ON SOME SETS OF INTEGERS WITH EQUAL SUMS OF LIKE POWERS

by

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I

Let us consider the Diophantine system

$$
\begin{aligned}
A_1 + A_2 + A_3 &= B_1 + B_2 + B_3 \\
A_1^3 + A_2^3 + A_3^3 &= B_1^3 + B_2^3 + B_3^3 \\
A_1 - A_2 &= A_2 - A_3 \\
B_1 - B_2 &= 2(A_1 - A_2),
\end{aligned}
\tag{1}
$$

where $A_i, B_{i\,(=1,2,3)}$ are the integers to be determined.

We see that equation (1_3) is satisfied if

$$A_1 = x + y, \quad A_2 = x, \quad A_3 = x - y \tag{2}$$

and thence from the equations (1_4) and (1_1)

$$B_1 = B_2 + 2y, \quad B_3 = 3x - 2y - 2B_2. \tag{3}$$

If in the equation (1_1) we substitute the values of $A_{i(=1,2,3)}$, B_1, B_3, from Eqs. (2) and (3), we have

$$A_2^3 - 3(2x - y)B_2^2 + (9x^2 - 12xy + 2y^2)B_2 - x(x - y)(4x - 5y) = 0,$$

or

$$(B_2 - x + y)[B_2^2 - (5x - 2y)B_2 + x(4x - 5y)] = 0,$$

We may exclude as trivial the case in which $B_2 = x - y$. Thus we have the equation

$$B_2^2 - (5x - 2y)B_2 + x(4x - 5y) = 0.$$

If B_2' is a root of the above equation, in which the coefficients are integers, then B_2' is either an integer or an irrational number.

But

$$B_2 = \frac{5x - 2y \pm \sqrt{9x^2 + 4y^2}}{2} \tag{4}$$

and, for excluding the case B_2 = irrational, it is necessary and sufficient to take:

$$9x^2 + 4y^2 = z^2. \tag{5}$$

Now it is clear that the problem of finding the general proper solution of (1) is reduced to finding the proper solutions of the equation (5),

For this we treat the following two cases.

In the first place suppose that the integer $x = A_2$ is odd. Then we have the following sets of values

$$y = \frac{a^2 - b^2}{\Delta}, \quad y = \frac{3ab}{\Delta}, \quad z = \frac{3(a^2 + b^2)}{\Delta},$$

where $\Delta = 3$ or 1, according as the relatively prime integers a, b (of which one is odd and the other even) are both prime to 3, or one of these is a multiple of 3.

If we substitute in the Eqs. (2), (4) and (3) we get the following solution of the system (1):

$$A_1 = \frac{1}{\Delta}(a + 3ab - b^2)$$

$$A_2 = \frac{1}{\Delta}(a^2 - b^2)$$

$$A_3 = \frac{1}{\Delta}(a^2 - 3ab - b^2)$$

$$B_1' = \frac{1}{\Delta}(a^2 + 3ab - 4b^2) \qquad B_1'' = \frac{1}{\Delta}(4a^2 + 3ab - b^2)$$

$$B_2' = \frac{1}{\Delta}(a^2 - 3ab - 4b^2) \qquad B_2'' = \frac{1}{\Delta}(4a^2 - 3ab - b^2)$$

$$B_3' = \frac{1}{\Delta}(a^2 + 5b^2) \qquad B_3'' = -\frac{1}{\Delta}(5a^2 + b^2)$$

Let us now consider the case in which $x = A_2$ is even. Then y is odd and the general proper solution of (5) is

$$x = \frac{4ab}{\Delta}, \quad y = \frac{3}{\Delta}(a^2 - b^2), \quad z = \frac{6}{\Delta}(a^2 + b^2),$$

where $\Delta = 1$ or 3 according as the relatively primes a, b (of which one is odd and the other even) are both prime to 3, or one of these is a multiple of 3.

