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ON THE SIGNLESS LAPLACIAN SPECTRAL RADIUS OF UNICYCLIC GRAPHS WITH FIXED MATCHING NUMBER

Jing-Ming Zhang, Ting-Zhu Huang, and Ji-Ming Guo

Abstract. We determine the graph with the largest signless Laplacian spectral radius among all unicyclic graphs with fixed matching number.

1. Introduction

Let $G = (V(G), E(G))$ be a simple connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. Its adjacency matrix $A(G) = (a_{ij})$ is defined as an $n \times n$ matrix (a_{ij}), where $a_{ij} = 1$ if v_i is adjacent to v_j; $a_{ij} = 0$, otherwise. Denote by $d(v_i)$ or $d_G(v_i)$ the degree of the vertex v_i $(i = 1, 2, \ldots, n)$. Let $Q(G) = D(G) + A(G)$ be the signless Laplacian matrix of a graph G, where $D(G) = \operatorname{diag}(d(v_1), d(v_2), \ldots, d(v_n))$ denotes the diagonal matrix of the vertex degrees of G. It is well known that $A(G)$ is a real symmetric matrix and $Q(G)$ is a positive semidefinite matrix. Hence, the eigenvalues of $A(G)$ and $Q(G)$ can be ordered as

$$\lambda_1(G) \geqslant \lambda_2(G) \geqslant \cdots \geqslant \lambda_n(G)$$

and

$$q_1(G) \geqslant q_2(G) \geqslant \cdots \geqslant q_n(G) \geqslant 0,$$

respectively. The largest eigenvalues of $A(G)$ and $Q(G)$ are called the spectral radius and the signless Laplacian spectral radius of G, denoted by $\rho(G)$ and $q(G)$, respectively. When G is connected, $A(G)$ and $Q(G)$ are nonegative irreducible matrix. By the Perron–Frobenius theory, $\rho(G)$ is simple and has a unique positive unit eigenvector, so does $q(G)$. We refer to such the eigenvector corresponding to $q(G)$ as the Perron vector of G.

Two distinct edges in a graph G are independent if they are not adjacent in G. A set of pairwise independent edges of G is called a matching in G. A matching of

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maximum cardinality is a maximum matching in G. The cardinality of a maximum matching of G is commonly known as its matching number, denoted by $\mu(G)$.

Denote by C_n and P_n the cycle and the path on n vertices, respectively. The characteristic polynomial of $A(G)$ is $\det(xI - A(G))$, which is denoted by $\Phi(G)$ or $\Phi(G, x)$. The characteristic polynomial of $Q(G)$ is $\det(xI-Q(G))$, which is denoted by $\Psi(G)$ or $\Psi(G, x)$.

A unicyclic graph is a connected graph in which the number of vertices equals the number of edges. Let $U_n(\mu)$ denote the set of all unicyclic graphs on n vertices with matching number μ.

The investigation on the spectral radius of graphs is an important topic in the theory of graph spectra, and some early results can go back to the very beginnings (see [**4**]). The recent developments on this topic also involve the problem concerning graphs with maximal or minimal spectral radius of a given class of graphs. In [**2**], Chang et al. gave the first two spectral radii of unicyclic graphs with perfect matchings. Recently, Yu et al. [**9**] gave the first two spectral radii of unicyclic graphs with a given matching number; and Guo [**13**] gave the first six spectral radii over the class of unicyclic graphs on a given number of vertices; Guo [**12**] gave the first ten spectral radii over the class of unicyclic graphs on a given number of vertices and the first four spectral radii of unicyclic graphs with perfect matchings. For more results on this topic, the reader is referred to [**1, 6, 14, 3**] and the references therein.

In this paper, we deal with the extremal signless Laplacian spectral radius problems for the unicyclic graphs with fixed matching number. The graph with the largest signless Laplacian spectral radius among all unicyclic graphs with a fixed matching number is obtained.

2. Lemmas

Let $G - u$ or $G - uv$ denote the graph obtained from G by deleting the vertex $u \in V(G)$ or the edge $uv \in E(G)$. A pendant vertex of G is a vertex with degree one. A path $P : vv_1v_2 \cdots v_k$ in G is called a pendant path if $d(v_1) = d(v_2) = \cdots = d(v_{k-1}) = 2$ and $d(v_k) = 1$. If $k = 1$, then we say vv_1 is a pendant edge of the graph G.

In order to complete the proof of our main result, we need the following lemmas.

Lemma 2.1. [**7, 15**] *Let G be a connected graph, and u, v be two vertices of G. Suppose that $v_1, v_2, \ldots, v_s \in N(v)\backslash\{N(u) \cup u\}$ $(1 \leqslant s \leqslant d(v))$ and $x = (x_1, x_2, \ldots, x_n)$ is the Perron vector of G, where x_i corresponds to the vertex v_i $(1 \leqslant i \leqslant n)$. Let G^* be the graph obtained from G by deleting the edges vv_i and adding the edges $uv_i (1 \leqslant i \leqslant s)$. If $x_u \geqslant x_v$, then $q(G) < q(G^*)$.*

From Lemma 2.1, we obtain the following results.

