W(\lambda_{n+1})}{(\lambda_{n+1} - \lambda_n)^{\alpha-\alpha'}}\right) \quad \textit{for} \quad \lambda_n \leqslant x < \lambda_{n+1}, \ \alpha' = 0, 1, \ldots r, \tag{5.2}$$

where r is the greatest integer in α.

Proof. We appeal to Theorem I with both its hypotheses (i), (ii*b*) in o-forms, i. e. with (ii*b*) a two-sided inequality just like (i) and m arbitrarily small for all large x. The conclusion of Theorem I in these special circumstances is (3.5*b*) in o-form. In Theorem III, hypothesis (i) of Theorem I appears in the o-form (5.1), while hypothesis (ii*b*) with $\alpha' = 0$ is present implicitly in the special o-form:

$$A_0(\lambda_n + t) - A_0(t) = 0 \quad \text{for} \quad 0 < t < \theta(\lambda_n) \equiv \lambda_{n+1} - \lambda_n.$$

Hence the conclusion of Theorem III follows in the o-form of (3.5*b*) which is now the same as (5.2) in the case $\alpha' = 0$.

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5) A more general case of Theorem I', viz. the case $\gamma = 1$, $\alpha' = k+1$, $\alpha = k+r+1$, where $k \geqslant 0, r > 0$, was treated earlier by Minakshisundaram and Rajagopal ([5], Corollary 2.1).

6) $R(\lambda_n, \alpha)$—summability of Σa_n is defined as (C, α)—summability of $\varphi(x)$ when $\varphi(x)$ is the λ_n—step function $A_0(x)$ of (1.2).

The remaining case $\alpha' = 1, 2, \dots r$ of (5.2) is proved by induction based on the case $\alpha' = 0$. For, if (5.2) holds for an integer $\alpha' = m$, $0 \leqslant m \leqslant r-1$, then it holds for $\alpha' = m+1$ as well by the following argument. For $\lambda_n \leqslant u < \lambda_{n+1}$ or $0 \leqslant u - \lambda_n < \theta(\lambda_n)$,

$$(5.3) \qquad A_{m+1}(u) - A_{m+1}(\lambda_n) =$$

$$= \int_{\lambda_m}^{u} A_m(t)\,dt = o\left((\lambda_{n+1} - \lambda_n)\frac{W(\lambda_{n+1})}{(\lambda_{n+1} - \lambda_n)^{\alpha - m}}\right) \equiv o\left(\frac{W(\lambda_{n+1})}{[\theta(\lambda_n)]^{\alpha - m + 1}}\right).$$

(5.3) is hypothesis (ii*b*) of Theorem I in *o*-form, with $\Phi_{\alpha'}(u) \equiv A_{m+1}(u)$, $x = \lambda_n$ and the immaterial difference that $W(x)$ in the hypothesis is now replaced by $W(x + \theta(x))$. Taking (5.3) along with (5.1), and appealing to the *o*-form of Theorem I already used, we establish (5.2) for $\alpha' = m+1$ and so complete the proof.

6. *AN EXTENSION OF A THEOREM OF KARAMATA*

We have seen that Theorem I with hypothesis (ii*a*) becomes trival for all x such that $\theta(x) > x > 0$. We shall therefore suppose in this section that

$$(6.1) \qquad 0 < \theta(x) \leqslant x,$$

and further that $\theta(x)$ belongs to the class of positive functions $f(x)$ of $x > 0$ such that constants H, H' exist, corresponding to constants h, h' satisfying the condition:

$$(6.2) \qquad 0 < H' < \frac{f(y)}{f(x)} < H < \infty \quad \text{when} \quad 0 < h' < \frac{y}{x} < h < \infty.$$ [7]

These suppositions about $\theta(x)$ result in a simplification of Theorem I in certain cases; because they make one alone of the Tauberian hypotheses

