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FINITE GROUPS WITH THREE CONJUGACY CLASS SIZES OF CERTAIN ELEMENTS

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Abstract. Let G be a finite group and m, n two positive coprime integers. We prove that the set of conjugacy class sizes of primary and biprimary elements of G is $\{1, m, n\}$ if and only if G is quasi-Frobenius with abelian kernel and complement.

1. Introduction

Throughout this paper all groups considered are finite and G always denotes a group. A primary element is an element of prime power order and a biprimary element is an element whose order is divisible by precisely two distinct primes. We will denote by x^G the conjugacy class containing x, and $|x^G|$ the conjugacy class size of x^G. A positive integer a is a Hall number of group G if a is a divisor of $|G|$ and $(a, |G|/a) = 1$. We say that G is quasi-Frobenius if $G/Z(G)$ is Frobenius. The inverse image in G of the kernel and a complement of $G/Z(G)$ are called the kernel and a complement of G. The other notation and terminology are standard, as in [**2**].

A well-known problem in group theory is to study the influence of conjugacy class sizes on the structure of a group. For instance, as regards groups whose set of conjugacy class sizes is $\{1, m\}$ with m an integer, Itô [**4**] proved that G is nilpotent, $m = p^a$ for some prime p, and $G = P \times A$, where P is a Sylow p-subgroup of G. Moreover, $A \leqslant Z(G)$. He also showed [**5**] that G is solvable if the set of conjugacy class sizes of G is $\{1, m, n\}$ with integers m and n. Furthermore, by defining a graph Γ of G, Bertram, Herzog and Mann [**1**] proved that a group whose set of conjugacy class sizes is $\{1, m, n\}$ with m and n coprime if and only if $G/Z(G)$ is Frobenius with abelian kernel and complement.

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On the other hand, it is interesting to investigate the structure of a group by some conjugacy class sizes. For instance, in a recent paper [**6**], Kong proved that: Let G be a p-solvable group for a fixed prime p. If the conjugacy class sizes of all primary and biprimary elements of G are $\{1, p^a, n\}$ with a and n two positive integers and $(p, n) = 1$,then G is p-nilpotent or G has abelian Sylow p-subgroups. Here, we work on groups whose set of conjugacy class sizes of primary and biprimary elements is $\{1, m, n\}$ with m and n coprime. Our main result is:

THEOREM 1.1. *Let G be a group. Let m and n be two positive coprime integers. The set of conjugacy class sizes of primary and biprimary elements of G is $\{1, m, n\}$ if and only if G is quasi-Frobenius with abelian kernel and complement.*

REMARK 1.1. This is a generalization of [**6**, Theorem].

2. Preliminaries

In this section we list some basic and known results which will be used in the sequel.

LEMMA 2.1. [**7**, Lemma 2.4] *Let G be a group. A prime p does not divide any conjugacy class size of primary element of G if and only if G has a central Sylow p-subgroup.*

LEMMA 2.2. [**2**, Theorem 6.4.3] *If $G = AB$, where A, B are two nilpotent subgroups of G, then G is solvable.*

LEMMA 2.3. [**8**, Lemma 2.1] *Let G be a π-separable group with π a subset of $\pi(G)$.*

(a) *$|x^G|$ is a π-number for every primary π'-element x if and only if G has an abelian Hall π'-subgroup.*

(b) *$|x^G|$ is a π'-number for every primary π'-element x if and only if $G = O_\pi(G) \times O_{\pi'}(G)$.*

3. Proof of Theorem 1.1

Assume first that G is quasi-Frobenius with an abelian kernel and complement. Write $G/Z(G) = K/Z(G) \rtimes H/Z(G)$, which is a Frobenius group with abelian kernel $K/Z(G)$ and an abelian complement $H/Z(G)$. For every non-central element $x \in G$, we have that $xZ(G)$ belongs to $K/Z(G)$ or some conjugate of $H/Z(G)$.

If $xZ(G) \in K/Z(G)$, then $K/Z(G) \leqslant C_{G/Z(G)}(xZ(G)) = K/Z(G)$, implying $|x^G| = |(xZ(G))^{G/Z(G)}| = |H/Z(G)|$ since $K/Z(G)$ is abelian and $G/Z(G)$ is a Frobenius group with kernel $K/Z(G)$ and a complement $H/Z(G)$; if $xZ(G) \in H/Z(G)$, then $|x^G| = |(xZ(G))^{G/Z(G)}| = |K/Z(G)|$ by the same reasoning above. Let $m := |H/Z(G)|$ and $n := |K/Z(G)|$. We clearly see that the sufficiency holds. Now we prove the necessity.

If a prime r does not divide m or n, by applying Lemma 2.1 we have that G has a central Sylow r-subgroup. Hence we may write $G = A \times B$, where $B \leqslant Z(G)$ and A is a $\pi(m) \cup \pi(n)$-group. Further, central factors are irrelevant in this context, so the set of conjugacy class sizes of primary and biprimary elements of A is $\{1, m, n\}$.