Corollary 2.1. *Let $G = (V, E)$ be a connected graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. Suppose that v_1v_2 is an edge of G which does not lie on a circuit of length three satisfying $d(v_1) \geqslant 2$ and $d(v_2) \geqslant 2$. Let $\tilde{G}$ be the graph obtained* *from* $G - v_1v_2$ *by amalgamating* v_1 *and* v_2 *to form a new vertex* w_1 *together with attaching a new pendant vertex* w_2 *to* w_1. *Then* $q(\tilde{G}) > q(G)$.

Corollary 2.2. *Let* w *and* v *be two vertices in a connected graph* G *and suppose that* s *paths* $\{ww_1w_1', ww_2w_2', \dots, ww_sw_s'\}$ *of length* 2 *are attached to* G *at* w *and* t *paths* $\{vv_1v_1', vv_2v_2', \dots, vv_tv_t'\}$ *of length* 2 *are attached to* G *at* v *to form* $G_{s,t}$. *Then either* $q(G_{s+i,t-i}) > q(G_{s,t})$ $(1 \leqslant i \leqslant t)$, *or* $q(G_{s-i,t+i}) > q(G_{s,t})$ $(1 \leqslant i \leqslant s)$ *and* $\mu(G_{0,s+t}) = \mu(G_{s+t,0}) = \mu(G_{s,t})$.

Corollary 2.3. *Let* w *and* v *be two vertices in a connected graph* G *and suppose that* s *paths* $\{ww_1, ww_2, \dots, ww_s\}$ *of length* 1 *are attached to* G *at* w *and* t *paths* $\{vv_1v_1', vv_2v_2', \dots, vv_tv_t'\}$ *of length* 2 *are attached to* G *at* v *to form* $H_{s,t}$. *Then either*

$q(H_{s,t} - ww_1 - \cdots - ww_i + vw_1 + \cdots + vw_i) > q(H_{s,t})$ $(1 \leqslant i \leqslant s)$ *or*

$q(H_{s,t} - vv_1 - \cdots - vv_i + wv_1 + \cdots + wv_i) > q(H_{s,t})$ $(1 \leqslant i \leqslant t)$.

Corollary 2.4. *Let* w *and* v *be two vertices in a connected graph* G *and suppose that* s *paths* $\{ww_1, ww_2, \dots, ww_s\}$ *of length* 1 *are attached to* G *at* w *and* t *paths* $\{vv_1, vv_2, \dots, vv_t\}$ *of length* 1 *are attached to* G *at* v *to form* $F_{s,t}$. *Then either*

$q(F_{s+i,t-i}) > q(F_{s,t})$ $(1 \leqslant i \leqslant t)$, *or* $q(F_{s-i,t+i}) > q(F_{s,t})$ $(1 \leqslant i \leqslant s)$.

Corollary 2.5. *Suppose* u *is a vertex of graph* G *with* $d(u) \geqslant 2$. *Let* $G : uv$ *be a graph obtained by attaching a pendant edge* uv *to* G *at* u. *Suppose* t *paths* $\{vv_1v_1', \dots, vv_tv_t'\}$ *of length* 2 *are attached to* $G : uv$ *at* v *to form* $L_{0,t}$. *Let*

$$M_{1,t} = L_{0,t} - vv_1 - \cdots - vv_t + uv_1 + \cdots + uv_t.$$

Then we have $\mu(M_{1,t}) = \mu(L_{0,t})$ *and* $q(M_{1,t}) > q(L_{0,t})$, $(t \geqslant 1)$.

An internal path of a graph G is a sequence of vertices $v_1, v_2, \dots, v_m$ with $m \geqslant 2$ such that:

(1) The vertices in the sequences are distinct (except possibly $v_1 = v_m$);

(2) v_i is adjacent to $v_{i+1}, (i = 1, 2, \dots, m-1)$;

(3) The vertex degrees $d(v_i)$ satisfy $d(v_1) \geqslant 3$, $d(v_2) = \cdots = d(v_{m-1}) = 2$ (unless $m = 2$) and $d(v_m) \geqslant 3$.

Lemma 2.2. *Suppose that* $P : v_1v_2 \cdots v_k$ $(k \geqslant 3)$ *is an internal path of the graph* G *and* $v_1v_k \notin E(G)$ *for* $k = 3$. *Let* G^* *be the graph obtained from* $G - v_iv_{i+1} - v_{i+1}v_{i+2}$ $(1 \leqslant i \leqslant k-2)$ *by amalgamating* v_i, v_{i+1} *and* v_{i+2} *to form a new vertex* w_1 *together with attaching a new pendant path* $w_1w_2w_3$ *of length* 2 *at* w_1. *Then* $q(G^*) > q(G)$ *and* $\mu(G^*) = \mu(G)$.