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[7] In the theorems of this section we suppose that $W(x)$, $\theta(x)$ belong to the class of function f restricted by (6.2), although (as it will be evident from the proofs of the theorems) W and θ in the theorems need only be of the less restricted class of functions g (associated with the particular θ in question) which satisfy:

$$0 < H' < \frac{g(y)}{g(x)} < H < \infty \text{ when } 0 < |y - x| < \eta\,\theta(x), \quad \eta < 1.$$

(ii$a$), (ii$b$) of Theorem I, in a particular form, sufficient for a conclusion which we expect to result from the two hypotheses together in the same form. Such a simplification of Theorem I is presented below as a new theorem.

THEOREM IV. *Suppose that* $\theta(x)$, $W(x)$ *are functions of the class* f *and the second function is monotonic increasing. Suppose that* α, α' *are given such that* $0 \leqslant \alpha' < \alpha$,

$$\Phi_\alpha(x) = o(W(x)), \quad x \to \infty, \tag{6.3}$$

and o n e o n l y *of the following conditions is fulfilled:*

$$\overline{\lim_{x\to\infty}} \quad \overline{\operatorname{bound}}_{y-\varepsilon\theta(y)<x<y} \frac{[\Phi_{\alpha'}(y)-\Phi_{\alpha'}(x)]}{W(y)/[\theta(y)]^{\alpha-\alpha'}} \leqslant \omega(\varepsilon) \to +0 \quad as \quad \varepsilon \to 0, \tag{6.4a}$$

$$\overline{\lim_{x\to\infty}} \quad \overline{\operatorname{bound}}_{x<y<x+\varepsilon\theta(x)} \frac{[\Phi_{\alpha'}(y)-\Phi_{\alpha'}(x)]}{W(x)/[\theta(x)]^{\alpha-\alpha'}} \leqslant \omega^*(\varepsilon) \to +0 \quad as \quad \varepsilon \to 0. \tag{6.4b}$$

Then

$$\Phi_{\alpha'}(x) = o\left(\frac{W(x)}{[\theta(x)]^{\alpha-\alpha'}}\right), \quad x \to \infty.$$

Proof. Condition (6.3) is the same as

$$|\Phi_\alpha(x)| < \varepsilon^{\alpha-\alpha'+1} W(x) \quad \text{for all large } x, \tag{6.5}$$

ε being any small positive number. Conditions (6.4a), (6.4b) can be written, when $x \to \infty$, in the forms

$$\Phi_{\alpha'}(y) - \Phi_{\alpha'}(x) < [\omega(\varepsilon) + o(1)] \frac{W(y)}{[\theta(y)]^{\alpha-\alpha'}} \quad \text{for } 0 < y - x < \varepsilon\,\theta(y), \tag{6.6a}$$

$$\Phi_{\alpha'}(y) - \Phi_{\alpha'}(x) < [\omega^*(\varepsilon) + o(1)] \frac{W(x)}{[\theta(x)]^{\alpha-\alpha'}} \quad \text{for } 0 < y - x < \varepsilon\,\theta(x). \tag{6.6b}$$

Now (6.5), (6.6a), (6.6b) are obvious variants of hypotheses (i), (ii$a$), (ii$b$) respectively of Theorem I. Hence, assuming that (6.4a) and (6.4b) are *both* fulfilled along with (6.3), we can conclude from (3.5a) and (3.5b) that, as $x \to \infty$,

$$-[K^* + o(1)] \frac{W(x+\varepsilon\theta(x))}{[\theta(x)]^{\alpha-\alpha'}} < \Phi_{\alpha'}(x) < [K + o(1)] \frac{W(x)}{[\theta(x)]^{\alpha-\alpha'}}, \tag{6.7}$$

where

$$\begin{cases} K = 2^{p+1} c (p+1)^{\alpha-\alpha'} \varepsilon + \omega(\varepsilon), \\ K^* = 2^{p+1} c (p+1)^{\alpha-\alpha'} \varepsilon + \omega^*(\varepsilon). \end{cases}$$

In (6.7) K, K^* can each be made arbitrarily small by choosing ε sufficiently small, and $W(x+\varepsilon\,\theta(x)) = O(W(x))$ on account of W being a function of the class f satisfying (6.2). Hence (6.7) establishes the conclusion of Theorem IV on the assumption that *both* (6.4a) and (6.4b) hold. To complete the proof we have to establish the same conclusion assuming that *either* (6.4a) *or* (6.4b) holds. For this purpose it is enough to appeal to the next lemma.

