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A NOTE ON RAKIĆ DUALITY PRINCIPLE FOR OSSERMAN MANIFOLDS

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Abstract. We prove that for a Riemannian manifold, the pointwise Osserman condition is equivalent to the Rakić duality principle.

1. Introduction

Let $\mathcal{R}$ be an algebraic curvature tensor on a Euclidean space $\mathbb{R}^n$ and let for $X \in \mathbb{R}^n$, $\mathcal{R}_X : Y \mapsto \mathcal{R}(Y,X)X$ be the corresponding Jacobi operator. An algebraic curvature tensor $\mathcal{R}$ is called *Osserman*, if the spectrum of the Jacobi operator $\mathcal{R}_X$ does not depend on the choice of a unit vector $X \in \mathbb{R}^n$.

Let M^n be a Riemannian manifold, R be its curvature tensor, and R_X be the corresponding Jacobi operator. It is well known that the properties of R_X are intimately related with the underlying geometry of the manifold. The manifold M^n is called *pointwise Osserman*, if R is Osserman at every point $p \in M^n$, and is called *globally Osserman*, if the spectrum of R_X is the same for all X in the unit tangent bundle of M^n. Locally two-point homogeneous spaces are globally Osserman, since the isometry group of each of these spaces acts transitively on the unit tangent bundle. Osserman [**8**] conjectured that the converse is also true. This gives a very nice characterization of local two-point homogeneous spaces in terms of the geometry of the Jacobi operator.

At present, the Osserman Conjecture is almost completely solved by the results of Chi [**3**], who proved the Conjecture in dimensions $n \neq 4k$, $k > 1$ and $n = 4$, and the first author [**5, 6, 7**], who proved it in all the remaining cases, except for some cases in dimension $n = 16$.

One of the crucial steps in the existing proofs of the Osserman Conjecture is the fact that any Osserman algebraic curvature tensor satisfies the Rakić duality principle [**9**]. We say that an algebraic curvature tensor $\mathcal{R}$ on a Euclidean space $\mathbb{R}^n$ satisfies the *Rakić duality principle*, if for any unit vectors $X, Y \in \mathbb{R}^n$, the vector Y

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is an eigenvector of $\mathcal{R}_X$ if and only if the vector X is an eigenvector of $\mathcal{R}_Y$ (with the same eigenvalue).

The duality principle is extended to the pseudo-Riemannian settings in [**1**].

2. Equivalence of duality principle and Osserman pointwise condition

Recently, for an algebraic curvature tensor in Riemannian signature, Brozos-Vázquez and Merino [**2**] proved the equivalence of the Osserman condition and the duality principle for spaces of dimension less than 5. We show that this holds in an arbitrary dimension.

Theorem 2.1. *The following two conditions for an algebraic curvature tensor $\mathcal{R}$ in the Riemannian signature are equivalent:*

(a) *$\mathcal{R}$ satisfies the duality principle;*
(b) *$\mathcal{R}$ is Osserman.*

Proof. The implication (b) $\implies$ (a) is proved in [**9**].

To establish the converse, consider the characteristic polynomial $\chi_X(t)$ of the Jacobi operator $\mathcal{R}_X$, where X is a unit vector. As the coefficients of χ_X are analytic functions on the unit sphere $S \subset \mathbb{R}^n$, there is an open and dense subset $S' \subset S$ such that for all $X \in S'$, the number and the multiplicity of the eigenvalues of $\mathcal{R}_X$ are constant, the eigenvalues are analytic functions of X, and the eigendistributions of $\mathcal{R}_X$ are analytic (viewed as submanifolds of the appropriate Grassmannians) [**4, 10**].

Let $X \in S'$ and let $Y \in S$ be orthogonal to X. Suppose λ_0 is an eigenvalue of $\mathcal{R}_X$ with a unit eigenvector e_0. For small ϕ, the vector $\cos\phi X + \sin\phi Y$ belongs to S', so there exist a differentiable (in fact, analytic) eigenvalue function $\lambda(\phi)$ of the operator $\mathcal{R}_{\cos\phi X+\sin\phi Y}$ such that $\lambda(0) = \lambda_0$, and a differentiable unit vector function $e(\phi)$, a section of the $\lambda(\phi)$-eigenspace of $\mathcal{R}_{\cos\phi X+\sin\phi Y}$, such that $e(0) = e_0$. Differentiating the equation

$$\mathcal{R}(e(\phi), \cos\phi X + \sin\phi Y, \cos\phi X + \sin\phi Y, e(\phi)) = \lambda(\phi)$$

at $\phi = 0$ we obtain

$$2\mathcal{R}(e_0, Y, X, e_0) + 2\mathcal{R}(e_0, X, X, e'(0)) = \lambda'(0).$$

But $\mathcal{R}(e_0, X, X, e'(0)) = \lambda_0\langle e_0, e'(0)\rangle = 0$ and $\mathcal{R}(e_0, Y, X, e_0) = \mathcal{R}(X, e_0, e_0, Y) = \lambda_0\langle X, Y\rangle = 0$, by duality. It follows that the eigenvalues of $\mathcal{R}_X$ are constant on every connected component of S'. Then the coefficients of $\chi_X(t)$ are constant on the whole unit sphere S, which implies that $\mathcal{R}$ is Osserman. □

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