PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 97(111) (2015), 1–10 DOI: 10.2298/PIM150107001M

WAVE FRONTS VIA FOURIER SERIES COEFFICIENTS

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Abstract. Motivated by the product of periodic distributions, we give a new description of the wave front and the Sobolev-type wave front of a distribution $f \in \mathscr{D}'(\mathbb{R}^d)$ in terms of Fourier series coefficients.

1. Introduction

In this article microlocal properties of a distribution f at $x_0 \in \mathbb{R}^d$ are detected through the Fourier series expansion of periodizations of φf, where φ is a cut-off function near x_0. In contrast to [**7**], where weighted type wave front sets have been discussed by the use of Gabor and dual Gabor frames depending on an additional continuous parameter $\varepsilon \to 0$, we shall show that the classical Fourier basis can be used for microlocal analysis. Our approach leads to discretized definitions of wave fronts in terms of Fourier coefficients. This is the main novelty of the paper, which also includes proofs for the equivalence between these discretized definitions and Hörmander's approach.

The space of periodic distributions is one of the basic Schwartz spaces and has been studied in many books and papers in the second half of the last century. We refer here just to few of them [**14, 17, 15, 1, 2, 8**] (see [**16**] for applications in summability of Fourier series). In the context of our paper, we mention the recent articles [**11, 12**] and book [**13**], where Ruzhansky and Turunen have studied generalized functions on a torus $\mathbb{T}^d$. Their interest there lies in pseudo-differential operators and microlocal analysis over $\mathbb{T}^d \times \mathbb{Z}^d$. On the other hand, at present time, Hörmander's notion of the wave front set attracts a lot of interest among mathematicians and there exists a vast literature related to this basic notion and its important role in the qualitative analysis of PDEs and ΨDOs. We mention the basic books of Hörmander [**5, 6**] as standard references for classical and Sobolev type wave front sets; the articles [**9, 7**] deal with weighted type wave fronts, while [**3, 4**] study extended wave fronts by considering local and global versions with

2010 *Mathematics Subject Classification:* Primary 35A18; Secondary 46F10.
Key words and phrases: wave fronts; Fourier series; multiplication of distributions.
Communicated by Gradimir Milovanović.

respect to various Banach and Fréchet spaces of functions over the configuration and the frequency domains.

It is well known that the product of two distributions can be defined if their wave fronts are in a "good" position with respect to each other. This motivates us to study the product and wave fronts via spaces of periodic distributions. In Subsection 1.3 below we discuss an elementary approach to local multiplication based on Fourier series. The main results of this article are presented in Sections 2 and 3. In Theorems 2.1 and 3.1 we characterize the wave front and the Sobolev type wave front of a distribution $f \in \mathscr{D}'(\mathbb{R}^d)$ by estimates of the Fourier coefficients of its localizations. It should be mentioned that the toroidal wave fronts have been studied in [**12, 13**] through Fourier series as in this paper, but our approach is quite different and is related to the Hörmander's wave fronts in a precise fashion. Let us note that Sobolev type wave fronts were not considered in [**12, 13**].

1.1. Notation. For $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, we write $|x| = \sqrt{x_1^2 + \cdots + x_d^2}$ and $\langle x \rangle = (1 + |x|^2)^{1/2}$. Let $0 < \eta \leqslant 1$. We will use the notation

$$I_{\eta,x} = \prod_{j=1}^{d} (x_j - \eta/2, x_j + \eta/2) \text{ and } I_\eta := I_{\eta,0}.$$

Throughout the article, the word *periodic* always refers to functions or distributions on $\mathbb{R}^d$ which are periodic of period 1 in each variable, i.e., $f(x+n) = f(x)$, $x \in \mathbb{R}^d$, $n \in \mathbb{Z}^d$. We also use the notation e_y for $e_y(x) = e^{2\pi i y \cdot x}$, $y \in \mathbb{R}^d$. We will consider periodic extensions of localizations of distributions around a point $x_0 \in \mathbb{R}^d$, so if a distribution g is supported by I_{η,x_0}, with $0 < \eta < 1$, we shall write $g_p(x) := \sum_{n\in\mathbb{Z}^d} g(x+n)$ for its periodic extension.

