CONVERGENCE THEOREMS OF A SCHEME FOR
I-ASYMPTOTICALLY QUASI-NONEXPANSIVE
TYPE MAPPING IN BANACH SPACE

Seyit Temir

Abstract. Let $X$ be a Banach space. Let $K$ be a nonempty subset of $X$. Let $T : K \to K$ be an $I$-asymptotically quasi-nonexpansive type mapping and $I : K \to K$ be an asymptotically quasi-nonexpansive type mappings in the Banach space. Our aim is to establish the necessary and sufficient conditions for the convergence of the Ishikawa iterative sequences with errors of an $I$-asymptotically quasi-nonexpansive type mapping in Banach spaces to a common fixed point of $T$ and $I$. Also, we study the convergence of the Ishikawa iterative sequences to common fixed point for nonself $I$-asymptotically quasi-nonexpansive type mapping in Banach spaces. The results presented in this paper extend and generalize some recent work of Chang and Zhou [1], Wang [19], Yao and Wang [20] and many others.

1. Introduction

Let $X$ be a real Banach space, $K$ be a nonempty subset of Banach space and $T, I : K \to K$. Let $F(T) = \{x \in K : Tx = x\}$ and $F(I) = \{x \in K : Ix = x\}$ denote the set of fixed points of mappings $T$ and $I$, respectively. Recall some definitions and notations. $T$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$. The quasi-nonexpansive mappings defined as the following were studied by Diaz and Metcalf [4] and Dotson [5] in Banach spaces. $T$ is called a quasi-nonexpansive mapping if $F(T) \neq \emptyset$ and $\|Tx - p\| \leq \|x - p\|$ for all $x \in K$ and $p \in F(T)$. The concept of asymptotically nonexpansiveness defined as the following was introduced by Goebel and Kirk [7]. $T$ is called asymptotically quasi-nonexpansive mapping if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n \to \infty} k_n = 1$ such that $\|T^nx - p\| \leq k_n \|x - p\|$ for all $x \in K$ and $p \in F(T)$ and $n \geq 1$. Let $X$ be a Banach space and $K$ be a nonempty subset of the Banach space. Let $T, I : K \to K$ be two mappings. $T$ is called $I$-nonexpansive if $\|Tx - Ty\| \leq \|Ix - Iy\|$ for all $x, y \in K$. $T$ is called $I$-quasi-nonexpansive if $\cap F(T) \neq \emptyset$ and $\|Tx - p\| \leq \|Ix - p\|$ for all $x \in K$ and $p \in F(T) \cap F(I)$.

2010 Mathematics Subject Classification: 47H09; 47H10.
Key words and phrases: $I$-asymptotically quasi-nonexpansive type mapping, nonself $I$-asymptotically quasi-nonexpansive type mapping, Ishikawa iterative schemes.
Communicated by Stevan Pilipović.
From the above definitions, it follows that if $F(T) \cap F(I)$ is nonempty, an I-nonexpansive mapping must be I quasi-nonexpansive, and linear I quasi-nonexpansive mappings are I-nonexpansive mappings. But it is easily seen that there exist nonlinear continuous $I$ quasi-nonexpansive mappings which are not I-nonexpansive. $T$ is called I-asymptotically quasi-nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n \to \infty} k_n = 1$ such that $\|T^n x - p\| \leq k_n\|P^n x - p\|$ for all $x \in K$ and $p \in F(T) \cap F(I)$ and $n \geq 1$. $T$ is called I-asymptotically nonexpansive type mapping if $\limsup_{n \to \infty} \{\sup\{\|T^n x - T^n y\| - \|P^n x - P^n y\|\}\} = 0$ for all $x, y \in K$.

$T$ is called I-asymptotically quasi-nonexpansive type if $F(T) \cap F(I) \neq \emptyset$ and
\[
\limsup_{n \to \infty} \{\sup\{\|T^n x - p\| - \|P^n x - p\|\}\} \leq 0
\]
for all $x \in K$ and $p \in F(T) \cap F(I)$.

$I$ is called asymptotically quasi-nonexpansive type if $F(I) \neq \emptyset$ and
\[
\limsup_{n \to \infty} \{\sup\{\|I^n x - p\| - \|x - p\|\}\} \leq 0
\]
for all $x \in K$ and $p \in F(I)$.

From the above definitions, it follows that if $F(I)$ is nonempty, quasi-nonexpansive mappings, asymptotically nonexpansive mappings, asymptotically quasi-nonexpansive mappings and asymptotically nonexpansive type mappings all are special cases of asymptotically quasi-nonexpansive type mappings.

Let $\{x_n\}$ be of the Ishikawa iterative scheme $[8]$ associated with $T$, $x_0 \in K$,
\[
y_n = (1 - \beta_n)x_n + \beta_nTx_n
\]
\[
x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTy_n
\]
for every $n \in \mathbb{N}$, where $0 \leq \alpha_n, \beta_n \leq 1$.

Let $S, T : K \to K$ be two mappings. In 2006, Lan $[9]$ introduced the following iterative scheme with errors. The sequence $x_n$ in $K$ defined by
\[
y_n = (1 - \beta_n)x_n + \beta_nT^n x_n + \psi_n
\]
\[
x_{n+1} = (1 - \alpha_n)x_n + \alpha_nS^n y_n + \varphi_n
\]
for every $n \in \mathbb{N}$, where $0 \leq \{\alpha_n\}, \{\beta_n\} \leq 1$ and $\{\varphi_n\}, \{\psi_n\}$ are two sequences in $K$.