Substituting the foregoing values in the Eqs. (2), (4) and (3) we have the following solution of the system (1):

$$A_1 = \frac{1}{\Delta}(3a^2 + 4ab - 3b^2)$$

$$A_2 = \frac{1}{\Delta} 4ab$$

$$A_3 = \frac{1}{\Delta}(-3a^2 + 4ab + 3b^2)$$

$$B_1' = \frac{1}{\Delta}(10ab - 6b^2) \qquad B_1'' = \frac{1}{\Delta}(10ab + 6a^2)$$

$$B_2' = \frac{1}{\Delta}(10ab - 6a^2) \qquad B_2'' = \frac{1}{\Delta}(10ab + 6b^2)$$

$$B_3' = \frac{1}{\Delta}(6a^2 - 8ab + 6b^2) \qquad B_3'' = -\frac{1}{\Delta}(6a^2 + 8ab + 6b^2).$$

Hence we conclude as follows:

"Proper solutions of the system

$$A_1, A_2, A_3, \overset{n}{=} B_1, B_2, B_3 \overset{n}{=} C_1, C_2, C_3,^{*} \quad (n = 1,3) \tag{6}$$

are given by means of the identities

1) $$\frac{1}{\Delta}(a^2 + 3ab - b^2), \ \frac{1}{\Delta}(a^2 - b^2), \ \frac{1}{\Delta}(a^2 - 3ab - b^2) \overset{n}{=}$$

$$\overset{n}{=} \frac{1}{\Delta}(a^2 + 3ab - 4b^2), \ \frac{1}{\Delta}(a^2 - 3ab - 4b^2), \ \frac{1}{\Delta}(a^2 + 5b^2) \overset{n}{=}$$

$$\overset{n}{=} \frac{1}{\Delta}(4a^2 + 3ab - b^2), \ \frac{1}{\Delta}(4a^2 - 3ab - b^2), \ \frac{1}{\Delta}(-5a^2 + b^2) \tag{7}$$

*) The separation of two sets of numbers by the symbol $\overset{n}{=}$ denote that they have the same sum of kth powers for $k = 1, \ldots, n$.

where $\Delta = 3$ or 1, according as the relatively prime integers a, b (of which one is odd and the other even) are both prime to 3, or one of these is a multiple of 3, and

$$2) \qquad \frac{1}{\Delta}(3a^2 + 4ab - 3b^2),\ \frac{1}{\Delta}4ab,\ \frac{1}{\Delta}(-3a^2 + 4ab + 3b^2) \overset{n}{=}$$

$$\overset{n}{=} \frac{1}{\Delta}(10ab - 6b^2),\ \frac{1}{\Delta}(10ab - 6a^2),\ \frac{1}{\Delta}(6a^2 - 8ab + 6b^2) \overset{n}{=}$$

$$\overset{n}{=} \frac{1}{\Delta}(10ab + 6a^2),\ \frac{1}{\Delta}(10ab + 6b^2),\ -\frac{1}{\Delta}(6a^2 + 8ab + 6b^2) \qquad (8)$$

where $\Delta = 1$ or 3, according as ab is prime to 3, or one of a, b is a multiple of 3."

Remark. The system (6) is equivalent to the system

$$(A_1 + A_2) + (A_1 + A_3) + (A_2 + A_3) = (B_1 + B_2) + (B_1 + B_3) + (B_2 + B_3) =$$
$$= (C_1 + C_2) + (C_1 + C_3) + (C_2 + C_3)$$

$$(A_1 + A_2)(A_1 + A_3)(A_2 + A_3) = (B_1 + B_2)(B_1 + B_3)(B_2 + B_3) =$$
$$= (C_1 + C_2)(C_1 + C_3)(C_2 + C_3).$$

Hence we have a process for finding solutions of the system

$$x_1 + y_1 + z_1 = x_2 + y_2 + z_2 = x_3 + y_3 + z_3$$

$$x_1 y_1 z_1 = x_2 y_2 z_2 = x_3 y_3 z_3.$$

Applications. 1) Interesting applications of the preceding results we may have as follows. Let us ask for instance the natural numbers a and b for which in identities (7) the $\Delta^{-1}(a^2 - b^2)$ has a given value. Let it be $\Delta^{-1}(a^2 - b^2) = 21$. Then for $\Delta = 3$ and ab prime to 3 we have $a = 8$, $b = 1$ and $a = 32$, $b = 31$. Thus from (7) we obtain the relations

$$29,\ 21,\ 13 \overset{n}{=} 28,\ 12,\ 23 \overset{n}{=} 93,\ 77,\ -107$$

and

$$1013,\ 21,971 \overset{n}{=} 52,\ -1932,\ 1943 \overset{n}{=} 2037,\ 53,\ -2027. \qquad (n = 1,3) \quad (9)$$