Proof. Let $G' = G^* - w_2 - w_3$. By similar reasoning as that of Theorem 3.1 of [**11**] and Theorem 4.11 of [**10**], we have $q(G') > q(G)$. From the well-known Perron–Frobenius theory, we have $q(G^*) > q(G')$. Thus we have $q(G^*) > q(G)$. Next,we prove $\mu(G^*) \geqslant \mu(G)$. Let M be a maximum matching of G. If $v_iv_{i+1} \in M$ or $v_{i+1}v_{i+2} \in M$, then $\{M - \{v_iv_{i+1}\}\} \cup \{w_2w_3\}$ or $\{M - \{v_{i+1}v_{i+2}\}\} \cup \{w_2w_3\}$ is a matching of G^*. Thus, $\mu(G^*) \geqslant \mu(G)$; if $v_iv_{i+1} \notin M$ and $v_{i+1}v_{i+2} \notin M$, then there exist two edges v_iu and $v_{i+2}v \in M$. Thus, $\{M - \{v_iu\}\} \cup \{w_2w_3\}$ is a matching of G^*. Hence, $\mu(G^*) \geqslant \mu(G)$. Let M_0 be a maximum matching of G^*. If there exists some vertex, say $u(\neq w_2)$, of G^* such that $uw_1 \in M_0$, then $w_2w_3 \in M_0$. Thus $\{M_0 - \{uw_1, w_2w_3\}\} \cup \{uv_i, v_{i+1}v_{i+2}\}$ is a matching of G; if there's no such a vertex, then we have either $w_1w_2 \in M_0$ or $w_2w_3 \in M_0$. Thus $\{M_0 - \{w_1w_2, w_2w_3\}\} \cup \{v_{i+1}v_{i+2}\}$ is a matching of G. So, $\mu(G) \geqslant \mu(G^*)$. Hence, $\mu(G) = \mu(G^*)$. □

Let $S(G)$ be the subdivision graph of G obtained by subdividing every edge of G.

LEMMA 2.3. [**5, 18**] *Let G be a graph on n vertices and m edges. Then $\Phi(S(G)) = x^{m-n}\Psi(G, x^2)$, where $\Phi(G)$ and $\Psi(G)$ are the characteristic polynomials of $A(G)$ and $Q(G)$, respectively.*

LEMMA 2.4. [**8**] *Let u be a vertex of a connected graph G with at least two vertices. Let $G_{k,l}$ $(k, l \geqslant 0)$ be the graph obtained from G by attaching two pendant paths of lengths k and l at u, respectively. If $k \geqslant l \geqslant 1$, then $q(G_{k,l}) > q(G_{k+1,l-1})$.*

COROLLARY 2.6. *Suppose that $P : v_1v_2 \cdots v_k$ $(k \geqslant 4)$ is a pendant path of the graph G with $d(v_1) \geqslant 3$. Let $G^* = G - v_{k-2}v_{k-1} + v_1v_{k-1}$. Then $q(G^*) > q(G)$ and $\mu(G^*) = \mu(G)$.*

PROOF. By Lemma 2.4, we have $q(G^*) > q(G)$. By similar reasoning as that of Lemma 2.2, we have $\mu(G^*) = \mu(G)$. □

LEMMA 2.5. [**17**] *Let $e = uv$ be an edge of G, and $C(e)$ be the set of all circuits containing e. Then $\Phi(G)$ satisfies*

$$\Phi(G) = \Phi(G - e) - \Phi(G - u - v) - 2\sum_{Z} \Phi(G - V(Z)),$$

where the summation extends over all $Z \in C(e)$.

From the Perron–Frobenius theory, we immediately have the following

LEMMA 2.6. (1) *Let $\triangle(G)$ be the maximum degree of G. Then $\rho(G) \geqslant \sqrt{\triangle(G)}$.*
(2) *Let G be a connected graph, and let G' be a proper spanning subgraph of G. Then $\rho(G) > \rho(G')$ and $q(G) > q(G')$.*