LEMMA 4. *If* $W(x)$, $\theta(x)$ *are any functions of the class* f, *then* (6.4a) *and* (6.4b) *are equivalent.*

Proof. It is proved below that (6.4a) follows from (6.4b). The proof that (6.4b) follows from (6.4a) is along the same lines.

If

(6.8) $$y-\varepsilon\theta(y) < x < y, \quad 0 < \varepsilon < {}^1/_2,$$

then

$$\frac{y}{x} < \left[1-\varepsilon\frac{\theta(y)}{y}\right]^{-1} < 1+2\varepsilon\frac{\theta(y)}{y},$$

i. e.

$$y < x + 2\varepsilon\frac{x}{y}\theta(y) < x+\varepsilon H\theta(x),$$

where H is a constant independent of ε since $x<y<2x$ and $\theta(x)$ is of the class f which satisfies (6.2). Hence, when (6.4b) is given, we obtain under the conditions (6.8) and $x\to\infty$,

$$\frac{\Phi_{\alpha'}(y)-\Phi_{\alpha'}(x)}{W(y)/[\theta(y)]^{\alpha-\alpha'}} = \frac{\Phi_{\alpha'}(y)-\Phi_{\alpha'}(x)}{W(x)/[\theta(x)]^{\alpha-\alpha'}}\cdot\frac{W(x)}{W(y)}\cdot\left[\frac{\theta(y)}{\theta(x)}\right]^{\alpha-\alpha'} <$$

$$< [\omega^*(\varepsilon H)+o(1)]\cdot O(1)\cdot O(1),$$

so that (6.4a) follows from (6.4b).

COROLLARY IV. *If, in Theorem IV,* $\theta(x)\equiv x^{1-(a-b)/\alpha}$, $W(x)\equiv x^{\alpha+b}$, $a\geqslant b\geqslant -\alpha$, $\alpha>\alpha'=0$, *we get the essentials of a theorem of* Karamata ([4], Théorème 1).[8]

The theorem which follows is a modification of Theorem IV resulting from an obvious modification in the proof of the latter theorem.

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[8] Karamata, assuming that α is an integer, avoids the use of Lemma 3, as explained in footnote[5]). In other words, he does not require the condition in the lemma that $W(x)$ is monotonic increasing, i. e. the condition $b\geqslant-\alpha$.

THEOREM V. *If, in Theorem IV, either of the alternative hypotheses* (6.4*a*), (6.4*b*) *is changed to* one *of the following:*

$$(6.9a)\qquad \varlimsup_{x\to\infty}\ \overline{\text{bound}}_{y-\varepsilon\theta(y)<x<y}\ [\Phi_{\alpha'}(y)-\Phi_{\alpha'}(x)]<o(1)\quad as\quad \varepsilon\to 0,$$

$$(6.9b)\qquad \varlimsup_{x\to\infty}\ \overline{\text{bound}}_{x<y<x+\varepsilon\theta(x)}\ [\Phi_{\alpha'}(y)-\Phi_{\alpha'}(x)]<o(1)\quad as\quad \varepsilon\to 0,$$

the conclusion will be changed to

$$\Phi_{\alpha'}(x)=o\left(\frac{W(x)}{[\theta(x)]^{\alpha-\alpha'}}\right)+o(1).$$

Karamata ([3], § 1.1) gives a version of one particular case of Theorem V, in which $\Phi_\alpha(x)\equiv A_1(x)$, $\Phi_{\alpha'}(x)\equiv A_0(x)$ and $A_0(x)$ is a series defined as in (1.2) but with $\lambda_n=n$. The next section contains similar results for series, but more general than Karamata's theorem just referred to.