The iterative approximation problems for nonexpansive mapping, asymptotically nonexpansive mapping and asymptotically quasi-nonexpansive mapping were studied Ghosh and Deb Nath $[6]$, Goebel and Kirk $[7]$, Liu $[10, 11]$, Petryshyn and Williamson $[13]$ in the settings of Hilbert spaces and uniformly convex Banach spaces. The strong and weak convergences of the sequence of Mann iterates to a fixed point of quasi-nonexpansive maps were studied by Petryshyn and Williamson $[13]$. Subsequently, the convergence of Ishikawa iterates of quasi-nonexpansive mappings in Banach spaces were discussed by Ghosh and Deb Nath $[6]$. The above results and some necessary and sufficient conditions for Ishikawa iterative sequences obtained to converge to a fixed point for asymptotically quasi-nonexpansive mappings were extended by Liu $[10]$. In $[11]$, the results of Liu $[10]$ were extended and some sufficient and necessary conditions for Ishikawa iterative sequences of
asymptotically quasi-nonexpansive mappings with error member to converge to fixed points were proved. Recently, Temir and Gul [17] obtained the weakly almost convergence theorems for I-asymptotically quasi-nonexpansive mapping in a Hilbert space. In [20], Yao and Wang established the strong convergence of an iterative scheme with errors involving I-asymptotically quasi-nonexpansive mappings in a uniformly convex Banach space. Temir [18], studied the convergence to common fixed point of Ishikawa iterative process of generalized I-asymptotically quasi-nonexpansive mappings to common fixed point in Banach space. In [1], the convergence theorems for Ishikawa iterative sequences with mixed errors of asymptotically quasi-nonexpansive type mappings in Banach spaces were studied.

2. Preliminaries and notations

We first recall the following definitions. A Banach space $X$ is said to satisfy Opial’s condition [12] if, for each sequence $\{x_n\}$ in $X$, the condition $x_n \to x$ implies

$$\lim \inf_{n \to \infty} \|x_n - x\| < \lim \inf_{n \to \infty} \|x_n - y\|$$

for all $y \in X$ with $y \neq x$. It is well known from [12] that all $l_r$ spaces for $1 < r < \infty$ have this property. However, the $L_r$ space do not have unless $r = 2$.

In order to prove the main results of this paper, we need the following lemmas.

**Lemma 2.1.** [16] Let $\{a_n\}, \{b_n\}$ be sequences of nonnegative real numbers satisfying the following conditions: $\forall n \geq 1$, $a_{n+1} \leq a_n + b_n$, where $\sum_{n=1}^{\infty} b_n < \infty$. Then $\lim_{n \to \infty} a_n$ exists.

**Lemma 2.2.** [15] Let $K$ be a nonempty closed bounded convex subset of a uniformly convex Banach space $X$ and $\{a_n\} \subseteq [\epsilon, 1 - \epsilon] \subset (0, 1)$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in $K$ such that $\limsup_{n \to \infty} \|x_n\| \leq c$, $\limsup_{n \to \infty} \|y_n\| \leq c$, and $\limsup_{n \to \infty} \|a_n x_n + (1 - a_n) y_n\| = c$ for some $c > 0$. Then $\lim_{n \to \infty} \|x_n - y_n\| = 0$.

**Lemma 2.3.** [2] Let $X$ be a uniformly convex Banach space, $K$ a nonempty closed convex subset of $X$ and $T : K \to K$ an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ and $\lim_{n \to \infty} k_n = 1$. Then $E - T$ is semi-closed (demi-closed) at zero, i.e., for each sequence $\{x_n\}$ in $K$, if $\{x_n\}$ converges weakly to $q \in K$ and $(E - T)\{x_n\}$ converges strongly to 0, then $(E - T)q = 0$.

3. Convergence theorems for I-asymptotically quasi-nonexpansive type mapping

In this section, $X$ is a Banach space and $K$ is its nonempty subset. Let $T, I : K \to K$ be two mappings, where $T$ is an $I$-asymptotically quasi-nonexpansive type mapping and $I : K \to K$ is an asymptotically quasi-nonexpansive type mapping. We study the strong and weak convergences of the sequence of Ishikawa iterates with mixed errors to a common fixed point of $T$ and $I$.

**Theorem 3.1.** Let $X$ be a Banach space, $K$ its nonempty subset, and $T, I : K \to K$ two mappings. Let $T$ be an $I$-asymptotically quasi-nonexpansive type and $I$ be an asymptotically quasi-nonexpansive type in the Banach space satisfying

$$(3.1) \quad \|Tx - p\| \leq L\|Ix - p\|$$
for all $x \in K$ and $p \in F(T) \cap F(I)$, where $L > 0$ is a constant and
\begin{equation}
\|Lx - p\| \leq \Gamma \|x - p\|
\end{equation}
for all $x \in K$ and $p \in F(I)$, where $\Gamma > 0$ is a constant. Write $I : K \to K$ instead of $S : K \to K$ in (3.2) and get
\begin{align*}
y_n &= (1 - \beta_n)x_n + \beta_nT^nx_n + \psi_n \\
x_{n+1} &= (1 - \alpha_n)x_n + \alpha_nI^ny_n + \varphi_n
\end{align*}
for every $n \in \mathbb{N}$, where $0 \leq \{\alpha_n\}, \{\beta_n\} \leq 1$ and $\{\varphi_n\}, \{\psi_n\}$ be two sequences in $K$ satisfying: (i) $\sum_{n=1}^{\infty} \alpha_n < \infty$; (ii) $\{\psi_n\}$ is bounded, $\varphi_n = \varphi'_n + \varphi''_n$, $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \|\varphi'_n\| < \infty$, $\|\varphi''_n\| = o(\alpha_n)$.