3. Main results

THEOREM 3.1. *Let $G = (V, E)$ be a connected graph with $n \geqslant 4$ vertices. Suppose that $v_1v_2 \in E(G)$, $v_1v_3 \in E(G)$, $v_1v_4 \in E(G)$, $d(v_3) \geqslant 2$, $d(v_4) \geqslant 2$, $d(v_1) = 3$, and $d(v_2) = 1$. Let $G_{v_1v_3}(G_{v_1v_4})$ be the graph obtained from $G - v_1v_3$ $(G - v_1v_4)$ by amalgamating v_1 and v_3 (v_4) to form a new vertex w_1 (w_3) together with subdivising the edge w_1v_2 (w_3v_2)* with a new vertex *$w_2(w_4)$. If $q = q(G) > 3 + \sqrt{5} \approx 5.23606$, then*
(1) *either $q(G_{v_1v_3}) > q(G)$ or $q(G_{v_1v_4}) > q(G)$;*
(2) *$\mu(G_{v_1v_3}) \geqslant \mu(G)$ and $\mu(G_{v_1v_4}) \geqslant \mu(G)$.*

PROOF. Let $X = (x_1, x_2, \ldots, x_n)^T$ be the Perron vector of G. Then from $(D(G) + A(G))X = q(G)X$, we have

$$qx_i = d_i x_i + \sum_{v_i v_j \in E} x_j. \tag{3.1}$$

We distinguish the following two cases:

Case 1: $x_3 \geqslant x_4$. From (3.1), we have

$$x_1 = (q-1)x_2, \tag{3.2}$$

$$qx_1 = 3x_1 + x_2 + x_3 + x_4. \tag{3.3}$$

Substituting (3.2) into (3.3), together with condition $x_3 \geqslant x_4$, we get

$$\Big(q - 3 - \frac{1}{q-1}\Big)x_1 \leqslant 2x_3. \tag{3.4}$$

From (3.4), we have if $q-3-\frac{1}{q-1} > 2$, namely, $q(G) > 3+\sqrt{5}$, then $x_3 > x_1$. Suppose that the vertices $w_1, w_2, v_2, v_4, \ldots, v_n$ of $G_{v_1v_3}$ are relabelled $v_3, v_1, v_2, v_4, \ldots, v_n$, respectively. Then

$$X^T Q(G_{v_1v_3})X - q(G) = X^T Q(G_{v_1v_3})X - X^T Q(G)X = x_3^2 - x_1^2 + 2x_4(x_3 - x_1) > 0.$$

Thus $q(G_{v_1v_3}) > q(G)$.

Case 2: $x_4 > x_3$. By similar reasoning as that of Case 1, we have $q(G_{v_1v_4}) > q(G)$.

Now, we prove that (2) holds. Let M be a maximum matching of G. If $v_1v_3 \in M$, then $\{M - v_1v_3\} \cup \{v_2w_2\}$ is a matching of $G_{v_1v_3}$. So, $\mu(G_{v_1v_3}) \geqslant \mu(G)$. If $v_1v_3 \notin M$, then $v_1v_2 \in M$ or $v_1v_4 \in M$. So $\{M - v_1v_2 - v_1v_4\} \cup \{v_2w_2\}$ is also a matching of $G_{v_1v_3}$. Thus, $\mu(G_{v_1v_3}) \geqslant \mu(G)$. By similar reasoning, we have $\mu(G_{v_1v_4}) \geqslant \mu(G)$. □

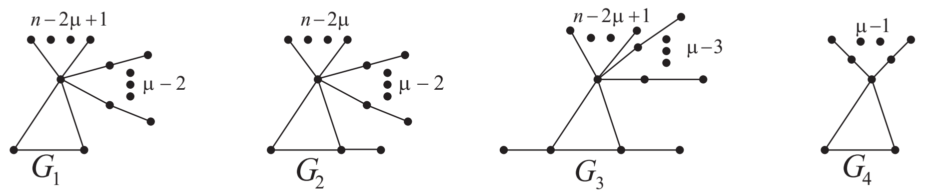

FIGURE 1. $G_1 - G_4$

LEMMA 3.1. *Let G_1, G_2 and G_3 be the graphs as given in Figure 1. Then for $n \geqslant 6$, we have $q(G_1) > q(G_2)$, $q(G_1) > q(G_3)$.*

PROOF. From Lemma 2.1, it is easy to see that $q(G_1) > q(G_2)$. Now we prove that $q(G_1) > q(G_3)$. From Lemma 2.5, we have

$$\begin{aligned}(3.5)\quad \Phi(S(G_3)) - \Phi(S(G_1)) = x^2(x^2-1)^{n-2\mu}(x^4-3x^2+1)^{\mu-4}[(n-\mu-3)x^{10} \\ + (20+6\mu-6n)x^8 + (11n-49-10\mu)x^6 \\ + (54-6n+3\mu)x^4 + (n-25)x^2+4].\end{aligned}$$

If $n \geqslant 10$, it is easy to prove that for $x \geqslant \sqrt{n-\mu+1}$, $x^2-1>0$, $x^4-3x^2+1>0$ and $(n-\mu-3)x^{10} + (20+6\mu-6n)x^8 + (11n-49-10\mu)x^6 + (54-6n+3\mu)x^4 + (n-25)x^2+4>0$. By Lemma 2.6, we know that $\rho(S(G_1)) \geqslant \sqrt{n-\mu+1}$. Thus, $\rho(S(G_1)) > \rho(S(G_3))$ $(n \geqslant 10)$. For $6 \leqslant n \leqslant 9$, by direct calculation, we have $\rho(S(G_1)) > \rho(S(G_3))$. By Lemma 2.3, we have $q(G_1) > q(G_3)$. □