Consequently, without loss of generality, we may assume that G is a $\pi(m) \cup \pi(n)$-group without central Sylow subgroups.

Let x be a primary or a biprimary element of conjugacy class size m in G. We assert that $C_G(x)$ is nilpotent. By considering the primary decomposition of x, we may consider x as a p-element for some prime p. Moreover, for every primary p'-element $z \in C_G(x)$, as $C_G(xz) = C_G(x) \cap C_G(z) \leqslant C_G(x)$, it implies that $m = |x^G| \mid |(xz)^G| \in \{1, m, n\}$. Easily, $C_G(x) = C_G(xz) \leqslant C_G(z)$ and thus $z \in Z(C_G(x))$, yielding $C_G(x) = C_G(x)_p \times C_G(x)_{p'}$, where $C_G(x)_p$ is the Sylow p-subgroup of $C_G(x)$ and $C_G(x)_{p'}$ is the abelian Hall p'-subgroup of $C_G(x)$. As a consequence, $C_G(x)$ is nilpotent.

Furthermore, by the symmetry of m and n, if y is a primary or a biprimary element of conjugacy class size n, we obtain that $C_G(y)$ is also nilpotent. Therefore, G is solvable by Lemma 2.2 because $G = C_G(x)C_G(y)$. Write $\pi := \pi(m)$. It is clear that G has nilpotent Hall π-subgroups and nilpotent Hall π'-subgroups. According to the symmetry of m and n, we divide the proof into three cases.

Case 1. Both m and n are Hall numbers of G.

Let K be a Hall π-subgroup of G and H be a Hall π'-subgroup of G. Since m is a Hall number of G, we obtain that every primary π-element has conjugacy class size 1 or n in G, by Lemma 2.3(a), it follows that K is abelian. Similarly, H is also abelian.

We prove that $Z(G) = 1$. If not, then either $Z(G)_\pi \neq 1$ or $Z(G)_{\pi'} \neq 1$. Suppose first that the former holds. Then for every primary element $v \in G$, it follows that $|G : C_G(v)|_\pi$ divides $|G : Z(G)|_\pi < m$ because m is a Hall number, indicating $|v^G| = 1$ or n. This shows that every primary element of G has conjugacy class size 1 or n, a contradiction to our assumption. Hence $Z(G)_{\pi'} \neq 1$. However, the same argument above leads to a contradiction. Thus $Z(G) = 1$, as required.

By symmetry, we can assume that $O_\pi(G) > 1$ or $O_{\pi'}(G) > 1$, by appealing to the π-separability of G. We assume for instance that $O_\pi(G) > 1$. Then for every noncentral primary element $a \in O_\pi(G)$, we obtain $|a^G| = n$ as K is abelian. Note that $O_{\pi'}(G) \leqslant C_G(a)$ and n is a Hall number of G. This implies that $O_{\pi'}(G) = 1$ and thus $F(G) = O_\pi(G)$ is abelian. Moreover, $K \leqslant C_G(F(G)) \leqslant F(G) = O_\pi(G)$ as G is solvable with K abelian, yielding $K \trianglelefteq G$. Write $G = K \rtimes H$. We show that H acts fixed-point-freely on K. Assume that there exists some $1 \neq y \in H$ and $1 \neq a \in K$ such that $y \in C_G(a)$. By considering the primary decomposition of x and y, we may assume that x and y are primary elements. As both m and n are Hall numbers, we obtain that $y \in Z(G)$, contradicting with the fact $Z(G) = 1$. Hence $C_G(K) \leqslant K$. By [**3**, Exercises 7.1(a)], we have that G is a Frobenius group. Therefore, G is a Frobenius group with abelian kernel K and an abelian complement H.

Case 2. Only one of m and n is a Hall number of G.

Without loss of generality, we consider that m is a Hall number of G while n is not. Further, G has an abelian Hall π-subgroup if we are applying the same argument as in the first paragraph of Case 1.