From the above we have

$$21 \overset{n}{=} 28,\ 12,\ 23,\ -13,\ -29 \qquad (\text{A})$$

$$21 \overset{n}{=} 93,\ 77,\ -107,\ -13,\ -29 \qquad (\text{B})$$

$$21 \overset{n}{=} 52,\ -1932,\ 1943,\ 971,\ -1013 \qquad (\Gamma)$$

$$21 \overset{n}{=} 2037,\ 53,\ -2027,\ 971,\ -1013. \qquad (\Delta)$$

Now we take:
from (A) and (Γ)

$$12,\ 23,\ 28,\ 1013,\ 1932 \overset{n}{=} 13,\ 29,\ 52,\ 971,\ 1943; \qquad (E)$$

from (A) and (Δ)

$$12,\ 23,\ 28,\ 1013,\ 2027 \overset{n}{=} 13,\ 29,\ 53,\ 971,\ 2037; \qquad (Z)$$

from (B) and (Γ)

$$93,\ 77,\ 1013,\ 1932 \overset{n}{=} 13,\ 29,\ 52,\ 107,\ 971,\ 1943; \qquad (H)$$

from (B) and (Δ)

$$93,\ 77,\ 1013,\ 2027 \overset{n}{=} 13,\ 29,\ 53,\ 107,\ 971,\ 2037. \qquad (\Theta)$$

But from (8) and for $a=2,\ b=3,\ \Delta=3$ we have the relations

$$3,\ 8,\ 13 \overset{n}{=} 2,\ 12,\ 10 \overset{n}{=} 28,\ 38,\ -42 \quad (n=1,3). \qquad (10)$$

Thus we have

$$\left.\begin{array}{l} 28 \overset{n}{=} 3,\ 8,\ 13,\ -38,\ 42 \\ 28 \overset{n}{=} 2,\ 12,\ 10,\ -38,\ 42 \end{array}\right. \quad (n=1,3)$$

and because from (9)

$$\left.\begin{array}{l} 28 \overset{n}{=} 29,\ 21,\ 13,\ -12,\ -23 \\ 28 \overset{n}{=} 93,\ 77,\ -107,\ -12,\ -23, \end{array}\right. \quad (n=1,3)$$

we are led to the relations

$$\left.\begin{array}{r} 3,\ 8,\ 12,\ 23,\ 42 \overset{n}{=} 21,\ 29,\ 38 \\ 3,\ 8,\ 12,\ 13,\ 23,\ 42,\ 107 \overset{n}{=} 38,\ 77,\ 93 \\ 2,\ 10,\ 12,\ 12,\ 23,\ 42 \overset{n}{=} 13,\ 21,\ 29,\ 38 \\ 2,\ 10,\ 12,\ 12,\ 23,\ 42,\ 107 \overset{n}{=} 38,\ 77,\ 93 \end{array}\right. \quad (n=1,3)$$

or from the first and third of the above to the equalities

$$13,\ 29 \overset{n}{=} 3,\ 8,\ 12,\ 13,\ 23,\ 42,\ -21,\ -38$$

$$13,\ 29 \overset{n}{=} 2,\ 10,\ 12,\ 12,\ 23,\ 42,\ -21,\ -38.$$

Adding each of them to each of (E), (Z), (H) and (Θ) we take new relations, for ex.

$$\left.\begin{array}{l} 12,\ 21,\ 28,\ 38,\ 1013,\ 1932 \overset{n}{=} 3,\ 8,\ 12,\ 13,\ 42,\ 52,\ 971,\ 1943 \\ 21,\ 28,\ 77,\ 93,\ 1013,\ 1932 \overset{n}{=} 3, 8, 12, 13, 23, 42, 52, 107, 971, 1943 \end{array}\right. \quad (n=1,3)$$

etc.

2) It is known that for any solution of the system

$$\sum_{i=1}^{3} A_i^n = \sum_{i=1}^{3} B_i^n \quad (n = 1, 3)$$

we have

$$\sum_{i=1}^{3} (t - A_i)^k + \sum_{i=1}^{3} (t + B_i)^k = \sum_{i=1}^{3} (t - B_i)^k + \sum_{i=1}^{3} (t + A_i)^k \quad (k = 1, 2, 3, 4)$$

t being arbitrary.

Applying this theorem, for ex., in the Eq., (10_1) and putting $t = 8$ we get

$$5,\ 0,\ -5,\ 10,\ 18,\ 20 \overset{n}{=} 6,\ -2,\ -4,\ 11,\ 16,\ 21$$

and from this

$$2,\ 4,\ 10,\ 18,\ 20 \overset{n}{=} 6,\ 11,\ 16,\ 21 \quad (n = 1, 3)$$

where the numbers in the second member are in arithmetical progression.