THEOREM 3.2. *Suppose u is a vertex of the unicyclic graph G with $d(u) \geqslant 2$. Let $G : uv$ be a graph obtained by attaching a pendant edge uv to G at u. Suppose that s paths $\{vw_1, \dots, vw_s\}$ of length 1 and t paths $\{vv_1v'_1, \dots, vv_tv'_t\}$ of length 2 are attached to $G : uv$ at v to form $L_{s,t}$. Let $M_{s-1,t+1} = L_{s,t} - vv_1 - \cdots - vv_t - vw_1 - \cdots - vw_{s-1} + uv_1 + \cdots + uv_t + uw_1 + \cdots + uw_{s-1}$. Then we have*
(1) $q(M_{s-1,t+1}) > q(L_{s,t})$, $(s \geqslant 2$ or $t \geqslant 1)$;
(2) $\mu(L_{0,t}) = \mu(M_{-1,t+1})$ and $\mu(L_{s,t}) \leqslant \mu(M_{s-1,t+1})$, $(s \geqslant 1)$.

PROOF. We distinguish the following four cases:
Case 1: $s = 0, t \geqslant 1$. Then we have $M_{s-1,t+1} = M_{-1,t+1}$ and $L_{s,t} = L_{0,t}$.
Since $M_{-1,t+1}$ $(t \geqslant 1)$ can also be obtained from $L_{0,t}$ by identifying u and v with subsequent removal of the loop, and adding a new pendant edge at this new vertex, it is easy to show that $\mu(M_{-1,t+1}) = \mu(L_{0,t})$ and from Corollary 2.1, we have $q(M_{-1,t+1}) > q(L_{0,t})$, $(t \geqslant 1)$.

Case 2: $s \geqslant 2$, $t \geqslant 1$. Suppose a new pendant edge vw is attached to $G : uv$ at v to form $G : uvw$. And then we subdivide every edge of G in $G : uvw$ to obtain the graph $S(G) : uvw$.
Suppose that s paths $\{ww_1w'_1, \dots, ww_sw'_s\}$ of length 2 and t paths

$$\{ww_{11}w_{12}w_{13}w_{14}, \dots, ww_{t1}w_{t2}w_{t3}w_{t4}\}$$

of length 4 are attached to $S(G) : uvw$ at w to form $\bar{L}_{s,t}$. Let

$$\begin{aligned}\bar{M}_{s-1,t+1} = \bar{L}_{s,t} - ww_1 - \cdots - ww_{s-1} - ww_{11} - \cdots - ww_{t1} \\ + uw_1 + \cdots + uw_{s-1} + uw_{11} + \cdots + uw_{t1}.\end{aligned}$$

Obviously, $\bar{M}_{s-1,t+1} \cong S(M_{s-1,t+1})$, $\bar{L}_{s,t} \cong S(L_{s,t})$ and $\bar{M}_{0,1} \cong \bar{L}_{1,0}$.
By Lemma 2.3, we only need to prove that $\rho(\bar{M}_{s-1,t+1}) > \rho(\bar{L}_{s,t})$. Obviously, P_4 is a proper subgraph of $S(L_{1,t})$. From Lemma 2.6, we have $\rho(S(L_{1,t})) > \rho(S(G)-u) > \rho(P_4) \approx 1.61803 > \sqrt{2}$.

From Lemma 2.5, we have

$$\begin{aligned}\Phi(\bar{L}_{s,t}) &= \Phi(\bar{L}_{s,t} - ww_1) - \Phi(\bar{L}_{s,t} - w - w_1)\\ &= (x^2-1)\Phi(\bar{L}_{s-1,t}) - x(x^2-1)^{s-1}(x^4-3x^2+1)^t\Phi(S(G):uv)\\ &\cdots\\ &= (x^2-1)^{s-1}\Phi(\bar{L}_{1,t}) - (s-1)x(x^2-1)^{s-1}(x^4-3x^2+1)^t\Phi(S(G):uv),\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\Phi(\bar{L}_{1,t}) &= \Phi(\bar{L}_{1,t} - ww_{11}) - \Phi(\bar{L}_{1,t} - w - w_{11})\\ &= (x^4-3x^2+1)\Phi(\bar{L}_{1,t-1}) - (x^2-1)(x^3-2x)\\ &\qquad \cdot (x^4-3x^2+1)^{t-1}\Phi(S(G):uv)\\ &\cdots\\ &= (x^4-3x^2+1)^t\Phi(\bar{L}_{1,0}) - t(x^2-1)(x^3-2x)\\ &\qquad \cdot (x^4-3x^2+1)^{t-1}\Phi(S(G):uv).\end{aligned} \tag{3.7}$$