1.2. Basic spaces. The space of periodic test functions $\mathscr{P} = \mathscr{P}(\mathbb{R}^d)$ consists of smooth periodic functions; its topology is given via the sequence of norms $\|\varphi\|_k = \sup_{x\in I_1, |\alpha|\leqslant k} |\varphi^{(\alpha)}(x)|$, $k \in \mathbb{N}$. Obviously, $\varphi \in \mathscr{P}$ if and only if $\sum_{n\in\mathbb{Z}^d} |\varphi_n|^2 \langle n \rangle^{2k} < \infty$ for every $k \in \mathbb{Z}$, where $\varphi_n = \int_{I_1} \varphi(x) e^{-2\pi i n\cdot x} dx = \langle \varphi, e_{-n} \rangle, n \in \mathbb{Z}^d$. The dual space of $\mathscr{P}$, the space of periodic distributions, is denoted by $\mathscr{P}'$. One has: $f = \sum_{n\in\mathbb{Z}^d} f_n e_n \in \mathscr{P}'$ if and only if $\sum_{n\in\mathbb{Z}^d} |f_n|^2 \langle n \rangle^{-2k_0} < \infty$, for some $k_0 > 0$. If $f = \sum_{n\in\mathbb{Z}^d} f_n e_n \in \mathscr{P}'$ and $\varphi = \sum_{n\in\mathbb{Z}^d} \varphi_n e_n \in \mathscr{P}$, then their dual pairing is given by $\langle f, \varphi \rangle = \sum_{n\in\mathbb{Z}^d} f_n \varphi_{-n}$.

Let ν, ω be positive functions over $\mathbb{Z}^d$. We call ω a ν-moderate weight if there is C such that

$$\omega(m+n) \leqslant C\omega(m)\,\nu(n), \quad \forall m, n \in \mathbb{Z}^d. \tag{1.1}$$

If we take ν to be a polynomial, we call ω polynomially moderate. The set of all polynomially moderate weights on $\mathbb{Z}^d$ will be denoted as $\mathcal{P}ol(\mathbb{Z}^d)$. For $\omega \in \mathcal{P}ol(\mathbb{Z}^d)$, we define

$$\mathscr{P}l^q_\omega = \left\{ f \in \mathscr{P}' : \{f_n \omega(n)\}_{n\in\mathbb{Z}^d} \in l^q, \text{ where } f_n = \langle f, e_{-n} \rangle \right\}$$

supplied with the norm $\|f\|_{\mathscr{P}l^q_\omega} = \|\{f_n\omega(n)\}\|_{l^q}$. We consider from now only values of $q \geqslant 1$. Clearly, $\mathscr{P}l^{q_1}_{\omega_1} \subseteq \mathscr{P}l^{q_2}_{\omega_2}$, if $q_1 \leqslant q_2$ and $\omega_2 \leqslant C\omega_1$.

We will also consider the local space $\mathscr{P}l^q_{\omega,\mathrm{loc}}$ consisting of distributions $f \in \mathscr{D}'(\mathbb{R}^d)$ such that the periodic extensions $(\varphi f)_p \in \mathscr{P}l^q_\omega$, for all $x_0 \in \mathbb{R}^d$ and $\varphi \in \mathscr{D}(I_{1,x_0})$. Its topology is defined via the family of seminorms $\|f\|_{x_0,\varphi} = \|(\varphi f)_p\|_{\mathscr{P}l^q_\omega}$, where $x_0 \in \mathbb{R}^d$ and $\varphi \in \mathscr{D}(I_{1,x_0})$. For the sake of completeness, we give the following elementary proposition (its proof also shows that the definition of $\mathscr{P}l^q_{\omega,\mathrm{loc}}$ is consistent).

Proposition 1.1. $\mathscr{P}l^q_\omega \subset \mathscr{P}l^q_{\omega,\mathrm{loc}}$.

Proof. Let $f \in \mathscr{P}l^q_\omega$ and $\varphi \in \mathscr{D}(I_{1,x_0})$, then $(\varphi f)_p = \varphi_p f$. Write $f = \sum_n f_n e_n$ and $\varphi_p = \sum_n \varphi_n e_n \in \mathscr{P}$. By (1.1) and the generalized Minkowski inequality, we have

$$\begin{aligned}\|\varphi_p f\|_{\mathscr{P}l^q_\omega} &\leqslant C\left(\sum_n \left(\sum_j |\varphi_j|\nu(j)|f_{n-j}|\omega(n-j)\right)^q\right)^{1/q}\\ &\leqslant C\|\varphi_p\|_{\mathscr{P}l^1_\nu}\|f\|_{\mathscr{P}l^q_\omega} < \infty.\end{aligned}$$

□

Set $\omega_s(n) = \langle n\rangle^s$, $s \in \mathbb{R}$. For convenience, we write $\mathscr{P}l^q_s := \mathscr{P}l^q_{\omega_s}$ and $\mathscr{P}l^q_{s,\mathrm{loc}} := \mathscr{P}l^q_{\omega_s,\mathrm{loc}}$. We clearly have

$$\mathscr{P} = \bigcap_{s\geqslant 0} \mathscr{P}l^q_s = \bigcap_{\omega\in\mathscr{P}ol(\mathbb{Z}^d)} \mathscr{P}l^q_\omega \text{ and } \mathscr{P}' = \bigcup_{s\leqslant 0} \mathscr{P}l^q_s = \bigcup_{\omega\in\mathscr{P}ol(\mathbb{Z}^d)} \mathscr{P}l^q_\omega.$$