Let $y \in G$ be a primary or a biprimary element of conjugacy class size n in G, which exists by the hypothesis. Further, by considering the primary decomposition of y, y can be assumed to be a p-element. Assume first that $p \in \pi'$. Then for every primary π-element $z \in C_G(y)$, we see that $|C_G(y) : C_G(y) \cap C_G(z)| = 1$, which follows that $C_G(y) = K_1 \times C_G(y)_{\pi'}$, where K_1 is an abelian Hall π-subgroup of G and $C_G(y)_{\pi'}$ is the Hall π'-subgroup of $C_G(y)$. Without loss of generality, we may assume that $y \in C_G(y)_{\pi'} \leqslant H$, where H is a Hall π'-subgroup of G. If $C_G(y)_{\pi'} = H$, then $C_G(y) = G$, yielding $y \in Z(G)$. This contradiction shows that $C_G(y)_{\pi'} < H$. Moreover, $|h^G| = n$ for every primary element $h \in H - C_G(y)_{\pi'}$. In fact, if $|h^G| = m$, then there exists some $g \in G$ such that $h \in H^g \leqslant C_G(h)$, indicating $h \in Z(H)^g$. Hence, $h \in C_G(y)^g = C_G(y)^g_{\pi'} \times K_1^g$. This implies $h \in C_G(y)^g_{\pi'}$ and thus $|h^G| = 1$ or n, a contradiction. Therefore, we conclude that every primary element $h_0 \in H$ has conjugacy class size 1 or n in G. By Lemma 2.3(b), we have $G = H \times K_1$. And then $K_1 \leqslant Z(G)$ because K_1 is abelian, which is again a contradiction. As a result, $p \in \pi$.

We claim that H is abelian. Let $v \in H$ be an arbitrary non-central primary element. If $|v^G| = n$, then $C_G(v) = C_G(v)_\pi \times C_G(v)_{\pi'}$, where $C_G(v)_{\pi'}$ is the Hall π'-subgroup of $C_G(v)$ and $C_G(v)_\pi$ is the Hall π-subgroup of $C_G(v)$, and therefore of G. On the other hand, there exists a primary or biprimary element $x \in G$ of conjugacy class size m. Without loss, we consider x as an s-element. We clearly see that $s \in \pi'$ since m is a Hall number of G. By the solvability of G, there exists some element $g \in G$ such that $v^{g^{-1}} \in C_G(x)$, showing $x^g \in C_G(v)_{\pi'}$. This gives that $|x^G|$ is a π'-number, a contradiction. Hence $|v^G| = 1$ or m. Moreover, by Lemma 2.3(a), H is abelian, as desired.

On the other hand, similarly as in Case 1, we obtain that $Z(G)_\pi = 1$. Assume first that $O_\pi(G) \neq 1$. Then $O_{\pi'}(G) \leqslant Z(G)$, which implies that $O_\pi(G) = K \trianglelefteq G$ as $C_G(F(G)) \leqslant F(G)$. Write $G = K \rtimes H$. Let $1 \neq v_1 \in K$ be a primary element and $u_1 \in H - O_{\pi'}(G)$ be a primary or biprimary element. If $[u_1, v_1] = 1$, then $|u_1^G| = 1$, a contradiction to the choice of u_1. Hence, $[u_1, v_1] \neq 1$, which implies that $[u_1, v_1] \notin O_{\pi'}(G)$. We see clearly that $[u_1, v_1] \notin Z(G)_{\pi'} = Z(G)$. For every $1 \neq aZ(G) \in KZ(G)/Z(G)$ and $1 \neq bZ(G) \in H/Z(G)$, by considering the primary decomposition of a and b, we may assume that a and b are primary elements, then $[a, b] \in K$. Further, $C_{G/Z(G)}(KZ(G)/Z(G)) \leqslant KZ(G)/Z(G)$. By [3, Exercises 7.1(a)], we get that $G/Z(G) = KZ(G)/Z(G) \rtimes H/Z(G)$ is a Frobenius group with abelian kernel $KZ(G)/Z(G)$ and a cyclic complement $H/Z(G)$. Moreover, $KZ(G)$ and H are abelian, so the theorem is established.

Suppose then that $O_\pi(G) = 1$, leading $O_{\pi'}(G) \neq 1$. Then we obtain that $F(G) = O_{\pi'}(G)$. Since G is solvable, we see that $H \leqslant C_G(F(G)) = F(G)$ as H is abelian. This shows $H \trianglelefteq G$. By considering the action of K on H by conjugation, we have $H = [H, K] \times C_H(K)$. In particular, $C_H(K) = Z(G)$. For every $1 \neq v_0 \in [H, K]$ and $u_0 \in K$, we see that $[v_0, u_0] \in [H, K]$. Hence $[v_0, u_0] = 1$, which implies that $|v_0^G| = 1$ and thus $v_0 \in C_H(K)$, a contradiction. Hence $[v_0, u_0] \neq 1$, which implies that $[v_0, u_0] \notin C_H(K) = Z(G)$. Similarly as in the above paragraph, we also obtain that $G/Z(G) = H/Z(G) \rtimes KZ(G))/Z(G)$ is a Frobenuis group with abelian kernel $H/Z(G)$ and a cyclic complement $KZ(G)/Z(G)$. Further, H and $KZ(G)$ are abelian.

Case 3. Neither m nor n is a Hall number of G.