7. SOME TAUBERIAN THEOREMS FOR RIESZ SUMS OF SERIES

The series of this section are all conceived as λ_n-step functions defined as in (1.2). One of the theorems for them, given below as part of Theorem VI, is analogous to Corollary IV and depends on Theorem I (like Theorem IV) as well as the lemma which follows.

LEMMA 5. *If* $A_0(x)$ *is defined as in* (1.2), *then the condition*

$$(7.1)\quad \varlimsup_{n\to\infty}\ \max_{\lambda_n\leqslant\lambda_m<\lambda_n+\varepsilon\theta(\lambda_n)}(a_n+a_{n+1}+\ldots a_m)/\lambda_n^a\leqslant\Omega(\varepsilon)\to+0\quad as\quad \varepsilon\to 0,$$

where

$$\theta(x)\equiv x^{1-(a-b)/\alpha},\quad a\geqslant b,\quad \alpha>0,$$

implies both *the following conditions:*

$$(7.2a)\qquad \overline{\text{bound}}_{\lambda_m-\varepsilon\theta(\lambda_m)<x<\lambda_m}\ \frac{A_0(\lambda_m)-A_0(x)}{\lambda_m^a}\leqslant\omega(\varepsilon)+o(1),\quad m\to\infty,$$

$$(7.2b)\qquad \overline{\text{bound}}_{\lambda_n<y<\lambda_n+\varepsilon\theta(\lambda_n)}\ \frac{A_0(y)-A_0(\lambda_m)}{\lambda_n^a}\leqslant\omega^*(\varepsilon)+o(1),\quad n\to\infty,$$

where $\omega(\varepsilon)$, $\omega^*(\varepsilon)$ *both tend to* $+0$ *as* $\varepsilon\to 0$.

(7.2*a*), (7.2*b*) *are, in fact, special cases of* (6.4*a*), (6.4*b*) *respectively in which*

$$\Phi_{\alpha'}(u)\equiv A_0(u),\quad \theta(u)\equiv u^{1-(a-b)/\alpha},\quad W(u)\equiv u^{\alpha+b}.$$

Proof. Assuming (7.1), we first establish (7.2*a*). Suppose that

$$\lambda_n \leqslant \lambda_m - \varepsilon\,\theta(\lambda_m) < \lambda_{n+1} \quad (n < m),$$

$$\lambda_\nu \leqslant x < \lambda_{\nu+1} \quad (n \leqslant \nu \leqslant m-1).$$

Then the left-hand member of (7.2*a*) is

$$\text{(7.3)} \quad \max \frac{a_{\nu+1} + a_{\nu+2} + \dots a_m}{\lambda_{\nu+1}^a}\left(\frac{\lambda_{\nu+1}}{\lambda_m}\right)^a \text{ for } \lambda_m - \varepsilon\,\theta(\lambda_m) < \lambda_{n+1} \leqslant \lambda_{\nu+1} \leqslant \lambda_m$$

where the relation between $\lambda_{\nu+1}$ and λ_m gives

$$1-\varepsilon \leqslant 1 - \varepsilon\frac{\theta(\lambda_m)}{\lambda_m} < \frac{\lambda_{\nu+1}}{\lambda_m} \leqslant 1, \quad \text{or} \quad \left(\frac{\lambda_{\nu+1}}{\lambda_m}\right)^a \leqslant \frac{1}{(1-\varepsilon)^{|a|}},$$

and

$$\lambda_{\nu+1} \leqslant \lambda_m < \lambda_{\nu+1} + \varepsilon\,\theta(\lambda_{\nu+1})\left(\frac{\lambda_m}{\lambda_{\nu+1}}\right)^{1-(a-b)/\alpha} \leqslant \lambda_{\nu+1} + \frac{\varepsilon\,\theta(\lambda_{\nu+1})}{(1-\varepsilon)^{|1-(a-b)/\alpha|}}.$$