Moreover,

$$\mathscr{E} = \bigcap_{s\geqslant 0} \mathscr{P}l^q_{s,\mathrm{loc}} = \bigcap_{\omega\in\mathscr{P}ol(\mathbb{Z}^d)} \mathscr{P}l^q_{\omega,\mathrm{loc}} \text{ and } \mathscr{D}'_F = \bigcup_{s\leqslant 0} \mathscr{P}l^q_{s,\mathrm{loc}} = \bigcup_{\omega\in\mathscr{P}ol(\mathbb{Z}^d)} \mathscr{P}l^q_{\omega,\mathrm{loc}},$$

where $\mathscr{E}$ is the space of all smooth functions and $\mathscr{D}'_F$ is the space of distributions of finite order on $\mathbb{R}^d$.

1.3. Multiplication. In this section we make some comments about the multiplication of distributions. Assume that the indices $q, q_1, q_2 \in [1,\infty]$ are such that $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q} + 1$. We fix two weight functions $\omega, \nu \in \mathcal{P}ol(\mathbb{Z}^d)$ and we assume ω is ν-moderate (cf. 1.1).

We start with products in the spaces of type $\mathscr{P}l^q_\omega$. Here we define the product via Fourier coefficients. Indeed, let $f_1 = \sum_{n\in\mathbb{Z}^d} f_{1,n}e_n \in \mathscr{P}l^{q_1}_\omega$ and $f_2 = \sum_{n\in\mathbb{Z}^d} f_{2,n}e_n \in \mathscr{P}l^{q_2}_\nu$. We define their product as $f := f_1 f_2 := \sum_{n\in\mathbb{Z}^d} f_n e_n$, where

$$f_n = \sum_{j\in\mathbb{Z}^d} f_{1,n-j}f_{2,j}, \quad n \in \mathbb{Z}^d.$$

(One can easily see that the formula for the Fourier coefficients f_n is just the convolution formula on the integer lattice.) We will check in Proposition 1.2 that $f \in \mathscr{P}l^q_\omega$.

The previous definition allows us to introduce multiplication in the local versions of these spaces. In fact, let now $f_1 \in \mathscr{P}l^{q_1}_{\omega,\mathrm{loc}}$ and $f_2 \in \mathscr{P}l^{q_2}_{\nu,\mathrm{loc}}$. To define their product $f = f_1 f_2$, we proceed locally. Let $x_0 \in \mathbb{R}^d$ and $0 < \eta < 1$. Let $\phi \in \mathscr{D}(I_{1,x_0})$ be such that $\phi(x) = 1$ for $x \in I_{\eta,x_0}$. We define $f_{I_{\eta,x_0}} \in \mathscr{D}'(I_{\eta,x_0})$

as the restriction of $(\phi f_1)_p\,(\phi f_2)_p$ to I_{η,x_0}. Note that different choices of ϕ lead to different Fourier coefficients but, by Proposition 1.1, we have $f_{I_{\eta,x_0}} = f_{I_{\eta',x_0'}}$ on $I_{\eta,x_0}\cap I_{\eta',x_0'}$. The $\{f_{I_{\eta,x_0}}\}$ thus gives rise to a distribution $f\in \mathscr{P}l^{q}_{\omega,\,\mathrm{loc}}$ and we define the product of $f_1 f_2 := f$.

Proposition 1.2. *The mappings*

$$\mathscr{P}l^{q_1}_{\omega}\times\mathscr{P}l^{q_2}_{\nu}\ni(f_1,f_2)\mapsto f_1f_2\in\mathscr{P}l^{q}_{\omega}, \tag{1.2}$$

$$\mathscr{P}l^{q_1}_{\omega,\,\mathrm{loc}}\times\mathscr{P}l^{q_2}_{\nu,\,\mathrm{loc}}\ni(f_1,f_2)\mapsto f_1f_2\in\mathscr{P}l^{q}_{\omega,\,\mathrm{loc}} \tag{1.3}$$

are continuous.