Then $\{x_n\}$ converges strongly to a common fixed point of $T$ and $I$ in $K$ iff
\begin{equation}
\liminf_{n \to \infty} d(x_n, F(T) \cap F(I)) = 0.
\end{equation}

**Lemma 3.1.** Suppose all conditions in Theorem 3.1 are satisfied; then for $\varepsilon > 0$, there exists a positive integer $n_0$ and $M > 0$ such that
\begin{equation}
\|x_{n+1} - p\| \leq \|x_n - p\| + \alpha_nM + \|\varphi'_n\|
\end{equation}
for all $p \in F(T) \cap F(I)$, $n \geq n_0$ and
\begin{equation}
\|x_{n+m} - p\| \leq \|x_n - p\| + M \sum_{i=n}^{n+m-1} \alpha_i + \sum_{i=n}^{n+m-1} \|\varphi'_i\|
\end{equation}
for all $p \in F(T) \cap F(I)$, $n \geq n_0$, $\forall m \geq 1$, where $M = \sup_{n \geq 0} \{\varepsilon_n + \|\psi_n\|\} + 3\varepsilon < \infty$, and $\varepsilon_n$ is a sequence with $\varepsilon_n > 0$ and $\varepsilon_n \to 0$ such that $\|\varphi''_n\| = \varepsilon_n\alpha_n$.

**Proof.** For $p \in F(T) \cap F(I)$, from (3.3), we have
\begin{equation}
\|x_{n+1} - p\| = \|(1 - \alpha_n)(x_n - p) + \alpha_n(I^ny_n - p) + \psi_n\|
\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n\|I^ny_n - p\| + \|\varphi_n\|
= (1 - \alpha_n)\|x_n - p\| + \alpha_n\|I^ny_n - p\| - \|y_n - p\|
+ \alpha_n\|y_n - p\| + \|\varphi_n\|
\end{equation}

Now we consider the second term on the right-hand side of (3.5). From (1.1) and (1.2), for any given $\varepsilon > 0$, there exists a positive integer $n_0$ such that $n \geq n_0$, so we have
\begin{align*}
\sup_{x \in K, p \in F(T) \cap F(I)} \{\|T^n x - p\| - \|I^n x - p\|\} &< \varepsilon, \\
\sup_{x \in K, p \in F(I)} \{\|I^n x - p\| - \|x - p\|\} &< \varepsilon.
\end{align*}
Therefore, in particular, we have
\begin{equation}
\{\|T^n x - p\| - \|I^n x - p\|\} < \varepsilon,
\end{equation}
for all $p \in F(T) \cap F(I)$ and $\forall n \geq n_0$.
\begin{equation}
\{\|I^n y - p\| - \|y - p\|\} < \varepsilon,
\end{equation}
for all \( p \in F(I) \) and \( \forall n \geq n_0 \). From (3.7), we have

\[
(3.8) \quad \|x_{n+1} - p\| \leq (1 - \alpha_n)\|x_n - p\| + \alpha_n\|y_n - p\| + \|\varphi_n\|
\]

Consider the third term on the right-hand side of (3.8). From (3.6) and (3.7), we get

\[
(3.9) \quad \|y_n - p\| = \|(1 - \beta_n)(x_n - p) + \beta_n(T^n x_n - p) + \psi_n\|
\]

\[
\leq (1 - \beta_n)\|x_n - p\| + \beta_n\|T^n x_n - p\| - \|I^n x_n - p\| + \|\psi_n\|
\]

\[
\leq (1 - \beta_n)\|x_n - p\| + 2\beta_n\|y_n - p\| + \|\psi_n\|
\]

\[
= \|x_n - p\| + 2\beta_n\|y_n - p\| + \|\psi_n\|
\]

Now consider the fourth term on the right side of (3.8): we have \( \|\varphi_n\| \leq \|\varphi'_n\| + \|\varphi''_n\| \), \( \forall n \geq 0 \). Substituting (3.9) into (3.8), we have

\[
\|x_{n+1} - p\| \leq (1 - \alpha_n)\|x_n - p\| + \alpha_n\|x_n - p\| + \|\varphi'_n\| + \|\varphi''_n\| + \|\psi_n\|
\]

Taking \( M = \sup_{n \geq 0} \{\psi_n + \|\psi_n\|\} + 3\varepsilon \) we obtain

\[
(3.10) \quad \|x_{n+1} - p\| \leq \|x_n - p\| + \alpha_nM + \|\varphi'_n\|
\]

for all \( p \in F(T) \cap F(I) \), \( n \geq n_0 \). Writing \( n + m - 1 \) instead of \( n \) in inequality (3.10), for \( m \geq 1 \), we get

\[
\|x_{n+m} - p\| \leq \|x_{n+m-1} - p\| + \alpha_{n+m-1}\|\varphi'_{n+m-1}\|
\]

\[
\leq \|x_{n+m-2} - p\| + (\alpha_{n+m-1} + \alpha_{n+m-2})M + \|\varphi'_{n+m-2}\| + \|\varphi''_{n+m-1}\|
\]

\[
\vdots
\]

\[
\leq \|x_n - p\| + M \sum_{i=n}^{n+m-1} \alpha_i + \sum_{i=n}^{n+m-1} \|\varphi'_i\|
\]

for all \( p \in F(T) \cap F(I) \), \( n \geq n_0 \). Thus Lemma 3.1 is proved.