Hence we can apply (7.1) to (7.3) and obtain (7.2*a*) in the form

$$\overline{\text{bound}}_{\lambda_m - \varepsilon\theta(\lambda_m) < x < \lambda_m} \frac{A_0(\lambda_m) - A_0(x)}{\lambda_m^a} \leqslant$$

$$\leqslant \left[\Omega\left(\frac{\varepsilon}{(1-\varepsilon)^{|1-(a-b)/\alpha|}}\right) + o(1)\right]\frac{1}{(1-\varepsilon)^{|a|}}.$$

Next, assuming (7.1), we prove (7.2*b*). Suppose that

$$\lambda_m \leqslant \lambda_n + \varepsilon\,\theta(\lambda_n) < \lambda_{m+1} \quad (n \leqslant m),$$

$$\lambda_\nu \leqslant y < \lambda_{\nu+1} \quad (n \leqslant \nu \leqslant m).$$

Omitting the case $\nu = n$ in which (7.2*b*) is trivial, we may suppose that $\nu \geqslant n+1$. Then the left-hand member of (7.4*b*) is

$$\text{(7.4)} \quad \max \frac{a_{n+1} + a_{n+2} + \dots a_\nu}{\lambda_{n+1}^a}\left(\frac{\lambda_{n+1}}{\lambda_n}\right)^a \text{ for } \lambda_n < \lambda_{n+1} \leqslant \lambda_\nu \leqslant \lambda_m \leqslant \lambda_n + \varepsilon\,\theta(\lambda_n)$$

where the relation between λ_{n+1} and λ_ν gives

$$1 < \frac{\lambda_{n+1}}{\lambda_n} < 1 + \varepsilon,$$

$$\lambda_{n+1} \leqslant \lambda_\nu \leqslant \lambda_n + \varepsilon\,\theta(\lambda_n) < \lambda_{n+1} + \varepsilon\,\theta(\lambda_{n+1}) \left(\frac{\lambda_n}{\lambda_{n+1}}\right)^{1-(a-b)/\alpha} \leqslant$$

$$\leqslant \lambda_{n+1} + \varepsilon\,(1+\varepsilon)^{|1-(a-b)/\alpha|}\,\theta(\lambda_{n+1}).$$

Therefore we can apply (7.1) to (7.4) and get (7.2*b*) in the form

$$\overline{\underset{\lambda_n < y < \lambda_n + \varepsilon\theta(\lambda_n)}{\text{bound}}} \frac{A_0(y) - A_0(\lambda_n)}{\lambda_n^a} <$$

$$< \Omega\,[(\varepsilon\,(1+\varepsilon)^{|1-(a-b)/\alpha|}) + o(1)]\,(1+\varepsilon)^{|a|}.$$

THEOREM VI. *If* $A_\alpha(x)$, $\alpha \geqslant 0$, *is defined as in* (1.2) *and if we assume*

(i) $$A_\alpha(x) = o(x^{\alpha+b}) \quad as \quad x \to \infty,$$

for a positive $a \geqslant -b$, *along with* one *of the following hypotheses in which* $\theta(x) \equiv x^{1-(a-b)/\alpha}$, $a \geqslant b$:

(ii) $$\overline{\lim_{n\to\infty}} \max_{\lambda_n < \lambda_m < \lambda_n + \varepsilon\theta(\lambda_n)} |a_{n+1} + a_{n+2} + \ldots a_m|/\lambda_n^a < o(1) \ as\ \varepsilon \to 0,$$

