FRULLANI INTEGRALS AND VARIANTS OF THE EGOROFF THEOREM ON ESSENTIALLY UNIFORM CONVERGENCE

by

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1. *INTRODUCTION.* A classic theorem of Egoroff [1911] says that if $f_1(x), f_2(x), \ldots$ is a sequence of Lebesgue measurable functions defined over a set E having positive finite Lebesgue measure $|E|$ and if

$$\lim_{n \to \infty} f_n(x) = f(x), \qquad x \in E, \tag{1.1}$$

where $f(x)$ is finite-valued, then the convergence is essentially uniform over E, that is, to each $\theta > 0$ corresponds a subset E_1 of E of measure $|E_1| > |E| - \theta$ such that $f_n(x) \to f(x)$ uniformly over E_1. This useful theorem was employed by the author [1942, p. 11] to obtain a theorem which, among other things, led to a simplified proof of a theorem of Iyengar [1940] on Frullani integrals; see also Agnew [1951]. A function $F(t)$ Lebesgue integrable over each finite interval was given, and properties of the function

$$f(A, x) = \int_A^{A+x} F(t)\, dt \tag{1.2}$$

were being obtained. It was known that there is a constant L such that $\lim_{A \to \infty} f(A, x) = Lx$ over the set E of x for which $-a \leqq x \leqq a$, a being a positive constant. Without further elaboration, it was asserted that "by a theorem of Egoroff", the convergence of $f(A, x)$ is essentially uniform over E. In treating the same subject, Ostrowski [1949] restated the theorem of Iyengar and said that the proof "is difficult and will be given in another publication". Referring to Iyengar's use of a theorem of Arzelà and the author's use of the theorem of Egoroff, Ostrowski said the following. "Iyengar's proof of his theorem and the simplified proof of it given by Agnew are, however, still difficult since in both proofs certain theorems about nonuniform convergence are used which, although usually proved only in the case of convergent *sequences*, have to be used in the case of continuous approximation. Our proof... makes use of the theorem of Osgood, but exactly in the form proved by Osgood, that is for the case of *sequences* of functions".

It is certainly true, as the remarks of Ostrowski suggest, that difficulties appear in attempts to generalize the theorem of Egoroff from the sequence $f_1(x)$, $f_2(x), \dots$ to very general classes of functions $f(A, x)$. Some results and references are given in Sections 3, 4, and 5. Meanwhile, it is our first object to show that the application involving the particular function $f(A, x)$ in (1.2) does in fact follow almost immediately from the Egoroff theorem on sequences. The function $f(A, x)$ in (1.2) is continuous over the whole plane $-\infty < A, x < \infty$ and the limit Lx is continuous over $-\infty < x < \infty$. Hence the functions $f_n(x)$ of (2.3) below are continuous, and our conclusion for the special case is then obtained from the following theorem which is, as we shall see without omission of details a corollary of Egoroff's theorem on sequences.

2. *A COROLLARY OF EGOROFF'S THEOREM.* — THEOREM 2.1. *Let* $f(A, x)$ *be a function defined for* $A > A_0$ *and* x *in a set* E *having finite positive measure* $|E|$. *Let*

$$\lim_{A \to \infty} f(A, x) = f(x), \qquad x \in E, \tag{2.2}$$

where $f(x)$ *is finite valued. For each integer* $n > A_0$, *let the function* $f_n(x)$ *defined by*

$$f_n(x) = \underset{n \leqq A \leqq n+1}{\text{l. u. b.}} |f(A, x) - f(x)|, \qquad x \in E, \tag{2.3}$$

be measurable over E. Then to each $\theta > 0$ *corresponds a subset* E_1 *of* E *having measure* $|E_1| > |E| - \theta$ *such that* $f(A, x) \to f(x)$ *uniformly over* E_1 *as* $A \to \infty$, *that is, to each* $\varepsilon > 0$ *corresponds a number* $A(\varepsilon)$ *such that*

$$|f(A, x) - f(x)| < \varepsilon, \qquad x \in E_1,\ A > A(\varepsilon). \tag{2.4}$$

To prove this theorem we observe that (2.2) implies that, for each $x \in E$, $\lim f_n(x) = 0$. Since $f_n(x)$ is, for each $n > A_0$, a measurable function of x, it follows from Egoroff's theorem that $f_n(x) \to 0$ essentially uniformly over E. If $\theta > 0$ and we choose a subset E_1 of E such that $|E_1| > |E| - \theta$ and $|f_n(x)| < \varepsilon$ when $x \in E_1$ and $n > N(\varepsilon)$, then (2.4) holds when $A(\varepsilon) = N(\varepsilon) + 1$. This proves the theorem.

In applying the Egoroff theorem and its corollary, one has considerable freedom that we did not need for our special application. The set E may be a Lebesgue measurable set in Euclidean space of any number of dimensions, and the functions may be regarded either as real valued or complex valued. While it is essential that the limit function $f(x)$ be finite-valued, it need not be *assumed* that the functions $f_n(x)$ and $f(A, x)$ are finite valued. It follows from (1.1) and (2.2) that, for each $x \in E$, $f_n(x)$ and $f(A, x)$ must be finite-valued when n and A are sufficiently great. It is a part of the conclusion of Egoroff's theorem that if E_1 is a subset of E over which the convergence is uniform, then $f_n(x)$ and $f(A, x)$ must be finite-valued over E_1 when n and A are sufficiently great.