II

Let us consider the Diophantine system

$$A_1 + A_2 + A_3 = B_1 + B_2$$
$$A_1^3 + A_2^3 + A_3^3 = B_1^3 + B_2^3$$
$$A_1 - A_2 = A_2 - A_3$$

where $A_{i(=1,2,3)}$, $B_{j(=1,2)}$ are the integers to be determined.

We put

$$A_1 = x + y, \quad A_2 = x, \quad A_3 = x - y, \quad B_1 = x + \omega, \quad B_2 = x + v$$

and we have the system

$$x = \omega + v$$
$$2\omega^2 + \omega v + 2v^2 = 2y^2.$$

The last equation shows that ωv = even and for $v = 2w$ we get

$$\omega^2 + \omega w + 4w^2 = y^2.$$

From this we find

$y = 4a^2 - ab + b^2$, $\omega = 4a^2 - b^2$, $v = 2w = 2(2ab - a^2)$, when $x = 2a^2 + 4ab - b^2$,

or

$y = 2a^2 - ab + 2b^2$, $\omega = 4ab - b^2$, $v = 2w = 2(a^2 - b^2)$, when $x = a^2 + 4ab - 2b^2$.

Thus we have the identities, in which $n = 1, 3$:

$$(6a^2 + 4ab - 2b^2),\ (8ab - b^2) \overset{n}{=} (6a^2 + 3ab),\ (2a^2 + 4ab - b^2),\ (-2a^2 + 5ab - 2b^2) \quad (1)$$

$$(8ab - 2b^2),\ (3a^2 + 4ab - 4b^2) \overset{n}{=} (-a^2 + 5ab - 4b^2),\ (a^2 + 4ab - 2b^2),\ (3a^2 + 3ab). \quad (2)$$

These identities afford the two-parameter solutions of the given system.

Applications. 1) A two parameter solution of the system

$$\sum_{i=1}^{3} X_i^k = \sum_{i=1}^{3} \psi_i^k \quad (k = 2, 4)$$

may be found from each of the identities (1) and (2).

In fact starting from the theorem mentioned in application 2 of the § I and putting $t = 6a^2 + 4ab - 2b^2$, we take from (1) the relations:

$$0, (6a^2 - 4ab - b^2), (12a^2 + 7ab - 2b^2), (8a^2 + 8ab - 3b^2), (4a^2 + 9ab - 4b^2) \stackrel{n}{=} \stackrel{n}{=} (ab - 2b^2), (4a^2 - b^2), (8a^2 - ab), (12a^2 + 8ab - 4b^2), (6a^2 + 12ab - 3b^2), \quad (3)$$

where

$$(4a^2+9ab-4b^2)+(4a^2-b^2) = (8a^2+8ab-3b^2)+a(b-2b^2) = 8a^2+9ab-5b^2.$$

But, it is known, that from

$$\sum_{i=1}^{5} X_i \stackrel{4}{=} \sum_{i=1}^{5} \psi_i$$

we have

$$\sum_{i=1}^{5} (X_i - s) \stackrel{4}{=} \sum_{i=1}^{5} (\psi_i - s). \qquad \text{(s, arbitrary)}$$

Multiplying the members of (3) by 2 and setting $s = 8a^2 + 9ab - 5b^2$, we are led to the following relations:

$$(-8a^2-9ab+5b^2), (4a^2-17ab+3b^2), (16a^2+5ab+b^2), (8a^2+7ab-b^2), (9ab-3b^2) \stackrel{4}{=} \stackrel{4}{=} (-8a^2 - 7ab + b^2), (-9ab + 3b^2), (8a^2 - 11ab + 5b^2), (16a^2 + 7ab - 3b^2), (4a^2 + 15ab - b^2),$$

in which:

$$(-8a^2 + 7ab - b^2)^{2m} = (-8a^2 - 7ab - b^2)^{2m} \text{ and } (9ab - 3b^2)^{2m} = (-9ab + 3b^2)^{2m}$$