Substituting (3.7) into (3.6), we have

$$\begin{aligned}\Phi(\bar{L}_{s,t}) &= (x^2-1)^{s-1}(x^4-3x^2+1)^t\Phi(\bar{L}_{1,0})\\ &\quad - t(x^3-2x)(x^2-1)^s(x^4-3x^2+1)^{t-1}\Phi(S(G):uv)\\ &\quad - (s-1)x(x^2-1)^{s-1}(x^4-3x^2+1)^t\Phi(S(G):uv),\end{aligned} \tag{3.8}$$

$$\begin{aligned}\Phi(\bar{M}_{s-1,t+1}) &= \Phi(\bar{M}_{s-1,t+1} - uw_1) - \Phi(\bar{M}_{s-1,t+1} - u - w_1)\\ &= (x^2-1)\Phi(\bar{M}_{s-2,t+1}) - x(x^2-1)^{s-2}\\ &\qquad \cdot (x^4-3x^2+1)^{t+1}\Phi(S(G)-u)\\ &\cdots\\ &= (x^2-1)^{s-1}\Phi(\bar{M}_{0,t+1}) - (s-1)x(x^2-1)^{s-2}\\ &\qquad \cdot (x^4-3x^2+1)^{t+1}\Phi(S(G)-u),\end{aligned} \tag{3.9}$$

$$\begin{aligned}\Phi(\bar{M}_{0,t+1}) &= \Phi(\bar{M}_{0,t+1} - uw_{11}) - \Phi(\bar{M}_{0,t+1} - u - w_{11})\\ &= (x^4-3x^2+1)\Phi(\bar{M}_{0,t}) - (x^3-2x)(x^4-3x^2+1)^t\\ &\qquad \cdot \Phi(S(G)-u)\\ &\cdots\\ &= (x^4-3x^2+1)^t\Phi(\bar{M}_{0,1}) - t(x^3-2x)(x^4-3x^2+1)^t\\ &\qquad \cdot \Phi(S(G)-u).\end{aligned} \tag{3.10}$$

Substituting (3.10) into (3.9), we have

$$\begin{aligned}\Phi(\bar{M}_{s-1,t+1}) &= (x^2-1)^{s-1}(x^4-3x^2+1)^t\Phi(\bar{M}_{0,1})\\ &\quad - t(x^3-2x)(x^2-1)^{s-1}(x^4-3x^2+1)^t\Phi(S(G)-u)\\ &\quad - (s-1)x(x^2-1)^{s-2}(x^4-3x^2+1)^{t+1}\Phi(S(G)-u).\end{aligned} \tag{3.11}$$

From (3.8) and (3.11), we have

$$\begin{aligned}(3.12)\quad & \Phi(\bar{L}_{s,t}) - \Phi(\bar{M}_{s-1,t+1}) \\ & = [t(x^3-2x)(x^2-1)^{s-1}(x^4-3x^2+1)^{t-1} + (s-1)x(x^2-1)^{s-2} \\ & \quad \cdot (x^4-3x^2+1)^t][(x^4-3x^2+1)\Phi(S(G)-u) - (x^2-1)\Phi(S(G):uv)].\end{aligned}$$

From Lemma 2.5, by simple calculation, we have

$$\begin{aligned}(3.13)\quad & (x^4-3x^2+1)\Phi(S(G)-u) - (x^2-1)\Phi(S(G):uv) \\ & = -x^2\Phi(S(G)-u) + x(x^2-1)\sum_y \Phi(S(G)-u-y) \\ & \quad + 2x(x^2-1)\sum_Z \Phi(S(G)-Z).\end{aligned}$$

Since $d_G(u) \geqslant 2$ and $S(G)$ is the subdivision graph of G, we have $d_{S(G)}(u) \geqslant 2$. Without loss of generality, we can suppose that y_1 and y_2 are two vertices of $S(G)$ such that $y_i u \in E(S(G))$ $(i=1,2)$. For $x > \rho(S(G)-u)$, we have from (3.13) that

$$\begin{aligned}(3.14)\quad & (x^4-3x^2+1)\Phi(S(G)-u) - (x^2-1)\Phi(S(G):uv) \\ & \geqslant -x^2\Phi(S(G)-u) + x(x^2-1)[\Phi(S(G)-u-y_1) + \Phi(S(G)-u-y_2)]\end{aligned}$$

and from [**16**], we have

$$(3.15)\qquad x\Phi(S(G)-u-y_i) > \Phi(S(G)-u) > 0, \quad (i=1,2).$$

Since $x > \rho(S(G)-u) > \sqrt{2}$, we have $2(x^2-1) > x^2$. Hence, we have from (3.15) that for $x > \rho(S(G)-u)$,

