THE BUCKLING OF ARCHES WITH HINGED ENDS

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As is customary in the discussion of the buckling of arches, we shall be considering an arch the center line of which is the funicular curve for the arbitrary given load. The cross section of the arch, symmetrical with respect to the plane of the arch, may be arbitrarily varying along the center line. The intensity of the loading in all points of the arch may be proportional to a common factor p so that a gradual increase of loading is obtained by an increase of this factor. It is usually supposed that the direction of the loading changes during the deformation so that the angle between it and the element of the center line remains unchanged all the time.

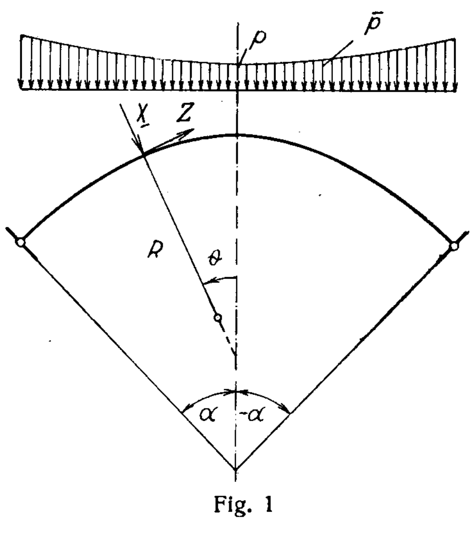

Fig. 1

Then, as it is known [1], the critical value of the pressure at which the buckling of the arch in its plane occurs is given with the smallest eigenvalue of the homogeneous linear differential equation:

$$\left(\frac{d^2}{d\varphi^2}+1\right)\left(\frac{1}{\bar{\rho}_0}\frac{dM}{d\varphi}\right)+\lambda\frac{d}{d\varphi}\left(\frac{\bar{\rho}_0^2\bar{X}}{\bar{B}}M\right)=0 \tag{1}$$

with homogeneous end conditions corresponding to the manner in which the ends of the arch are fixed.

In the following we shall consider a symmetrical arch with hinged ends. If we assume that the deformation is antisymmetrical, what in this case gives always the smallest critical load, then the conditions under which the equation (1) should be integrated are:

$$M \text{ must be an odd function of } \varphi \text{ and } M(o)=M(\varphi)=0\,. \tag{2}$$

In (1) and (2) there are:

φ – the angle between the normal on the center line (before the buckling) and the normal in the key of the arch;

$M(\varphi)$ – the bending moment in a point of the arch;

$\lambda=\frac{pa^3}{B_0}$ – the unknown eigenvalue;

p – the component of the loading normal to the center line in the key;

$\bar{X}(\varphi)$ – the dimensionless part of the component of the loading normal to the center line, so that the component itself is $X=p\bar{X}$;

a – radius of curvature in the key;

$\bar{\rho}_0(\varphi)$ – the dimensionless part of the radius of curvature (before the buckling), so that the radius itself is $\rho_0=a\bar{\rho}_0$;

$B_0=EJ_0$ – the stiffness of the arch in the key;

$\bar{B}(\varphi)$ – the dimensionless part of the stiffness in a point, so that $B=B_0\bar{B}$.

The solutions of the differential equation (1) for some forms of the arch and for various $\bar{B}(\varphi)$ are given mainly by means of numerical integration [**1, 2**]. We give here a general solution of the problem for a two hinged arch, applicable also in cases where the distribution of the loading (and consequently the form of the arch) and the variation of $\bar{B}(\varphi)$ are not expressed in analytical terms of the parameter φ. This solution is based on a more general solution of some eigenvalue problems proposed by M. Djurić [3].

Introducing the new variable

$$Z = \frac{\bar{\rho}_0^2 \bar{X}}{\bar{B}} M \tag{3}$$

in the equation (1) and integrating it, considering the conditions (2), which are the same for Z and M, we obtain

$$\left(\frac{d^2}{d\varphi^2} + 1\right)\left[\frac{\bar{B}}{\bar{\rho}_0^3 \bar{X}} Z + \int \frac{\bar{B}}{\bar{\rho}_0^4 \bar{X}} \frac{d\bar{\rho}_0}{d\varphi} Z d\varphi\right] + \lambda Z = 0 . \tag{1a}$$

The terms

$$\frac{\bar{B}}{\bar{\rho}_0^3 \bar{X}} \quad \text{and} \quad \frac{\bar{B}}{\bar{\rho}_0^4 \bar{X}} \frac{d\bar{\rho}_0}{d\varphi} \tag{4}$$

in this equation are the given functions of φ that in all cases can be developed in Fourier series. The first of them can be represented as an even periodic function with the period 2φ:

$$\frac{\bar{B}}{\bar{\rho}_0^3 \bar{X}} = \sum_{m=0}^{\infty} \alpha_m \cos \frac{m\pi\varphi}{\varphi_o} \tag{5}$$

and the second as an odd function with the same period:

$$\frac{\bar{B}}{\bar{\rho}_0^4 \bar{X}} \frac{d\bar{\rho}_0}{d\varphi} = \sum_{\mu=1}^{\infty} \beta_\mu \sin \frac{\mu\pi\varphi}{\varphi_0} . \tag{6}$$

We shall represent the solution in the form

$$Z = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi\varphi}{\varphi_0} . \tag{7}$$

As it is seen from (2) and (3) the end conditions have already been fulfilled in (7).