From these relations we conclude that:

$$(-8a^2 - 9ab + 5b^2), (4a^2 - 17ab + b^2), (16a^2 + 5ab + b^2) \stackrel{k}{=} \stackrel{k}{=} (8a^2 - 11ab + 5b^2), (16a^2 + 7ab - 3b^2), (4a^2 + 15ab - b^2), \quad (k = 2,4)$$

in which

$$(4a^2 - 17ab + 3b^2) - (-8a^2 - 9ab + 5b^2) = (16a^2 + 7ab - 3b^2) - (4a^2 + 15ab - b^2).$$

In a similar way one may obtain from (2) and for $t = 3a^2 + 4ab - 4b^2$, $s = 8a^2 + 7ab - 6b^2$ the identities

$$(8a^2 + 7ab - 6b^2), (2a^2 + 15ab - 2b^2), (4a^2 - 11ab + 10b^2) \stackrel{k}{=} \stackrel{k}{=} (8a^2 + 5ab + 2b^2), (2a^2 - 17ab + 6b^2), (-4a^2 - 9ab + 10b^2), \quad (k = 2,4)$$

in which

$$(8a^2+7ab-6b^2)-(2a^2+15ab-2b^2)=(3a^2-17ab+6b^2)-(-4a^2-9ab+10b^2).$$

If we put $a=3$, $b=1$, we have the particular illustrative relation

$$87,\ 61,\ 13 \stackrel{k}{=} 89,\ 27,\ 53. \qquad (k=2,4)$$

Making use of the preceding identities (for $k=2,4$) we take identities of the form

$$\sum_{i=1}^{14} X_i \stackrel{9}{=} \sum_{i=1}^{14} \psi_i.$$

2) Starting from the equalities $42,\ 45 \stackrel{n}{=} 6,\ 29,\ 52$ $(n=1,3)$, which arise from (2) for $a=-1$, $b=3$, and putting $t=53$ we get from the theorem mentioned in application 2 of the § I the Eqs.

$$8,\ 11,\ 59,\ 82,\ 105 \stackrel{4}{=} 98,\ 95,\ 47,\ 24,\ 1. \qquad (4)$$

But, it is known, that, for h arbitrary, $\sum X = \sum \psi$ imply

$$\sum X + \sum(\psi+h) \stackrel{n+1}{=} \sum \psi + \sum(x+h).$$

By the use of this lemma, from (4) and for $h=29-6=23$, we find

$$8,\ 11,\ 59,\ 70,\ 118,\ 121 \stackrel{5}{=} 1,\ 31,\ 34,\ 95,\ 98,\ 128, \qquad (5)$$

where

$$121+8=118+11=70+59=128+1=98+31=95+34$$

and

$$11-8=121-118=34-31=98-95=3.$$

If one applies this method to (5) and for $h=3$ one finds that

$$4,\ 8,\ 37,\ 59,\ 70,\ 101,\ 118,\ 131 \stackrel{6}{=} 1,\ 14,\ 31,\ 62,\ 73,\ 95,\ 124,\ 128.$$

Finally* form (5) we find

$$(121-8),\ (118-11),\ (70-59) \stackrel{k}{=} (128-1),\ (98-31),\ (95-34) \qquad (k=2,4)$$

or

$$113,\ 107,\ 11 \stackrel{k}{=} 127,\ 67,\ 61, \qquad (k=2,4)$$

where the integers in both members are prime numbers.

III

Let us consider the system

$$\sum_{j=1}^{2} A_j = \sum_{i=1}^{3} B_i, \quad (n=1,3) \qquad (1)$$

where A_j, B_i are the integers to be determined.

* See: *The Tôhoku Mathematical Journal*, **39**, 35—39.

If we set

$$A_2 = A_1 + y, \quad B_1 = A_1 + w, \quad B_2 = y + v, \quad B_3 = A_1 - \varphi \tag{2}$$

and substitute in Eqs. (1), we have the system:

$$\varphi = \omega + v$$

$$(y + v) A_1^2 + (y^2 - 2\omega^2 - 2\omega v^2 - v^2) A_1 - yv(y + v) + \omega v(\omega + v) = 0. \tag{3}$$

This system we shall now consider, supposing that both the roots A_1', A_1'', of the last equation are integers.

Then from the relations

$$A_1' A_1'' = \frac{\omega v(\omega + v) - yv(y + v)}{y + v} = \frac{\omega v(\omega + v)}{y + v} - yv,$$

$$A_1' + A_1'' = \frac{2\omega(\omega + v) - (y + v)(y - v)}{y + v} = \frac{2\omega(\omega + v)}{y + v} - (y - v)$$

we conclude that we may write

$$\omega(\omega + v) = k(y + v),$$

where k is an integer.