$$-x^2\Phi(S(G)-u) + x(x^2-1)[\Phi(S(G)-u-y_1) + \Phi(S(G)-u-y_2)] > 0.$$

So, from (3.14), we have for $x > \rho(S(G)-u)$,

$$(x^4-3x^2+1)\Phi(S(G)-u) - (x^2-1)\Phi(S(G):uv) > 0.$$

Combined with (3.12), we have for $x > \rho(S(G)-u)$, $\Phi(\bar{L}_{s,t}) - \Phi(\bar{M}_{s-1,t+1}) > 0$. Since $\rho(\bar{L}_{s,t}) > \rho(S(G)-u)$, we have $\rho(\bar{M}_{s-1,t+1}) > \rho(\bar{L}_{1,t})$. Hence, $q(M_{s-1,t+1}) > q(L_{s,t})$ by Lemma 2.3.

Case 3: $s=1, t \geqslant 1$. Then we have $\bar{M}_{s-1,t+1} = \bar{M}_{0,t+1}$, $\bar{L}_{s,t} = \bar{L}_{1,t}$. From (3.7) and (3.10), we have

$$\begin{aligned}\Phi(\bar{L}_{1,t}) - \Phi(\bar{M}_{0,t+1}) &= t(x^3-2x)(x^4-3x^2+1)^{t-1} \\ & \quad [(x^4-3x^2+1)\Phi(S(G)-u) - (x^2-1)\Phi(S(G):uv)].\end{aligned}$$

By the similar reasoning as that of Case 2, we have $q(M_{0,t+1}) > q(L_{1,t})$.

Case 4: $s \geqslant 2, t=0$. Then we have $\bar{M}_{s-1,t+1} = \bar{M}_{s-1,1}$ and $\bar{L}_{s,t} = \bar{L}_{s,0}$. From (3.6) and (3.9), we have

$$\begin{aligned}\Phi(\bar{L}_{s,0}) - \Phi(\bar{M}_{s-1,1}) &= (s-1)x(x^2-1)^{s-2} \\ & \quad \cdot [(x^4-3x^2+1)\Phi(S(G)-u) - (x^2-1)\Phi(S(G):uv)].\end{aligned}$$

By the similar reasoning as that of Case 2, we have $q(M_{s-1,1}) > q(L_{s,0})$.

In the end, we prove that $\mu(L_{s,t}) \leqslant \mu(M_{s-1,t+1})$ for $s \geqslant 1$. Let M be a maximum matching of $L_{s,t}$. If $uv \in M$, then $\{M - \{uv\}\} \cup \{vw_s\}$ is a matching of $M_{s-1,t+1}$. So, $\mu(L_{s,t}) \leqslant \mu(M_{s-1,t+1})$. If $uv \notin M$, then there exists some edge, say vw_s, of $L_{s,t}$ such that $vw_s \in M$. Then M is also a matching of $M_{s-1,t+1}$. Thus, $\mu(L_{s,t}) \leqslant \mu(M_{s-1,t+1})$. □

The length of the shortest path between vertices u and v is defined as the distance of u and v, denoted by $d(u, v)$. Let G_1, G_2, G_3 and G_4 be the graphs as Figure 1. Let B, $\overline{G}_1$, $\overline{G}_2$ and $\overline{G}_3$ be the graphs as given in Figure 2.

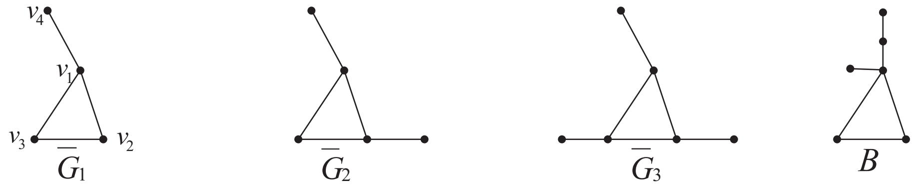

FIGURE 2. $\overline{G}_1 - \overline{G}_3,\ B$

Denote by $U_n^g(\mu)$ the set of all unicyclic graphs on n vertices with matching number μ and girth g ($\geqslant 3$).

THEOREM 3.3. *If $G \in U_n(\mu)$, ($n \geqslant 6$), then $q(G) \leqslant q(G_1)$, with equality if and only if $G = G_1$.*

PROOF. Let $X = (x_1, x_2, \dots, x_n)^T$ be the Perron vector of G. From Lemma 2.6 and by direct calculations, we have for $\mu \geqslant 3$, $q(G_1) \geqslant q(B) \approx 5.38 > 3 + \sqrt{5} > 4 \geqslant q(C_n)$. So, in the following, we can suppose that $q(G) > 3 + \sqrt{5}$ and $G \neq C_n$.