(ii′) $$\overline{\lim_{n\to\infty}} \max_{\lambda_n \leqslant \lambda_m < \lambda_n + \varepsilon\theta(\lambda_n)} (a_n + a_{n+1} + \ldots a_m)/\lambda_n^a < o(1) \ as\ \varepsilon \to 0,$$

then

$$A_0(\lambda_n) = o(\lambda_n^a) \quad as \quad n \to \infty.$$

Proof. To prove Theorem VI with hypothesis (ii), we merely appeal to a case of Theorem IV in which

$$\Phi_\alpha(x) \equiv A_\alpha(x), \quad \Phi_{\alpha'}(x) \equiv A_0(x), \quad \theta(x) \equiv x^{1-(a-b)/\alpha}, \quad W(x) \equiv x^{\alpha+b},$$

and the two-sided form of (6.4*b*) is given with $x = \lambda_n$ instead of either (6.4*a*) or (6.4*b*).

To establish Theorem VI with hypothesis (ii′) we argue as in the proof of Theorem IV, having chosen $\Phi_\alpha(x)$, $\Phi_{\alpha'}(x)$, $\theta(x)$, $W(x)$ as before, but use Lemma 5 in the place where Lemma 4 is used in the said proof.

The following corollary to Theorem VI originated with Ananda Rau ([2], Corollary 1.81) in its *O*-form (ii), its O_R-form (ii′) being due to Minakshisundaram and Rajagopal ([5], Corollaries 1.1, 1.2).

COROLLARY VI. *If* $A_\alpha(x)$, $\alpha \geqslant 0$, *is defined as in* (1.2) *and*

(i) $A_\alpha(x) = o(x^q)$ *as* $x \to \infty$, *for an* $\alpha > 0$, $q \geqslant 0$,

the one alone *of the following two conditions in which* $p \geqslant q - (\alpha + 1)$ *and* $n \to \infty$:

(ii) $$a_n = O(\lambda_n^p (\lambda_n - \lambda_{n-1})),$$

(ii′) $$a_n = O_R(\lambda_n^p (\lambda_n - \lambda_{n-1}))$$

with

either $\lambda_n - \lambda_{n-1} = o(\lambda_n^{(q-p)/(\alpha+1)})$ or $a_n = o_R(\lambda_n^{(p\alpha+q)/(\alpha+1)})$

ensures

$$A(\lambda_n) = o(\lambda_n^{(p\alpha+q)/(\alpha+1)}), \quad n \to \infty.$$

The proof of Corollary VI consists in showing that its conditions (ii), (ii′) yield conditions (ii), (ii′) respectively of Theorem VI with $a = (p\alpha + 1)/(\alpha + 1)$, $b = q - \alpha$.

The next corollary is a formal generalization of the preceding in the sense that the latter is the limiting case $k^{-1} = 0$, $k'^{-1} = 1$ of the former.

COROLLARY VI′. *Corollary VI can be restated with either* (ii) *or* (ii′) *replaced by* one *of the following conditions in which* $p \geqslant q - (\alpha + 1)$ *and* k, k *are constants such that*

$$k > 1, \quad k^{-1} + k'^{-1} = 1:$$

(ii) $$\sum_{\nu=1}^{n} |a_\nu|^k \lambda_\nu^k (\lambda_\nu - \lambda_{\nu-1})^{1-k} = O\left(\lambda_n^{k(p+1)+1}\right),$$

(ii′) $$\sum_{\nu=1}^{n} (|a_\nu| + a_\nu)^k \lambda_\nu^k (\lambda_\nu - \lambda_{\nu-1})^{1-k} = O\left(\lambda_n^{k(p+1)+1}\right)$$

with

either $\lambda_n - \lambda_{n-1} = o(\lambda^{n(q-p-k^{-1})/(\alpha+k'^{-1})})$ or $a_n = o_R\left(\lambda_n^{\frac{\alpha(p+k^{-1})+qk'^{-1}}{\alpha+k'^{-1}}}\right)$.