Proof. The continuity of (1.3) follows at once from that of (1.2). For (1.2), Young's inequality and (1.1) yield

$$\|f_1f_2\|_{\mathscr{P}l^{q}_{\omega}}\leqslant C\|f_1\|_{\mathscr{P}l^{q_1}_{\omega}}\|f_2\|_{\mathscr{P}l^{q_2}_{\nu}}\,. \qquad \square$$

In particular, we have

Corollary 1.1. *Let $s,s_1,s_2\in\mathbb{R}$ be such that $s_1+s_2\geqslant 0$, $s\leqslant\min\{s_1,s_2\}$. Then the two mappings*

$$\mathscr{P}l^{q_1}_{s_1}\times\mathscr{P}l^{q_2}_{s_2}\ni(f_1,f_2)\mapsto f_1f_2\in\mathscr{P}l^{q}_{s},$$
$$\mathscr{P}l^{q_1}_{s_1,\,\mathrm{loc}}\times\mathscr{P}l^{q_2}_{s_2,\,\mathrm{loc}}\ni(f_1,f_2)\mapsto f_1f_2\in\mathscr{P}l^{q}_{s,\,\mathrm{loc}}$$

are continuous.

Proof. We may assume $s_1\geqslant 0$ and $s=s_2$. It is obvious that $s_1\geqslant|s_2|$ has to hold in order to have $s_1+s_2\geqslant 0$. The result then follows from Proposition1.2 upon setting $\omega(n)=\langle n\rangle^{s_2}$ and $\nu(n)=\langle n\rangle^{s_1}$ because (1.1) holds for them. $\square$

Concerning the local products from Corollary 1.1, exactly the same method from the proof of Theorem 3.1 below applies to show that the local space $\mathscr{P}l^{2}_{s,\,\mathrm{loc}}$ coincides with the local Sobolev space $H^s_{\mathrm{loc}}(\mathbb{R}^d)$. Therefore, the multiplicative product for the local spaces in Corollary 1.1 agrees with the one defined by Hörmander in [**6**, Sect. 8.2]. Moreover, it is also worth mentioning that our results from the next sections imply that one can go beyond local products and in fact define the multiplicative product by *microlocalization* as in [**6**, Sect. 8.3]. We leave the formulation of such definitions to the reader. Theorem 3.1 below shows that the microlocal version of our multiplication also agrees with Hörmander's one.

2. Wave front

Our goal in this section is to describe the wave front of $f\in\mathscr{D}'(\mathbb{R}^d)$ via the Fourier series coefficients of the periodic extension of an appropriate localization of f around $x_0\in\mathbb{R}^d$, as explained in the previous section. Recall $(x_0,\xi_0)\notin WF(f)$ if there exist $\psi\in\mathscr{D}(\mathbb{R}^d)$ with $\psi\equiv 1$ in a neighborhood of x_0 and an open cone $\Gamma\subset\mathbb{R}^d$ containing ξ_0 such that

$$(\forall N>0)(\exists C_N>0)(\forall\xi\in\Gamma)\big(\big|\widehat{f\psi}(\xi)\big|\leqslant C_N\langle\xi\rangle^{-N}\big). \tag{2.1}$$

The following theorem tells that we can discretize (2.1).

THEOREM 2.1. *Let* $f \in \mathscr{D}'(\mathbb{R}^d)$ *and* $(x_0, \xi_0) \in \mathbb{R}^n \times (\mathbb{R}^d \setminus \{0\})$. *The following conditions are equivalent:*

(i) *There exist* $\phi \in \mathscr{D}(I_{\varepsilon,x_0})$, *with* $\varepsilon \in (0,1)$ *and* $\phi \equiv 1$ *in a neighborhood of* x_0, *and an open cone* Γ *containing* ξ_0 *such that*

$$(\forall N \in \mathbb{N})(\exists C_N > 0)(\forall n \in \Gamma \cap \mathbb{Z}^d)(|\widehat{f\phi}(n)| \leqslant C_N \langle n\rangle^{-N}).$$

(ii) $(x_0, \xi_0) \notin WF(f)$.

PROOF. It is well known that, by shrinking the conic neighborhood of ξ_0, one may choose ψ in (2.1) with arbitrarily small support around x_0. Thus, (ii) implies (i). So, it is enough to show that (i) implies (ii). Assume (i). We divide the proof in two steps. We first prove that there are ε' and an open cone $\xi_0 \in \Gamma_1$ such that

$$(\forall B \text{ bounded set in } \mathscr{D}(I_{\varepsilon',x_0}))(\forall N > 0)(\exists C'_N > 0) \quad (\forall n \in \Gamma_1 \cap \mathbb{Z}^d)\Big(\sup_{\varphi \in B} \big|\widehat{f\varphi}(n)\big| \leqslant \frac{C'_N}{\langle n\rangle^N}\Big). \tag{2.2}$$