Choose $G^* \in U_n(\mu)$ such that $q(G^*)$ is as large as possible. Then G^* consists of a cycle C_g ($g \geqslant 3$) as a subgraph. Let T be a tree attached at some vertex, say z, of C_g, $|V(T)|$ is the number of vertices of T including the vertex z. In the following, we prove that T is formed by attaching paths of length at most 2 at z.

Suppose that $P : v_0v_1 \dots v_k$ is a pendant path of G^* and v_k is a pendant vertex. If $k \geqslant 3$, let $H_1 = G^* - v_{k-2}v_{k-1} + v_0v_{k-1}$. From Corollary 2.6, we have $H_1 \in U_n^g(\mu)$ and $q(H_1) > q(G^*)$, a contradiction. Hence, the pendant paths of G^* have length at most 2.

For each vertex $u \in V(T - z)$, we prove that $d(u) \leqslant 2$. Otherwise, there must exist some vertex u_0 of $T - z$ such that $d(z, u_0) = max\{d(z, v) | v \in V(T), d(v) \geqslant 3\}$. By similar reasoning as that of above, the pendant paths attached at u_0 have length at most 2 by Corollary 2.6. Furthermore, there exists an internal path between u_0 and some vertex w of T, denoted by $\bar{P} : u_0w_1 \dots w_m$ ($w_m = w$). If $m \geqslant 2$, let H_2 be the graph obtained from $G^* - u_0w_1 - w_1w_2$ by amalgamating u_0, w_1 and w_2 to form a new vertex s_1 together with attaching a new pendant path $s_1s_2s_3$ of length 2 at s_1. From Lemma 2.2, we have $H_2 \in U_n^g(\mu)$ and $q(H_2) > q(G^*)$, a contradiction. If $m = 1$, by Theorem 3.2, Lemma 2.4 and Corollary 2.5, we can get a new graph $H_3 \in U_n^g(\mu)$ and $q(H_3) > q(G^*)$, a contradiction.

Thus, we have that the tree T is obtained by attaching some pendant paths of length at most 2 at z.

From Corollary 2.2, we have that all the pendant paths of length 2 in G^* must be attached at the same vertex of C_g.

Now we prove that for cycle C_g of G^*, $g = 3$. Assume that the cycle C_g of G^* with length at least 4. From Lemma 2.2, we have that each internal path of G^* has length 1. From Corollary 2.4, there exists at most one vertex of C_g such that it is attached more than one path of length 1. Then there must exist edges $v_1v_2 \in E(G^*)$, $v_1v_3 \in E(C_g)$, $v_1v_4 \in E(C_g)$ and $d(v_1) = 3$, $d(v_2) = 1$, $d(v_3) \geqslant 3$ and $d(v_4) \geqslant 3$. Let H_4 (H_5) be the graph obtained from $G^* - v_1v_3$ ($G^* - v_1v_4$) by amalgamating v_1 and v_3 (v_4) to form a new vertex y_1 (y_3) together with subdivising the edge y_1v_2 (y_3v_2) with a new vertex y_2 (y_4). From Theorem 3.1 and Lemma 2.4, we have either $q(H_4 - y_2v_2 + y_1v_2) > q(H_4) > q(G^*)$ or $q(H_5 - y_4v_2 + y_3v_2) > q(H_5) > q(G^*)$ and $\mu(H_4 - y_2v_2 + y_1v_2) \leqslant \mu(G^*) \leqslant \mu(H_4)$, $\mu(H_5 - y_4v_2 + y_3v_2) \leqslant \mu(G^*) \leqslant \mu(H_5)$, a contradiction. Hence, $g = 3$.

Thus, by Corollaries 2.3 and 2.4, we have G^* is the graph obtained by attaching the pendant paths of length at most 2 at the same vertex of C_3 of $\overline{G}$, where $\overline{G}$ is one of the graphs C_3, $\overline{G}_1$, $\overline{G}_2$, and $\overline{G}_3$ (see Figure 2). Then G^* is isomorphic to one of graphs G_1, G_2, G_3 and G_4. From Lemma 3.1, we know $q(G_1) > q(G_2)$ and $q(G_1) > q(G_3)$. If $G^* = G_4$ (in this case, $n = 2\mu + 1$), by Lemma 2.4, we have $q(G_1) > q(G_4)$. Thus, $G^* = G_1$, and the result follows. □

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School of Mathematical Sciences
University of Electronic Science and Technology of China
Sichuan, Chengdu, 611731
China
Zhangjm7519@126.com
tingzhuhuang@126.com

College of Science
China University of Petroleum
Shandong, QingDao, 266580
China
Zhangjm7519@126.com

College of Science
East China University of Science and Technology
Shanghai, 200237
China
(Corresponding author)
jimingguo@hotmail.com