We can deduce Corollary VI′ from Theorem VI by using Hölder's inequality (as in [6], -(A), (B), proofs of Lemmas 2′, III) and proving that (ii), (ii′) of the former lead to (ii), (ii′) respectively of the latter with $a = [\alpha(p + k^{-1}) + qk'^{-1}]/(\alpha + k'^{-1})$, $b = q - \alpha$.

The case $a = b = 0$ of Theorem VI and the corresponding case $p = -1$, $q = \alpha$ of Corollary VI are classical Tauberian theorems for series summable-$R(\lambda_n, \alpha)$. The case $p = -1$, $q = \alpha$ of Corollary VI′ with hypothesis (ii′) is due to Szász ([8], implication of Theorem 4), while the same case of the same corollary with hypothesis (ii) originated with Ganapathy Iyer ([2], Corollary 1.82).

The next theorem is the analogue of Theorem V for a series Σa_n and its Riesz sum, $A_\alpha(x)$, defined by (1.2).

THEOREM VII. *If $W(x)$ and $\theta(x)$ are functions of the class f subject to* (6.2), *the former being monotonic increasing, then, from* either (i) *and* (ii) *below* or (i) *and* (ii′) *below*:

(i) $$A_\alpha(x) = o(W(x)) \quad as \quad x \to \infty, \quad for\ an\ \alpha > 0,$$

(ii) $$\overline{\lim_{n\to\infty}} \max_{\lambda_n < \lambda_m < \lambda_n + \varepsilon\theta(\lambda_n)} |a_{n+1} + a_{n+2} + \dots a_m| = o(1) \quad as \quad \varepsilon \to 0,$$

(ii′) $$\overline{\lim_{n\to\infty}} \max_{\lambda_n \leqslant \lambda_m < \lambda_n + \varepsilon\theta(\lambda_n)} (a_n + a_{n+1} + \dots a_m) < o(1) \quad as \quad \varepsilon \to 0,$$

there follows:

$$A_0(\lambda_n) = o\left(\frac{W(\lambda_n)}{[\theta(\lambda_n)]^\alpha}\right) + o(1), \quad n \to \infty.$$

Proof. In case (ii) is assumed along with (i), we appeal to a version of Theorem V for $\Phi_\alpha(x) \equiv A_\alpha(x)$, $\Phi_{\alpha'}(x) \equiv A_0(x)$ with a two-sided form of (6.9*b*) in which $x = \lambda_n$, instead of either (6.9*a*) or (6.9*b*). In case (ii′) is assumed along with (i), we proceed as we would to prove Theorem V for $\Phi_\alpha(x) \equiv A_\alpha(x)$, $\Phi_{\alpha'}(x) \equiv A_0(x)$ and use a lemma (proved just like Lemma 5) to the effect that (ii′) ensures the *two* conditions:

(7.5*a*) $$\overline{\lim_{m\to\infty}} \overline{\mathrm{bound}}_{\lambda_m - \varepsilon\theta(\lambda_m) < x < \lambda_m} [A_0(\lambda_m) - A_0(x)] \leqslant \omega(\varepsilon) \to +0 \quad as \quad \varepsilon \to 0.$$

(7.5*b*) $$\overline{\lim_{n\to\infty}} \overline{\mathrm{bound}}_{\lambda_n < y < \lambda_n + \varepsilon\theta(\lambda_n)} [A_0(y) - A_0(\lambda_n)] \leqslant \omega^*(\varepsilon) \to +0 \quad as \quad \varepsilon \to 0,$$

which are special cases of (6.9*a*), (6.9*b*) respectively.