Of course measurability of the function $f(A, x)$, and measurability of the functions $f_n(x)$ in (2.3), are not necessary in order that $f(A, x)$ converge essentially uniformly over E as $A \to \infty$. This is shown by such trivial examples as that in which $g(x)$ is a bounded nonmeasurable function over E and $f(A, x) = g(x)/A$ when $A \geqq 1$ and $x \in E$. Whether the functions are measurable or not, the convergence in (2.2) is essentially uniform over E if and only if the sequence $f_n(x)$ of functions, which are defined by (2.3) and which may now have values $+\infty$, is such that $\lim f_n(x) = 0$ essentially uniformly over E.

3. *NONMEASURABLE FUNCTIONS* $f_n(x)$. It has for long time been standard procedure for books treating the theory of measure to prove the theorem of Egoroff. Nevertheless it seems to be difficult to find an explicit statement that the conclusion of Egoroff's theorem may fail if the functions $f_n(x)$ are nonmeasurable. We now give an example. Let E_1, E_2, $E_3, \ldots$ be a sequence of mutually exclusive nonmeasurable subsets of E such that for each $n = 1, 2, 3, \ldots$ the set E_n has lower (or inner) measure $\underline{m}(E_n) = 0$ and upper (or outer) measure $\overline{m}(E_n) = 1$. The classic congruent nonmeasurable sets constructed by Vitali in 1905 have this property; a paper of F. B. Jones [1942] is of interest in this connection. For each $n = 1, 2, 3, \ldots$ let

$$f_n(x) = 1, \qquad x \in E_n, \tag{3.1}$$

and $f_n(x) = 0$ otherwise. For each x in E we then have $f_n(x) \neq 0$ for at most one n; hence $\lim_{n \to \infty} f_n(x) = 0$. But if $\theta < 1$ and E^* is a measurable subset of E such that $|f_n(x)| < 1$ when $x \in E^*$ and $n \geqq N$, then E^* and E_N must be mutually exclusive and hence the measure $|E^*|$ of E^* must be zero. Therefore $f(x)$ does not converge essentially uniformly over E.

4. *MEASURABLE FUNCTIONS* $f(A, x)$. An example given by Tolstoff [1939] shows that the conclusion of Theorem 2.1 will fail to hold if we delete the hypothesis that the functions $f_n(x)$ in (2.3) are measurable and insert the hypothesis that $f(A, x)$ is a measurable function of the two variables A and x as well as a measurable function of x for each A and of A for each x. The following example is somewhat simpler than that of Tolstoff; the context of the Tolstoff example indicates that $(x_0 \varepsilon_n x)$ is a disturbing misprint of $(x, \varepsilon_n x)$. Let E be the interval $0 \leq x \leq 1$, and let $E_1, E_2, E_3, \ldots$ be the sequence of nonmeasurable subsets of E introduced in Section 3. For each $n = 1,2,3,\ldots$ let $f(n+x, x) = 1$ when $x \in E_n$; and let $f(A, x) = 0$ otherwise. That $f(A, x)$ is measurable in the three senses follows from the fact that (i) $f(A, x) = 0$ except for values of A and x representing coordinates of points on a countable set of line segments in the (A, x) plane, (ii) for each A, $f(A, x) \neq 0$ for at most one value of x and (iii) for each x, $f(A, x) \neq 0$ for at most one value of A. For this function $f(A, x)$, the functions $f_n(x)$ defined by (2.3) are precisely the elements of the sequence $f_n(x)$ treated in section 2. Thus the sequence $f_n(x)$ is not essentially uniformly convergent, and therefore (2.4) cannot hold.

5. *GENERALIZATIONS OF THE EGOROFF THEOREM.* Without going into details, we remark that use of Theorem 2.1 should produce very substantial simplifications of proofs of generalizations of the Egoroff theorem. Let F denote a class of functions $f(A, x)$ such that (2.2) and the hypothesis that $f \in F$ imply the hypothesis H of the sentence containing (2.3). Then a variant of Theorem 2.1 is obtained by replacing H by the hypothesis that $f \in F$. To prove a variant of this type, it is not necessary to formulate a proof in which questions of measurability mingle with the details of a proof modeled after a standard proof of Egoroff's theorem. It is much simpler to prove that F has the property noted above and then use Theorem 2.1.

The trivial case in which F is the class of continuous functions covers the application described in Section 1. The significantly more general case in which F is the class of Baire functions associated with Borel measurability (see Banach [1932]) was treated by Tolstoff [1939]. For modifications of the Egoroff theorem involving functions (and sequences of such functions) $f(A, x)$ which are continuous functions of A when x is fixed and measurable functions of x when A is fixed, see Stampacchia [1949] and authors noted there.

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