We choose ε' in such a way that $\phi(x) = 1$ for every $x \in I_{\varepsilon',x_0}$. For the cone, we select Γ_1 an open cone with $\xi_0 \in \Gamma_1$ and $\overline{\Gamma}_1 \subset \Gamma \cup \{0\}$. Let us show that (2.2) holds with these choices. Let $0 < c < 1$ be a constant smaller than the distance between $\partial\Gamma$ and the intersection of $\overline{\Gamma}_1$ with the unit sphere. Clearly, $\{y \in \mathbb{R}^d : (\exists \xi \in \Gamma_1)(|\xi - y| \leqslant c|\xi|)\} \subset \Gamma$. Let $B \subset \mathscr{D}(I_{\varepsilon',x_0})$ be a bounded set. We have that $\phi\varphi = \varphi$, $\forall \varphi \in B$. Moreover, note that $\widehat{f\varphi}(n)$ are precisely the Fourier coefficients of the periodic distribution $(f\phi)_p(\varphi)_p$. Therefore, for $\varphi \in B$ and $n \in \Gamma_1 \cap \mathbb{Z}^d$,

$$\big|\widehat{f\varphi}(n)\big| = \Big|\sum_{j\in\mathbb{Z}^d} \widehat{f\phi}(n-j)\widehat{\varphi}(j)\Big| \leqslant \Big(\sum_{|j|\leqslant c|n|} + \sum_{|j|>c|n|}\Big)\big|\widehat{f\phi}(n-j)\widehat{\varphi}(j)\big| =: I_1(n) + I_2(n)$$

Further on,

$$I_1(n) = \sum_{|n-j|\leqslant c|n|} \big|\widehat{f\phi}(j)\big|\big|\widehat{\varphi}(n-j)\big| \leqslant C \sup_{|n-j|\leqslant c|n|} \big|\widehat{f\phi}(j)\big|,$$

where C only depends on B. Since $|n-j| \leqslant c|n|$ implies $|j| \geqslant (1-c)|n|$,

$$\begin{aligned} \sup_{\varphi\in B,\, n\in\Gamma_1} \langle n\rangle^N I_1(n) &\leqslant C \sup_{n\in\Gamma_1} \langle n\rangle^N \sup_{|n-j|\leqslant c|n|} \big|\widehat{f\phi}(j)\big| \\ &\leqslant C \sup_{j\in\Gamma_{\xi_0}} (1-c)^{-N}\langle j\rangle^N \big|\widehat{f\phi}(j)\big| = C(1-c)^{-N} C_N. \end{aligned} \tag{2.3}$$

For the estimate of I_2 we use that $|n-j| \leqslant (1+c^{-1})|j|$ if $|j| \geqslant c|n|$. Moreover, by the Paley–Wiener theorem, there are $M, D > 0$ such that

$$\big|\widehat{f\phi}(n-j)\big| \leqslant D\langle n-j\rangle^M, \quad n, j \in \mathbb{Z}^d.$$

Due to the boundedness of $B \subset \mathscr{D}(\mathbb{R}^d)$,

$$\sup_{\varphi\in B}\sum_{j\in\mathbb{Z}^d}\langle j\rangle^{M+N}|\widehat{\varphi}(j)| =: K_N < \infty.$$

Thus, for the second term, we have

$$\begin{aligned}\sup_{n\in\Gamma_1}\langle n\rangle^N I_2(n) &\leqslant D\sup_{n\in\Gamma_1}\langle n\rangle^N \sum_{|j|\geqslant c|n|}\langle n-j\rangle^M\big|\widehat{\varphi}(j)\big| \\ &\leqslant Dc^{-N}(1+c^{-1})^M K_N, \quad \forall\varphi\in B.\end{aligned} \tag{2.4}$$

Combining (2.3) and (2.4), we get (2.2).

We now deduce that $(x_0,\xi_0)\notin WF(f)$ with the aid of (2.2). Let $\psi\in\mathscr{D}(I_{\varepsilon',x_0})$ be equal to 1 in a neighborhood of x_0. Then, the set $B=\{\varphi_t := \psi e_{-t} : t\in[0,1)^d\}$ is a bounded subset of $\mathscr{D}(I_{\varepsilon',x_0})$. So,

$$\sup_{t\in[0,1)^d}\big|\widehat{f\psi}(n+t)\big| = \sup_{t\in[0,1)^d}\big|\widehat{f\varphi_t}(n)\big| \leqslant \frac{C'_N}{\langle n\rangle^N}, \quad \forall n\in\Gamma_1\cap\mathbb{Z}^d,$$

that is,

$$\sup_{\xi\in(\Gamma_1\cap\mathbb{Z}^d)+[0,1)^d}\langle\xi\rangle^N\big|\widehat{f\psi}(\xi)\big| \leqslant (1+4d)^{N/2}C'_N.$$