Since \( \{\psi_n\} \) is bounded, \( \varphi_n = \varphi'_n + \varphi''_n \), \( n \in \mathbb{N} \) and \( \sum_{n=0}^{\infty} \|\varphi'_n\| < \infty \), \( \|\varphi''_n\| = o(\alpha_n) \), then we have \( \sum_{n=0}^{\infty} M\alpha_n + \|\varphi'_n\| < \infty \). From Lemma 2.1, we take \( \{a_n\} = \{x_n - p\} \) and \( \{b_n\} = M\alpha_n + \|\varphi'_n\| \). This implies that \( \lim_{n \to \infty} \|x_n - p\| \) exists.

**Proof of Theorem 3.1.** We only prove the sufficiency of Theorem 3.1. Suppose that (3.3) is satisfied; then \( \lim_{n \to \infty} d(x_n, F(T) \cap F(I)) = 0 \).

First we show that \( \{x_n\} \) is a Cauchy sequence in \( K \). For \( \varepsilon > 0 \) and \( n \geq n_1 \) there exists \( n_1 \geq n_0 \) such that \( d(x_n, F(T) \cap F(I)) < \varepsilon \). Then \( \sum_{n=n_1}^{\infty} \alpha_n < \frac{\varepsilon}{M} \sum_{n=n_1}^{\infty} \|\varphi'_n\| < \varepsilon \). By the definition of infimum and \( d(x_n, F(T) \cap F(I)) < \varepsilon \) there exists \( p_0 \in F(T) \cap F(I) \), we have

\[
\|x_n - p_0\| \leq \|x_n - p\| + \|p - p_0\| < \varepsilon
\]
$F(I)$ such that $d(x_{n_1}, p) < 2\varepsilon$. Furthermore, for $n \geq n_1 \geq n_0$ and $\forall m \geq 1$

$$\|x_{n+m} - x_n\| \leq \|x_{n+m} - p_0\| + \|x_n - p_0\|$$

$$\leq \|x_{n_1} - p_0\| + M \sum_{i=n_1}^{n+m-1} \alpha_i + \sum_{i=n_1}^{n+m-1} \|\varphi_i\|$$

$$+ \|x_{n_1} - p_0\| + M \sum_{i=n_1}^{n-1} \alpha_i + \sum_{i=n_1}^{n-1} \|\varphi_i\|.$$ 

Then for $n \geq n_1 \geq n_0$ and $\forall m \geq 1$ we have $\|x_{n+m} - x_n\| \leq 8\varepsilon$. Since $\varepsilon$ is arbitrary, then $\{x_n\}$ is a Cauchy sequence in $K$. Since $X$ is a Banach space, let $\{x_n\} \to p^*$ as $n \to \infty$. We prove that $p^* \in F(T) \cap F(I)$. We have $\{x_n\} \to p^*$ as $n \to \infty$ and $\lim_{n \to \infty} d(x_n, F(T) \cap F(I)) = 0$, for $\varepsilon > 0$, there exists a positive integer $n_1 \geq n_0$ and $n \geq n_1$ we have $\|x_n - p^*\| < \varepsilon$, $d(x_n, F(T) \cap F(I)) < \varepsilon$. Then there exists $q \in F(T) \cap F(I)$ such that $d(x_{n_1}, q) < 2\varepsilon$. Furthermore, for $n \geq n_2$

$$\|T^n p^* - q\| \leq \{(\|T^n p^* - q\| - \|p^* - q\|) + 2\|p^* - q\|$$

$$\leq \{(\|T^n p^* - q\| - \|I^n p^* - q\|) + \|I^n p^* - q\| + \|p^* - q\|\} + 3\|p^* - q\|$$

$$< 2\varepsilon + 3\varepsilon = 11\varepsilon.$$ 

Since $T$ is $I$-asymptotically quasi nonexpansive type and $I$ is asymptotically quasi nonexpansive type, this implies that $\{T^n p^*\} \to p^*$ as $n \to \infty$. Furthermore,

$$\|T^n p^* - Tp^*\| \leq \{(\|T^n p^* - q\| - \|p^* - q\|) + \|p^* - q\| + \|Tp^* - q\|.$$ 

Then for $n \geq n_2$ by $13.4$, $13.2$, $13.6$ and $13.7$ we have

$$\|T^n p^* - Tp^*\| \leq \{(\|T^n p^* - q\| - \|I^n p^* - q\|) + \|I^n p^* - q\| - \|p^* - q\|\}$$

$$+ 2\|p^* - q\| + L\|I p^* - q\|$$

$$\leq 2\varepsilon + 2\|p^* - q\| + L\|p^* - q\|$$

$$\leq 2\varepsilon + (2 + L\varepsilon)\{\|x_{n_2} - p^*\| + \|x_{n_2} - q\|\}$$

$$< 2\varepsilon + (2 + L\varepsilon)\varepsilon < \varepsilon(8 + 3L\varepsilon)$$

Since $\varepsilon$ is arbitrary, $\{T^n p^*\} \to Tp^*$ as $n \to \infty$, implying $Tp^* = p^* \in F(T) \cap F(I)$.