If there is a primary π-element f with conjugacy class size m, then $C_G(f) = C_G(f)_\pi \times H$, where H is an abelian Hall π'-subgroup of G. Let K be a Hall π-subgroup of G containing $C_G(f)_\pi$. We trivially have $C_G(f)_\pi < K$ since otherwise f would be central in G. Let $d \in K - C_G(f)_\pi$ be a primary element such that $|d^G| = n$. Since G is solvable, there exists some element $g \in G$ such that $K \leqslant C_G(d)^g$. Moreover, the solvability of $C_G(d)$ implies that there exists some element $t \in C_G(d)^g$ such that $d^g = (d^g)^t \in K$. Consequently, $d^g \leqslant K \leqslant C_G(d)^g$, yielding $d^g \in Z(K)$. This shows that $d^g \in C_G(f)_\pi$ and thus $d \in Z(G)$, contrary to the choice of d.

Consequently, we conclude that every primary π-element has conjugacy class size 1 or n. Symmetrically, every primary π'-element has size of 1 or m. Let x be a primary π'-element with $|x^G| = m$. For every primary element $z \in C_G(x)$, we see that $|C_G(x) : C_G(x) \cap C_G(z)| = 1$, it follows that $C_G(x) = C_G(x)_\pi \times H$, where $C_G(x)_\pi$ is abelian and H is an abelian Hall π'-subgroup of G. If $C_G(x)_\pi \nleqslant Z(G)$, then there is a primary element $u \in C_G(x)_\pi - Z(G)$ and thus $|u^G| = m$, which contradicts the argument above. Hence $C_G(x)_\pi = Z(G)_\pi$. We observe that K and H are abelian.

Since G is solvable, we have that $F(G) \neq 1$. By the symmetry of m and n, we may assume that $F(G)_\pi \nleqslant Z(G)$. We prove that $K \trianglelefteq G$. Let $v \in K - Z(G)$ be a primary element. Then $|v^G| = n$. Moreover, $C_G(v) = K \times C_G(v)_{\pi'}$, where $C_G(v)_{\pi'}$ is the abelian Hall π'-subgroup of $C_G(v)$ and K is the Hall π-subgroup of $C_G(v)$. Similarly, $C_G(v)_{\pi'} = Z(G)_{\pi'}$ by a similar argument in the paragraph above. Moreover, $|H : Z(G)_{\pi'}| = n$, indicating that $O_{\pi'}(G) = Z(G)_{\pi'}$. In fact, if there exists some primary element $z \in O_{\pi'}(G) - Z(G)_{\pi'}$, then $C_G(z) \geqslant \langle O_\pi(G), z\rangle > O_\pi(G) \geqslant Z(G)_\pi = C_G(x)_\pi$, giving $|z^G| = n$, a contradiction to the argument above. As a result, $K \leqslant C_G(F(G)) \leqslant F(G) = O_\pi(G) \times O_{\pi'}(G)$ since K is abelian, showing $K \trianglelefteq G$.

Now consider the action of H on K by conjugation. Since K is abelian, we have $K = [K, H] \times C_K(H)$. Moreover, $C_K(H)$ is an abelian direct factor of G. As a consequence, $m = |K : C_G(x)_\pi| = |K : Z(G)_\pi| = |K : C_K(H)| = |[K, H]|$. For every primary element $v \in H - Z(G)$ and $1 \neq u \in [K, H]$, it follows that $[u, v] \neq 1$. Further, $[u, v] \notin Z(G)$ since, otherwise, $[u, v] \in C_K(H) \cap [K, H] = 1$, also a contradiction. By the same argument as in Case 2, we get that $G/Z(G)$ is a Frobenius group with abelian kernel $KZ(G)/Z(G)$ and a cyclic complement $HZ(G)/Z(G)$. Moreover, $KZ(G)$ and $HZ(G)$ are abelian, and the theorem is established.

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References

1. E. A. Bertram, M. Herzog, A. Mann, *On a graph related to conjugacy classes of groups*, Bull. London Math. Soc. **22** (1990), 569–575.
2. B. Huppert, *Endliche Gruppen I*, Springer, Berlin, 1967.
3. I. M Isaacs, *Character Theory of Finite Groups*, Academic Press, New York, 1976.
4. N. Itô, *On finite groups with given conjugate types I*, Nagoya Math. J. **6** (1953), 17–28.

5. ______, *On finite groups with given conjugate types II*, Osaka J. Math. **7** (1970), 231–251.
6. Q. J. Kong, X. Y. Guo, *On an extension of a theorem on conjugacy class sizes*, Israel J. Math. **179** (2010), 279–284.
7. Q. J. Kong, *Finite groups with three conjugacy class sizes of some elements*, Proc. Indian Acad. Sci. **122**(3) (2012), 335–337.
8. C. G. Shao, Q. H. Jiang, *On conjugacy class sizes of primary and biprimary elements of a finite group*, Sci. China, Math. **57**(3) (2014), 491–498.

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