But from the last equation we have

$$\frac{\omega}{y + v} = \frac{k}{\omega + v} = \frac{r}{s},$$

where r/s is a fraction in its lowest terms.

Hence $\omega = rp, \quad y + v = sp, \quad k = rq, \quad \omega + v = sq, \quad (p, q$ integers)

or:

$$\omega = rp, \quad v = sq - rp, \quad y = sp + rp - sq. \tag{4}$$

If in the equation (3_2) we substitute the above values of ω, y, v, we have

$$A_1^2 + [(2r + s)p - 2(r + s)q] A_1 + (rp - sq)(r + s)(p - q) = 0$$

and

$$A_1 = \frac{2(r + s)q - (2r + s)p \pm \sqrt{s^2p^2 - 4r(r + s)pq + 4r(r + s)q^2}}{2}. \tag{5}$$

For finding the general solutions in integer of the Eq.

$$s^2p^2 - 4r(r + s)pq + 4r(r + s)q^2 = z^2, \tag{6}$$

where r, s considered parameters, we search, to begin with, a particular solution of this.

For this, let us write z in the form $z = sp - 2tq$, where t is a rational number subject to our choice, and seek to determine p, q and corresponding to them the value of z satisfying Eq. (6).

We have

$$(ts - r^2 - rs)\, pq = (t^2 - r^2 - rs)\, q^2.$$

This equation will be satisfied if we put

$$p = t^2 - r^2 - rs, \quad q = ts - r^2 - rs.$$

Thus

$$z = -st^2 + r(r + s)(2t - s).$$

For $t = s/2$ and after the omission of the common factor $-{}^1/_4$ we have the particular solution of (6):

$$p = (2r + s)^2 - 2s^2, \quad q = (2r + s)^2 - 3s^2, \quad z = s^3.$$

Now we shall obtain the general solution of (6) by means similar to that employed by A. Deshoves.*

For this, starting from the above particular solution, we put in (6):

$$p = \rho\,[(2r + s)^2 - 2s^2], \quad q = \rho\,[(2r + s)^2 - 3s^2] + a, \quad z = \rho s^3 + b.$$

Then (6) gives ρ as a rational function of r, s, a, b, so that

$$p = [b^2 - 4r(r + s)a^2]\,[(2r + s)^2 - 2s^2]$$

$$q = -4rs^2(r + s)a^2 + [(2r + s)^2 - 3s^2]\,b^2 - 2s^3ab$$

$$z = -4rs^3(r + s)a^2 - s^3b^2 + 4r(r + s)\,[(2r + s)^2 - 4s^2]\,ab.$$

These formulae afford the general solution of Eq. (6) except for an arbitrary constant myltiplying the second member of each formula.

Hence each set of integral solution of (6) leads, with the use of the Eqs. (5), (4), (3_1) and (2), to a set of solution of the system (1).

Example: 1) For

$a = 0$, $b = r = s = 1$ we get $p = 7$, $q = 6$, $z = -1$, $A_1 = 1$ or 2.

Hence we have:

$$7,\ 8 \overset{n}{=} 1,\ 5,\ 9 \text{ and } 7,\ 9 \overset{n}{=} 2,\ 4,\ 10. \quad (n = 1, 3)$$

From the above we get:

$$2,\ 4,\ 8,\ 10 \overset{n}{=} 1,\ 5,\ 9,\ 9. \quad (n = 1, 3)$$

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* *Nouv. Ann. Math.*, (3), **3**, (1884), 225—39.

2) For $a = b = r = s = 1$ we get

$$49, 81 \overset{n}{=} 13, 32, 85 \text{ and } 49, 50 \overset{n}{=} 1, 44, 54$$

and from the above:

$$1, 44, 54, 81 \overset{n}{=} 13, 32, 50, 85 \quad (n = 1, 3).$$

Hence we have a process for finding solutions of the system

$$\sum_{i=1}^{4} X_i^k \overset{n}{=} \sum_{i=1}^{4} \psi_i^k \text{ for } k = 1, 3.$$

IV

Let $A_{i(=1,2,3)}$ and B_i be six integers such that

$$A_1, A_2, A_3 \overset{n}{=} B_1, B_2, B_3 \qquad (n = 1,3)$$
$$A_1 - A_2 = B_1 - B_2.$$