Further we apply for $I : K \to K$ asymptotically quasi nonexpansive type mapping. Then for $n \geq n_2$ we have

$$\|I^n p^* - p^*\| \leq \{(\|I^n p^* - q\| - \|q - p^*\|) + 2\|p^* - q\|$$

$$\leq \varepsilon + 2\{\|x_{n_2} - p^*\| + \|x_{n_2} - q\|\} < \varepsilon + 2\{\varepsilon + 2\varepsilon\} = 7\varepsilon$$

This implies that $\{I^n p^*\} \to p^*$ as $n \to \infty$. Furthermore,

$$\|I^n p^* - Ip^*\| \leq \{(\|I^n p^* - q\| - \|p^* - q\|) + \|p^* - q\| + \|Ip^* - q\|.$$ 


Then for \( n \geq n_2 \) by (3.2) and (3.7) we have
\[
\|T^np^* - Ip^*\| \leq \{\|T^np^* - q\| - \|p^* - q\|\} + \|p^* - q\| + \Gamma\|Ip^* - q\|
\leq \varepsilon + \|p^* - q\| + \Gamma\|Ip^* - q\|
\leq \varepsilon + (1 + \Gamma)\{\|xn_2 - p^*\| + \|x_{n_2} - q\|\}
\leq \varepsilon + (1 + \Gamma)3\varepsilon < \varepsilon + (4 + 3\Gamma).
\]
Since \( \varepsilon \) is arbitrary, \( \{T^np^*\} \to p^* \) as \( n \to \infty \). Also
\[
\|T^np^* - Ip^*\| \leq \|T^np^* - q\| + \|Ip^* - q\| < 2\varepsilon
\]
Since \( \varepsilon \) is arbitrary, \( \{T^np^*\} \to Ip^* \) as \( n \to \infty \). This shows that \( Ip^* = p^* \in F(I) \).
From this we obtain \( p^* \in F(T) \cap F(I) \).
Thus \( \{x_n\} \) converges strongly to a common fixed point of \( T \) and \( I \) in \( K \), subset of \( X \) Banach space.

Now we establish the weak convergence theorem for Ishikawa iterates of \( I \)-asymptotically quasi-nonexpansive type mappings in Banach spaces. First, we prove the following lemma.

**Lemma 3.2.** Let \( X \) be a uniformly convex Banach space and \( K \) be a nonempty closed convex subset of \( X \). Let \( T, I \) and \( \{x_n\} \) be the same as in Lemma 3.1. If \( F = F(T) \cap F(I) \neq \emptyset \), then \( \lim_{n \to \infty} \|Tx_n - x_n\| = \lim_{n \to \infty} \|Ix_n - x_n\| = 0 \).

**Proof.** By Lemma 3.1, for any \( p \in F(T) \cap F(I) \), \( \lim_{n \to \infty} \|x_n - p\| \) exists. Let \( \lim_{n \to \infty} \|x_n - p\| = c \). If \( c = 0 \), then the proof is completed.

Now suppose \( c > 0 \). From (3.7), we have \( \|y_n - p\| \leq \|x_n - p\| + 2\beta_n\varepsilon + \|\psi_n\| \).
Taking \( \limsup \) on both sides in the above inequality,
\[
\lim_{n \to \infty} \sup_{\langle 3.11 \rangle} \|y_n - p\| \leq c.
\]
Since \( I \) is asymptotically nonexpansive type self-mappings on \( K \), from (3.7), which is on taking \( \limsup_{n \to \infty} \) and using (3.11), then we get \( \limsup_{n \to \infty} \|T^ny_n - p\| \leq c \).
Further, \( \lim_{n \to \infty} \|x_{n+1} - p\| = c \) means that
\[
\lim_{n \to \infty} \|\alpha_nT^nx_n + (1 - \alpha_n)x_n - p\| = c
\]
\[
\lim_{n \to \infty} (1 - \alpha_n)\|x_n - p\| + \alpha_n\|T^nx_n - p\| = c.
\]
It follows from Lemma 2.2
\[
\lim_{n \to \infty} \|T^nx_n - x_n\| = 0.
\]
Further,
\[
\lim_{n \to \infty} \|\alpha_n(T^nx_n - p) + (1 - \alpha_n)(x_n - p)\| = \lim_{n \to \infty} \|y_n - p\| = c.
\]
By Lemma 2.2, we have
\[
\lim_{n \to \infty} \|T^nx_n - x_n\| = 0.
\]
From (3.12) and (3.13), we have
\[
\lim_{n \to \infty} \|I^n x_n - x_n\| = 0.
\]
Using (3.1), (3.2), (3.3), (3.13) and (3.14), it is easy to show that

\[
\lim_{n \to \infty} \|Tx_n - x_n\| = 0
\]
\[
\lim_{n \to \infty} \|Ix_n - x_n\| = 0.
\]

Then the proof is completed. \(\Box\)

Theorem 3.2. Let \(X\) be a uniformly convex Banach space which satisfies Opial’s condition, \(K\) be a nonempty closed convex subset of \(X\). Let \(T, I\) and \(\{x_n\}\) be the same as in Lemma 3.1. If \(F(T) \cap F(I) \neq \emptyset\), the mappings \(E - T\) and \(E - I\) are semi-closed at zero, then \(\{x_n\}\) converges weakly to a common fixed point of \(T\) and \(I\).