Select now an open conic neighborhood Γ_2 of ξ_0 such that $\overline{\Gamma}_2\subset\Gamma_1\cup\{0\}$ and find c' such that $\{y\in\mathbb{R}^d : (\exists\xi\in\Gamma_2)(|\xi-y|\leqslant c'|\xi|)\}\subset\Gamma_1$. The latter condition implies that $\Gamma_2\cap\{\xi\in\mathbb{R}^d : |\xi|c'\geqslant 1\}\subset(\Gamma_1\cap\mathbb{Z}^d)+[0,1)^d$ and hence

$$\sup_{\xi\in\Gamma_2}\langle\xi\rangle^N\big|\widehat{f\psi}(\xi)\big| \leqslant \max\left\{C''_N, (1+4d)^{N/2}C'_N\right\} = C_N < \infty,$$

where $C''_N = \sup_{\xi\in\Gamma_2,\,|\xi|<1/c'}\langle\xi\rangle^N\big|\widehat{f\psi}(\xi)\big|$. This shows that $(x_0,\xi_0)\notin WF(f)$, as claimed. □

We mention that Theorem 2.1 also follows from the relation between discrete and Hörmander's wave front sets proved in Theorem 7.4 in [**12**], once one observes that the notion is local and so it does not depend of a particular parametrization.

3. Sobolev wave front

In this section we deal with wave fronts of Sobolev type. We slightly reformulate Hörmander's definition [**6**].

Definition 3.1. Let $f\in\mathscr{D}'(\mathbb{R}^d)$, $(x_0,\xi_0)\in\mathbb{R}^d\times(\mathbb{R}^d\setminus\{0\})$, and $s\in\mathbb{R}$. We say that f is Sobolev microlocally regular at (x_0,ξ_0) of order s, that is $(x_0,\xi_0)\notin WF_s(f)$, if there exist an open cone Γ around ξ_0 and $\psi\in\mathscr{D}(\mathbb{R}^d)$ with $\psi\equiv 1$ in a neighborhood of x_0 such that $\int_\Gamma\big|\widehat{\psi f}(\xi)\big|^2\langle\xi\rangle^{2s}d\xi<\infty$.

We shall now refine Theorem 2.1.

THEOREM 3.1. *Let $f \in \mathscr{D}'(\mathbb{R}^d)$. The following two conditions are equivalent:*

(i) *There exist an open cone Γ around ξ_0, $\phi \in \mathscr{D}(I_{\eta,x_0})$, $\eta \in (0,1)$, with $\phi \equiv 1$ in a neighborhood of x_0, such that*

$$\sum_{n\in\Gamma\cap\mathbb{Z}^d} |a_n|^2\langle n\rangle^{2s} < \infty, \quad \textit{where } (f\phi)_p = \sum_{n\in\mathbb{Z}^d} a_n e_n. \tag{3.1}$$

(ii) $(x_0,\xi_0) \notin WF_s(f)$.

PROOF. (i) $\Rightarrow$ (ii). Assume (3.1). Choose an open cone Γ_1 so that $\overline{\Gamma_1} \subset \Gamma \cup \{0\}$ and $\xi_0 \in \Gamma_1$. Find $0 < \varepsilon < \eta$ such that $\phi(x) = 1$ for all $x \in I_{\varepsilon,x_0}$. We first prove that: For every bounded set $B \subset \mathscr{D}(I_{\varepsilon,x_0})$

$$\sup_{\varphi\in B} \sum_{n\in\Gamma_1\cap\mathbb{Z}^d} \big|\widehat{f\varphi}(n)\big|^2\langle n\rangle^{2s} < \infty. \tag{3.2}$$

Fix a bounded subset $B \subset \mathscr{D}(I_{\varepsilon,x_0})$. In view of the choice of ε, we have that $f\varphi = f\varphi\phi$ and so $\widehat{f\varphi}(n) = \sum_{j\in\mathbb{Z}^d} a_j\widehat{\varphi}(n-j)$, for every $\varphi \in B$. We fix a constant $0 < c < 1$ that is smaller than the distance between $\partial\Gamma$ and the intersection of $\overline{\Gamma}_1$ with the unit sphere, and also smaller than the distance between $\partial\Gamma_1$ and the intersection of $\overline{\mathbb{R}^d \smallsetminus \Gamma}$ with the unit sphere. One has that $\xi \in \Gamma_1$ and $y \notin \Gamma$ imply $|\xi - y| > c\max\{|\xi|,|y|\}$. We keep $\varphi \in B$. By Peetre's inequality, we have