8. *CONCLUDING REMARKS*

In all the theorems of this paper the conditions on $\varphi(x)$, the function studied, involve two other functions, either $W(x)$ and $\theta(x)$ or $W(x)$ and

$V(x)$. $W(x)$ is introduced in an attempt to generalize, as in (6.3), the idea of (C, α)-summability of $\varphi(x)$ to sum s in the principal case $s=0$, while $\theta(x)$ is needed to extend and weaken the condition of slow increase of $\varphi(x)$ as in either (6.9*a*) or (6.9*b*). The type of a general Tauberian theorem for $\Phi_\alpha(x)$ is Theorem V which answers the question as to what (6.3) along with either (6.9*a*) or (6.9*b*) implies. Regarding all the other Tauberian theorems of this paper as variants of Theorem V, we naturally explain the presence in them of $W(x)$ and $\theta(x)$ but not that of $V(x)$.[9]

To explain the presence of $V(x)$ in some forms of our Tauberian theorems, we have to keep to the fore the general Hardy-Landau Tauberian condition on the terms of a series rather than the Schmidt condition (of slow increase or decrease) on the sequence of partial sums of the series. It was an attempt to study the combined effect on series $A_0(x)$ defined as in (1.2), of an extension of the Hardy-Landau condition and (5.1) in O-form, that originally led Minakshisundaram and Rajagopal to formulate the following theorem ([5], Theorem 1).

THEOREM VIII. *Let $W(x)$, $V(x)$ be positive functions of $x>0$, of which the former is monotonic increasing, satisfying the following conditions:*

$$\text{(i)}\qquad \begin{cases} \theta(x) \equiv [W(x)/V(x)]^{1/(\alpha+1)} \leqslant x \quad \text{for an } \alpha>0, \\ W(y)/W(x) = O_R(1) \quad \text{for } 0<y-x<\eta\,\theta(x), \quad \eta<1, \\ V(y)/V(x) = O_R(1) \quad \text{for } 0<|y-x|<\eta\,\theta(x), \quad \eta<1. \end{cases}$$

Let $A_\alpha(x)$ be defined as in (1.2) *and let*

$$\text{(ii)}\qquad A_\alpha(x) = O(W(x)), \quad x\to\infty,$$

$$\text{(iii)}\qquad a_n/(\lambda_n-\lambda_{n-1}) = O_R(V(\lambda_n)), \quad n\to\infty,$$

$$\text{(iv)}\qquad \lambda_n-\lambda_{n-1} \leqslant \eta\,\theta(\lambda_n).$$

Then

$$A_0(\lambda_n) = O\,[\theta(\lambda_n)\,V(\lambda_n)] \equiv O\,([V(\lambda_n)]^{\alpha/(\alpha+1)}\,[W(\lambda_n)]^{1/(\alpha+1)}).$$

If hypothesis (ii) *is postulated with o instead of O and* (iv) *with η arbitrarily small for all large n, the conclusion will have o in place of O.*

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[9] The entire material of this paper can be presented (and, in fact, had been, in a manuscript completed before the publication of the book by Chandrasekharan and Minakshisundaram [2]), in a form involving only $W(x)$ and $\theta(x)$, the latter function being defined as $[m\,W/l\,V]^{1/\alpha}$ in a deduction of Theorem II directly from Theorem I.

Theorem VIII is essentially the case $\alpha' = 0$, $\gamma = 1$ of Theorem I'. For, as shown elsewhere ([5], Lemmas 2,3), (iii), (iv) and the part of (i) relating to V in Theorem VIII together imply that

$$\left.\begin{array}{ll} \overline{\text{bound}}_{\lambda_m - \eta\theta(\lambda_m) < x < \lambda_m} & [A_0(\lambda_m) - A_0(x)] \\ \overline{\text{bound}}_{\lambda_m < y < \lambda_m + \eta\theta(\lambda_m)} & [A_0(y) - A_0(\lambda_m)] \end{array}\right\} = O_R[\eta\theta(\lambda_m)V(\lambda_m)], \quad m \to \infty,$$

which are precisely the implications of hypotheses (ii*a*), (ii*b*) of Theorem I when in these hypotheses we choose $\Phi_{\alpha'}(u) \equiv A_0(u)$, $\gamma = 1$, $x = \lambda_n$.

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