Proof. By assumption, \(F(T) \cap F(I)\) is nonempty. Take \(p \in F(T) \cap F(I)\). It follows from Lemma 3.1 that the limit \(\lim_{n \to \infty} \|x_n - p\|\) exists. Therefore, \(\{x_n - p\}\) is a bounded sequence in \(X\). Since \(X\) is a uniformly convex Banach space and \(K\) is a nonempty closed convex subset of \(X\), then \(K\) is weakly compact. This implies that there exists a subsequence \(\{x_{n_k}\}\) of \(\{x_n\}\) such that \(\{x_{n_k}\}\) converges to a point \(p \in w(\{x_n\})\), where \(w(\{x_n\})\) denotes the weak limit set of \(\{x_n\}\), which shows that \(w(\{x_n\})\) is nonempty. For any \(p \in w(\{x_n\})\), there exists a subsequence \(\{x_{n_k}\}\) of \(\{x_n\}\) such that \(x_{n_k} \to p\) weakly. Hence, it follows from (3.15) and (3.16) in Lemma 3.2 that \(Tp = p\) and \(Ip = p\). By Opial’s condition, \(\{x_n\}\) has only one weak limit point, i.e., \(\{x_n\}\) converges weakly to a common fixed point of \(T\) and \(I\). \(\Box\)

4. Convergence for nonself \(I\)-asymptotically quasi-nonexpansive type mappings

In this section, the convergence of the Ishikawa iterative sequences to common fixed point for nonself \(I\)-asymptotically quasi-nonexpansive type mappings is obtained in Banach spaces.

A subset \(K\) of \(X\) is called a retract of \(X\) if there exists a continuous map \(P : X \to K\) such that \(Px = x\) for all \(x \in K\). A map \(P : X \to K\) is called a retraction if \(P^2 = P\). In particular, a subset \(K\) is called a nonexpansive retract of \(X\) if there exists a nonexpansive retraction \(P : X \to K\) such that \(Px = x\) for all \(x \in K\).

Next, we introduce the following concepts for nonself mappings. Let \(X\) be a real Banach space. A subset \(K\) of \(X\) be nonempty nonexpansive retraction of \(X\) and \(P\) be nonexpansive retraction from \(X\) onto \(K\). A nonself mapping \(T : K \to X\) is called asymptotically nonexpansive if there exists a sequence \(\{v_n\} \subset [1, \infty)\) with \(\lim_{n \to \infty} v_n = 1\) such that

\[
\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq v_n\|x - y\|
\]

for all \(x, y \in K\) and \(n \geq 1\). \(T\) is called uniformly \(L\)-Lipschitzian if there exists a constant \(L > 0\) such that

\[
\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq L\|x - y\|
\]

for all \(x, y \in K\) and \(n \geq 1\). From the above definition, it is obvious that nonself asymptotically nonexpansive mappings is uniformly \(L\)-Lipschitzian.
Let $I : K \rightarrow X$ be a nonself asymptotically quasi-nonexpansive type mappings and $T : K \rightarrow X$ be a nonself $I$-asymptotically quasi-nonexpansive type mappings with $F(T) \cap F(I) = \{ x \in K : Tx = x = Ix \} \neq \emptyset$. A mapping $T : K \rightarrow X$ is called $\Lambda$-Lipschitzian if there exists constant $\Lambda > 0$ such that

$$
\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq \Lambda\|I(PI)^{n-1}x - I(PI)^{n-1}y\|
$$

for all $x, y \in K$ and $n \geq 1$.

Iterative techniques for converging fixed points of nonexpansive non-self mappings have been studied by many authors (see, for example, [3, 19, 14]). The concept of nonself asymptotically nonexpansive mappings was introduced in [3] as a generalization of asymptotically nonexpansive self-mappings and some strong and weak convergence theorems for such mappings were obtained. The sequence \( \{x_n\}_{n \geq 1} \) generated as follows: \( x_1 \in K \),

$$
y_n = P(\alpha_n T(PT)^{n-1}x_n + \beta_n x_n),
$$

$$
x_{n+1} = P(\alpha'_n I(PI)^{n-1}y_n + \beta'_n x_n), \quad \forall n \geq 1,
$$

where \( \{\alpha_n\}, \{\beta_n\}, \{\alpha'_n\}, \{\beta'_n\} \in (0, 1) \).

Let $T : K \rightarrow X$ be a nonself asymptotically quasi-nonexpansive type mapping and $I : K \rightarrow X$ be a nonself asymptotically quasi-nonexpansive type mapping.