$$\begin{aligned}\Bigg(\sum_{n\in\Gamma_1\cap\mathbb{Z}^d} \big|\widehat{f\varphi}(n)\big|^2\langle n\rangle^{2s}\Bigg)^{1/2} &\leqslant C\Bigg(\sum_{n\in\Gamma_1\cap\mathbb{Z}^d}\Bigg(\sum_{j\in\mathbb{Z}^d}|a_j|\langle j\rangle^s\big|\widehat{\varphi}(n-j)\big|\langle n-j\rangle^{|s|}\Bigg)^2\Bigg)^{1/2}\\ &\leqslant C(I_1(\varphi)+I_2(\varphi)),\end{aligned}$$

where

$$I_1(\varphi) = \Bigg(\sum_{n\in\Gamma_1\cap\mathbb{Z}^d}\Bigg(\sum_{j\in\Gamma\cap\mathbb{Z}^d}|a_j|\langle j\rangle^s\big|\widehat{\varphi}(n-j)\big|\langle n-j\rangle^{|s|}\Bigg)^2\Bigg)^{1/2},$$

$$I_2(\varphi) = \Bigg(\sum_{n\in\Gamma_1\cap\mathbb{Z}^d}\Bigg(\sum_{j\notin\Gamma\cap\mathbb{Z}^d}|a_j|\langle j\rangle^s\big|\widehat{\varphi}(n-j)\big|\langle n-j\rangle^{|s|}\Bigg)^2\Bigg)^{1/2}.$$

By Young's inequality and the fact that B is a bounded set,

$$\sup_{\varphi\in B} I_1(\varphi) \leqslant \Bigg(\sum_{n\in\Gamma\cap\mathbb{Z}^d}|a_n|^2\langle n\rangle^{2s}\Bigg)^{1/2}\sup_{\varphi\in B}\sum_{n\in\mathbb{Z}^d}|\varphi(n)|\langle n\rangle^{|s|} < \infty.$$

We now estimate $I_2(\varphi)$. Since $f\phi$ is compactly supported, $\langle j\rangle^s|a_j| = \langle j\rangle^s|\widehat{f\phi}(j)| \leqslant D\langle j\rangle^k$, $\forall n \in \mathbb{Z}^d$, for some $D > 0$ and $k > 0$. The fact that B is bounded yields the existence of $C' > 0$ such that $|\widehat{\varphi}(j)| \leqslant C'\langle j\rangle^{-k-|s|-3(d+1)/2}$. Because of the choice of Γ_1, we have

$$\sup_{\varphi\in B}(I_2(\varphi))^2 \leqslant (DC')^2 \sum_{n\in\Gamma_1\cap\mathbb{Z}^d}\Bigg(\sum_{j\notin\Gamma\cap\mathbb{Z}^d}\langle j\rangle^k\langle n-j\rangle^{-k-3(d+1)/2}\Bigg)^2$$
$$\leqslant (DC')^2 c^{-2k-3(d+1)} \sum_{n\in\Gamma_1\cap\mathbb{Z}^d}\langle n\rangle^{-d-1}\Bigg(\sum_{j\notin\Gamma\cap\mathbb{Z}^d}\langle j\rangle^{-d-1}\Bigg)^2.$$

Thus (3.2) has been established. We now deduce (ii) from (3.2). Once again we shrink the conic neighborhood of ξ_0. So let Γ_2 be an open cone such that $\overline{\Gamma_2} \subset \Gamma_1\cup\{0\}$ and $\xi_0\in\Gamma_2$. Let $\psi\in\mathscr{D}(I_{\varepsilon,x_0})$ be equal to 1 in a neighborhood of x_0. Find $r>0$ such that $\Gamma_2\cap\{\xi\in\mathbb{R}^d:\ |\xi|\geqslant r\}\subset(\Gamma_1\cap\mathbb{Z}^d)+[0,1)^d$. For each $n\in\Gamma_1\cap\mathbb{Z}^d$, write $\Lambda_n=n+[0,1]^d$. Then, by (3.2) and Peetre's inequality,

$$\int_{\substack{\xi\in\Gamma_2\\|\xi|\geqslant r}}|\widehat{\psi f}(\xi)|^2\langle\xi\rangle^{2s}d\xi \leqslant C\sum_{n\in\Gamma_1\cap\mathbb{Z}^d}\langle n\rangle^{2s}\int_{\Lambda_n}|\widehat{\psi f}(\xi)|^2d\xi$$
$$= C\int_{[0,1]^d}\sum_{n\in\Gamma_1\cap\mathbb{Z}^d}\langle n\rangle^{2s}|\widehat{\psi f}(n+t)|^2dt$$
$$\leqslant C\sup_{t\in[0,1]^d}\sum_{n\in\Gamma_1\cap\mathbb{Z}^d}\langle n\rangle^{2s}|\widehat{e_{-t}\psi f}(n)|^2<\infty.$$

Therefore, $(x_0,\xi_0)\notin WF(f)$.