Then, if we put

$$A_1 = A_2 + \omega, \quad B_1 = B_2 + \omega, \tag{1}$$

we have to determine A_2, A_3, B_2, B_3 and ω so as to satisfy the system:

$$A_2 + \omega, A_2, A_3 \overset{n}{=} B_2 + \omega, B_2, B_3. \qquad (n = 1,3) \tag{2}$$

Now if we put:

$$B_2 = A_2 - \nu, \tag{3}$$

the first of the first of the above Eqs. gives

$$B_3 = A_3 + 2\nu, \tag{4}$$

whereas the second becomes:

$$6\nu A_2^2 - 6\nu^2 A_2 + 6\omega\nu A_2 + 3\omega^2\nu - 3\omega\nu^2 - 6\nu A_3^2 - 12\nu^2 A_3 - 6\nu^3 = 0,$$

or

$$2A_2^2 - 2(\nu - \omega) A_2 + \omega(\omega - \nu) - 2(A_3 + \nu)^2 = 0. \tag{5}$$

Therefore

$$A_2 = \frac{\nu - \omega \pm \sqrt{\nu^2 - \omega^2 + 4(A_3 + \nu)^2}}{2}.$$

But, if A_2', A_2'' are the two roots of (5) from the relation $A_2' + A_2'' = \nu - \omega$, we have $A_2' = -A_2'' + \nu - \omega$ and hence the equations:

$$A_2' + \omega, \quad A_2', \quad A_3 \overset{n}{=} A_2' - \nu + \omega, \quad A_2' - \nu, \quad A_3 + 2\nu,$$

which arise from (2), with the use of the Eqs. (3), (4) and for $A_2 = A_2'$ become:

$$-A_2'' + \nu,\ -A_2'' + \nu - \omega,\ A_3 \stackrel{n}{=} -A_2'',\ -A_2'' - \omega,\ A_3 + 2\nu,$$

or

$$A_2'' + \omega,\ A_2'',\ A_3 \stackrel{n}{=} A_2'' - \nu + \omega,\ A_2'' - \nu,\ A_3 + 2\nu.$$

From the above result we conclude that we may consider only the root

$$A_2' = \frac{\nu - \omega + \sqrt{\nu^2 - \omega^2 + 4(A_3 + \nu)^2}}{2}$$

Putting $\nu^2 - \omega^2 + 4(A_3 + \nu)^2 = t^2$ we find that

$$\nu = 2ac - 2bd,\quad \omega = 2bc - 2ad,\quad A_3 = bc + ad - 2ac + 2bd$$

and $t = 2ac + 2bd$.

Hence $A_2' = 2ac - bc + ad$ and from (1), (3) and (4) we take:

$$A_1 = 2ac + bc - ad,\quad B_2 = 2bd - bc + ad,$$

$$B_1 = 2bd + bc - ad,\quad B_3 = bc + ad + 2ac - 2bd.$$

Thus we have the identities:

$$[a(2c - d) + bc],\ [a(2c + d) - bc],\ [a(-2c + d) + b(c + 2d)] \stackrel{n}{=}$$
$$\stackrel{n}{=} [-ad + b(c + 2d)],\ [ad - b(c - 2d)],\ [a(2c + d) + b(c - 2d)],\quad (n = 1,3)$$

As an illustrative example, for $a = 5$, $b = 4$, $c = 2$, $d = 1$ we get

$$23,\ 17,\ 1 \stackrel{n}{=} 11,\ 5,\ 25,\quad (n = 1,3).$$

V

SUMMARY

The authors in the parts I and IV of this paper give solutions of the system $\sum_{i=1}^{3} A_i^n = \sum_{i=1}^{3} B_i^n$ $(n = 1,3)$ under certain additional conditions. The solutions given in part I lead to solutions of the system $A_1, A_2, A_3 \stackrel{n}{=} B_1, B_2, B_3 \stackrel{n}{=} C_1, C_2, C_3$, $(n = 1,3)$.

In part II and III of this paper are given solutions of the system $\sum_{i=1}^{2} A_j \stackrel{n}{=} \sum_{i=1}^{3} B_i$ $(n = 1,3)$ of which in part II arise solutions of the system: $\sum_{i=1}^{3} X_i^k = \sum_{i=1}^{3} \psi_i^k$ $(k = 2,4)$ and in part III of the system $\sum_{i=1}^{4} X_i^n = \sum_{i=1}^{4} \psi_i^n$ $(n = 1,3)$.

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