Now we define an \( \{x_n\}_{n \geq 1} \) sequence as follows:

$$
y_n = P(\alpha_n T(PT)^{n-1}x_n + \beta_n x_n + \gamma_n \psi_n),
$$

$$
x_{n+1} = P(\alpha'_n I(PI)^{n-1}y_n + \beta'_n x_n + \gamma'_n \psi_n + \phi_n), \quad \forall n \geq 1,
$$

where \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}, \{\gamma'_n\} \) are sequences in \( (0, 1) \) with \( \alpha_n + \beta_n + \gamma_n = 1 = \alpha'_n + \beta'_n + \gamma'_n \) and \( \{\psi_n\}, \{\phi_n\} \) are bounded sequences in $K$.

$$
\limsup_{n \rightarrow \infty} \left( \sup_{x \in X, p \in F(T) \cap F(I)} \{ \|T(PT)^{n-1}x - p\| - \|I(PI)^{n-1}x - p\| \} \right) \leq 0.
$$

Observe that

$$
\limsup_{n \rightarrow \infty} \left( \sup_{x \in X, p \in F(T) \cap F(I)} \{ \|T(PT)^{n-1}x - p\| - \|I(PI)^{n-1}x - p\| \} \right) \times \limsup_{n \rightarrow \infty} \left( \sup_{x \in X, p \in F(T) \cap F(I)} \{ \|T(PT)^{n-1}x - p\| + \|I(PI)^{n-1}x - p\| \} \right)
$$

$$
= \limsup_{n \rightarrow \infty} \left( \sup_{x \in X, p \in F(T) \cap F(I)} \{ \|T(PT)^{n-1}x - p\|^2 - \|I(PI)^{n-1}x - p\|^2 \} \right) \leq 0.
$$

Therefore we have

$$
\limsup_{n \rightarrow \infty} \left( \sup_{x \in X, p \in F(T) \cap F(I)} \{ \|T(PT)^{n-1}x - p\| - \|I(PI)^{n-1}x - p\| \} \right) \leq 0.
$$

This implies that for any given $\varepsilon > 0$, there exists a positive integer $n_0$ such that for $n \geq n_0$ we have

$$
\left( \sup_{x \in X, p \in F(T) \cap F(I)} \{ \|T(PT)^{n-1}x - p\| - \|I(PI)^{n-1}x - p\| \} \right) \leq 0.
$$
THEOREM 4.1. Let $X$ be a Banach space and $K$ be a nonempty subset of the Banach space. Let $T, I : K \to X$ be two nonself mappings. Let $T$ be a nonself $I$-asymptotically quasi-nonexpansive type and $I$ be a nonself asymptotically quasi-nonexpansive type in Banach space with $F(T) \cap F(I) \neq \emptyset$. Let the sequence $\{x_n\}$ be defined by (4.1) and for every $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}, \{\gamma'_n\}$ are sequences in $(0, 1)$ with $\alpha_n + \beta_n + \gamma_n = 1 = \alpha'_n + \beta'_n + \gamma'_n$, $\sum_{i=1}^{\infty} \gamma_i < \infty$, $\sum_{i=1}^{\infty} \gamma'_i < \infty$, and $\{\psi_n\}, \{\varphi_n\}$ are bounded sequences in $K$.

Then $\{x_n\}$ converges strongly to a common fixed point of $T$ and $I$ in $K$ if

\begin{equation}
\liminf_{n \to \infty} d(x_n, F(T) \cap F(I)) = 0. \tag{4.2}
\end{equation}

PROOF. The necessity of condition (4.2) is obvious. Next we prove the sufficiency of condition (4.2). Let the sequence $\{x_n\}$ be defined by (4.1). Let $p \in F(T) \cap F(I)$, by boundedness of the sequences $\{\psi_n\}, \{\varphi_n\}$, so we can put

$$M = \max\{\sup_{n \geq 1} \|\psi_n - p\|, \sup_{n \geq 1} \|\varphi_n - p\|\}.$$ 

For any given $\varepsilon > 0$, there exists a positive integer $n_0$ such that $n \geq n_0$

$$\sup_{x \in K, p \in F(T) \cap F(I)} \{\|T(PT)^{n-1}x - p\| - \|I(IP)^{n-1}x - p\|\} < \varepsilon.$$

$$\sup_{x \in K, p \in F(I)} \{\|I(IP)^{n-1}x - p\| - \|x - p\|\} < \varepsilon.$$

Therefore, in particular, we have

\begin{equation}
\{\|T(PT)^{n-1}x_n - p\| - \|I(IP)^{n-1}x_n - p\|\} < \varepsilon, \tag{4.3}
\end{equation}

for all $p \in F(T) \cap F(I)$ and $\forall n \geq n_0$.\n
\begin{equation}
\{\|I(IP)^{n-1}y_n - p\| - \|y_n - p\|\} < \varepsilon, \tag{4.4}
\end{equation}

for all $p \in F(I)$ and $\forall n \geq n_0$. Thus for each $n \geq 1$ and for any $p \in F(T) \cap F(I)$, using (4.1), (4.3) and (4.4), we have

\begin{equation}
\|x_{n+1} - p\| = \|P(\alpha'_n x_n + \beta'_n I(IP)^{n-1}y_n + \gamma'_n \varphi_n - p)\| \\
\leq \alpha'_n \|x_n - p\| + \beta'_n \|I(IP)^{n-1}y_n - p\| + \gamma'_n \|\varphi_n - p\| \\
= \alpha'_n \|x_n - p\| + \beta'_n (\|I(IP)^{n-1}y_n - p\| - \|y_n - p\|) \\
+ \beta'_n \|y_n - p\| + \gamma'_n \|\varphi_n - p\| \\
\leq \alpha'_n \|x_n - p\| + \beta'_n (\varepsilon) + \beta'_n \|y_n - p\| + \gamma'_n M \tag{4.5}
\end{equation}
and
\[
\|y_n - p\| = \|P(\alpha_n x_n + \beta_n T(PT)^{n-1} x_n + \gamma_n \psi_n - p)\|
\]
\[
\leq \alpha_n \|x_n - p\| + \beta_n \|T(PT)^{n-1} x_n - p\| + \gamma_n \|\psi_n - p\|
\]
\[
\leq \alpha_n \|x_n - p\| + \beta_n \|T(PT)^{n-1} x_n - p\| + \|I(PT)^{n-1} x_n - p\|
\]
\[
+ \beta_n \|I(PT)^{n-1} x_n - p\| - \|x_n - p\|) + \beta_n \|x_n - p\| + \gamma_n M
\]
\[
\leq \alpha_n \|x_n - p\| + 2\beta_n \|\epsilon\| + \beta_n \|x_n - p\| + \gamma_n M
\]
\[
\leq (1 - \gamma_n) \|x_n - p\| + 2\beta_n \|\epsilon\| + \gamma_n M
\]
\[
\leq \|x_n - p\| + D_n
\]