(ii) $\Rightarrow$ (i). A variant of the argument employed above, but with integrals instead of sums, applies to show that $(x_0,\xi_0)\notin WF_s(f)$ implies the following property: There exist an open cone Γ and $\varepsilon\in(0,1)$ such that for every bounded set $B\subset\mathscr{D}(I_{\varepsilon,x_0})$

$$\sup_{\psi\in B}\int_\Gamma|\widehat{f\psi}(\xi)|^2\langle\xi\rangle^{2s}d\xi<\infty. \tag{3.3}$$

We leave such details to the reader. So, assume that (3.3) holds. Let Γ_1 be an open cone containing ξ_0 such that $\overline{\Gamma}_1\subset\Gamma\cup\{0\}$. Then, there exists some $r>0$ such that $(\Gamma_1+[0,1]^d)\cap\{\xi\in\mathbb{R}^d:\ |\xi|\geqslant r\}\subset\Gamma$. Let $\phi\in\mathscr{D}(I_{\varepsilon,x_0})$ such that $\phi\equiv1$ in a neighborhood of x_0. Consider a measurable function $t:\Gamma\to[0,1]^d$. Taking the bounded set

$$B=\left\{\psi_{j,t}\in\mathscr{D}(I_{\varepsilon,x_0}):\ \psi_{j,t}(x)=x_je^{-2\pi ix\cdot t(\xi)}\phi(x),\ \xi\in\Gamma,\ j=1,\dots,d\right\}$$

in (3.3), we obtain that there is a constant $C>0$ such that

$$\int_\Gamma|\nabla(\widehat{\phi f})(\xi+t(\xi))|^2\langle\xi\rangle^{2s}d\xi<C. \tag{3.4}$$

The constant C is actually independent of t. For each $n\in\Gamma_1\cap\mathbb{Z}^d$, let Λ_n be the unit cube $n+[0,1]^d=\prod_{j=1}^d[n_j,n_j+1]$. Note that $\Lambda_n\subset\Gamma$ if $|n|\geqslant r$. Then

$$\Bigg(\sum_{n\in\Gamma_1\cap\mathbb{Z}^d}|\widehat{\phi f}(n)|^2\langle n\rangle^{2s}\Bigg)^{1/2}=\Bigg(\sum_{n\in\Gamma_1\cap\mathbb{Z}^d}\int_{\Lambda_n}|\widehat{\phi f}(n)|^2\langle n\rangle^{2s}d\xi\Bigg)^{1/2}\leqslant I_1^{1/2}+I_2^{1/2},$$

where $I_1 := \sum_{n\in\Gamma_1\cap\mathbb{Z}^d}\int_{\Lambda_n}|\widehat{\phi f}(n)-\widehat{\phi f}(\xi)|^2\langle n\rangle^{2s}d\xi$ and

$$\begin{aligned}I_2 &:= \sum_{n\in\Gamma_1\cap\mathbb{Z}^d}\int_{\Lambda_n}|\widehat{\phi f}(\xi)|^2\langle n\rangle^{2s}d\xi\\ &\leqslant \sum_{|n|\leqslant r}\int_{\Lambda_n}|\widehat{\phi f}(\xi)|^2\langle n\rangle^{2s}d\xi + C'\int_\Gamma|\widehat{\phi f}(\xi)|^2\langle \xi\rangle^{2s}d\xi<\infty.\end{aligned}$$

It remains to show that I_1 is finite. Given $\theta>0$, define $t_\theta:\Gamma\to[0,1]^d$ as

$$t_\theta(\xi)=\begin{cases}\theta(n-\xi) & \text{if } \xi\in\Lambda_n \text{ and } |\xi|\geqslant r,\\ 0 & \text{otherwise.}\end{cases}$$

We now make use of (3.4). Since

$$|\widehat{\phi f}(\xi)-\widehat{\phi f}(n)|^2\leqslant|n-\xi|\int_0^1|\nabla(\widehat{\phi f})(\xi+\theta(n-\xi))|^2d\theta,$$

we have

$$\begin{aligned}I_1 &\leqslant \sum_{\substack{|n|\leqslant r\\ n\in\Gamma_1\cap\mathbb{Z}^d}}\int_{\Lambda_n}|\widehat{\phi f}(n)-\widehat{\phi f}(\xi)|^2\langle n\rangle^{2s}d\xi\\ &\quad + C'\sup_{\theta\in[0,1]}\int_\Gamma|\nabla(\widehat{\phi f})(\xi+t_\theta(\xi))|^2\langle\xi\rangle^{2s}d\xi<\infty.\end{aligned}$$

This completes the proof. □

Acknowledgement. This work is supported by the Project 174024 of the Serbian Ministry of Science.

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(Received 26 11 2014)

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