where \(D_n = 2\beta_n \|\epsilon\| + \gamma_n M\). Then \(\sum_{n=1}^{\infty} D_n < \infty\) since \(\sum_{n=1}^{\infty} \gamma_n < \infty\).

Substituting (4.6) into (4.5), we have
\[
\|x_{n+1} - p\| \leq \alpha_n' \|x_n - p\| + \beta_n' \|\epsilon\| + \beta_n' \|x_n - p\| + D_n) + \gamma_n' M
\]
\[
\leq (\alpha_n' + \beta_n') \|x_n - p\| + \beta_n' \|\epsilon\| + D_n) + \gamma_n' M
\]
\[
\leq (1 - \gamma_n') \|x_n - p\| + G_n
\]
\[
\leq \|x_n - p\| + G_n
\]

where \(G_n = \beta_n' \|\epsilon\| + D_n) + \gamma_n' M\). Then \(\sum_{n=1}^{\infty} G_n < \infty\) since \(\sum_{n=1}^{\infty} \gamma_n' < \infty\) and \(\sum_{n=1}^{\infty} D_n < \infty\).

It follows from (4.7) that \(d(x_{n+1}, F(T) \cap F(I)) \leq d(x_n, F(T) \cap F(I)) + G_n\).

By Lemma 2.1, we can get that \(\lim_{n \to \infty} d(x_n, F(T) \cap F(I))\) exists. By condition \(\lim \inf_{n \to \infty} d(x_n, F(T) \cap F(I)) = 0\), we have
\[
\lim_{n \to \infty} d(x_n, F(T) \cap F(I)) = 0.
\]

Next we prove that \(\{x_n\}\) is a Cauchy sequence in \(X\). In fact, for any \(n \geq n_0\), any \(m \geq n_1\) and any \(p \in F(T) \cap F(I)\) we have
\[
\|x_{n+m} - p\| \leq \|x_{n+m-1} - p\| + G_{n+m-1}
\]
\[
\leq \|x_{n+m-2} - p\| + G_{n+m-1} + G_{n+m-2}
\]
\[
\leq \ldots \leq \|x_n - p\| + \sum_{k=n}^{\infty} G_k.
\]

So by (4.8), we have
\[
\|x_{n+m} - x_n\| \leq \|x_{n+m} - p\| + \|x_n - p\| \leq 2\|x_n - p\| + \sum_{k=n}^{\infty} G_k.
\]

By the arbitrariness of \(p \in F(T) \cap F(I)\) and (4.10), we have
\[
\|x_{n+m} - p\| \leq 2d(x_n, F(T) \cap F(I)) + \sum_{k=n}^{\infty} G_k \quad \forall n \geq n_0.
\]
For any given \( \varepsilon > 0 \), there exists a positive integer \( n_1 \geq n_0 \), such that for any \( n \geq n_1 \),
\[ d(x_n, F(T) \cap F(I)) < \frac{\varepsilon}{2} \text{ and } \sum_{k=n}^{\infty} G_k < \frac{\varepsilon}{2}, \]
we have \( \|x_{n+m} - x_n\| < \varepsilon \), and so for any \( m \geq 1 \)
\[ \lim_{n \to \infty} \|x_{n+m} - x_n\| = 0. \]
This shows that \( \{x_n\} \) is a Cauchy sequence in \( X \). Since \( X \) is complete, there exists a \( p^* \in X \) such that \( x_n \to p^* \) as \( n \to \infty \).

Finally, by the routine method, we have to prove that \( p^* \in F(T) \cap F(I) \). By contradiction, we assume that \( p^* \) is not in \( F(T) \cap F(I) \). Since \( F(T) \cap F(I) \) is a closed set, \( d(p^*, F(T) \cap F(I)) > 0 \). Hence for all \( p \in F(T) \cap F(I) \), we have
\[ \|p^* - p\| \leq \|x_n - p^*\| + \|x_n - p\|. \]
This implies that
\[ d(p^*, F(T) \cap F(I)) \leq \|x_n - p^*\| + d(x_n, F(T) \cap F(I)). \]
Letting \( n \to \infty \) in \( (4.1) \) and noting \( (4.8) \), we have \( d(p^*, F(T) \cap F(I)) \leq 0 \). This is a contradiction. Hence \( p^* \in F(T) \cap F(I) \). This completes the proof of Theorem 4.1. \( \square \)

References


Department of Mathematics
Art and Science Faculty
Harran University
Sanliurfa
Turkey
temirezit@harran.edu.tr