Introducing (5), (6), and (7) in the equation (1a) we obtain

$$\left(\frac{d^2}{d\varphi}+1\right)\left[\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\alpha_m c_n \cos\frac{m\pi\varphi}{\varphi_o}\sin\frac{n\pi\varphi}{\varphi_o}+\right.$$
$$\left.+\int\sum_{\mu=1}^{\infty}\sum_{n=1}^{\infty}\beta_\mu c_n \sin\frac{\mu\pi\varphi}{\varphi_o}\sin\frac{n\pi\varphi}{\varphi_o}\,d\varphi\right]+\lambda\sum_{n=1}^{\infty}c_n\sin\frac{n\pi\varphi}{\varphi_o}=0\,. \tag{8}$$

The terms of the first double sum can be represented as

$$\alpha_m c_n \cos\frac{m\pi\varphi}{\varphi_o}\sin\frac{n\pi\varphi}{\varphi_o}=\frac{1}{2}\alpha_m c_n\left[\sin\,(n+m)\frac{\pi\varphi}{\varphi_o}+\sin\,(n-m)\frac{\pi\varphi}{\varphi_o}\right],$$

so that the whole sum arranged according to the $\sin\frac{\nu\pi\varphi}{\varphi_o}$ where $\nu=1, 2, 3\ldots$ takes the form

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\alpha_m c_n\cos\frac{m\pi\varphi}{\varphi_0}\sin\frac{n\pi\varphi}{\varphi_0}=\frac{1}{2}\sum_{\nu=1}^{\infty}\sum_{n=1}^{\infty}c_n\left(\alpha_{\nu-n}+\alpha_{n-\nu}-\alpha_{n+\nu}\right)\sin\frac{\nu\pi\varphi}{\varphi_0}, \tag{9}$$

wherein $\alpha_{-\nu}=0$.

In the same manner the second double sum can be written as

$$\sum_{\mu=1}^{\infty}\sum_{n=1}^{\infty}\beta_\mu c_n\sin\frac{\mu\pi\varphi}{\varphi_0}\sin\frac{n\pi\varphi}{\varphi_0}=$$
$$=\frac{1}{2}\sum_{n=1}^{\infty}\beta_n c_n+\frac{1}{2}\sum_{\nu=1}^{\infty}\sum_{n=1}^{\infty}c_n\left(-\beta_{\nu-n}+\beta_{n-\nu}+\beta_{n+\nu}\right)\cos\frac{\nu\pi\varphi}{\varphi_0}\,, \tag{10}$$

wherein $\beta_0=\beta_{-\nu}=0$.

With (9) and (10) the equation (8) becomes:

$$\frac{1}{2}\sum_{\nu=1}^{\infty}\left\{\left[1-\left(\frac{\nu\pi}{\varphi_0}\right)^2\right]\sum_{n=1}^{\infty}c_n\left[\left(\alpha_{\nu-n}+\alpha_{n-\nu}-\alpha_{n+\nu}\right)+\right.\right.$$
$$\left.\left.+\frac{\varphi_0}{\nu\pi}\left(-\beta_{\nu-n}+\beta_{n-\nu}+\beta_{n+\nu}\right)\right]\sin\frac{\nu\pi\varphi}{\varphi_0}\right\}+\frac{1}{2}\varphi\sum_{n=1}^{\infty}c_n\beta_n+\lambda\sum_{\nu=1}^{\infty}c_n\sin\frac{\nu\pi\varphi}{\varphi_0}=0\,. \tag{8a}$$

If we represent φ as a periodic function with the period $2\varphi_0$ in the form of a Fourier series, then the first simple sum in (8a) can be written in the form:

$$\frac{1}{2}\varphi\sum_{n=1}^{\infty} c_n\beta_n = \frac{1}{2}\sum_{\nu=1}^{\infty}\sum_{n=1}^{\infty} c_n\beta_n(-1)^{\nu+1}\frac{2\varphi_0}{\nu\pi}\sin\frac{\nu\pi\varphi}{\varphi_0}. \tag{11}$$

Putting (11) in (8a), dividing the whole equation by $\frac{1}{2}\left[1-\left(\frac{\nu\pi}{\varphi_0}\right)^2\right]$ and introducing the notations

$$a_\nu=\frac{\varphi_0}{\nu\varphi}, \qquad k_\nu=\frac{2}{1-\left(\frac{\nu\pi}{\varphi_0}\right)^2}, \tag{12}$$

we get

$$\sum_{\nu=1}^{\infty}\Big\{k_\nu\lambda c_\nu+\sum_{n=1}^{\infty} c_n\left[(\alpha_{\nu-n}+\alpha_{n-\nu}-\alpha_{n+\nu})+\right.$$
$$\left.+a_\nu(-\beta_{\nu-n}+\beta_{n-\nu}+\beta_{n+\nu}+k_\nu(-1)^{\nu+1}\beta_n)\right]\sin\frac{\nu\pi\varphi}{\varphi_0}\Big\} = 0. \tag{13}$$

This equation can be fulfilled only if the coefficient by $\sin\frac{\nu\pi\varphi}{\varphi_0}$ vanishes for every $\nu\,(\nu=1,2,3\ldots)$ i. e. if

$$k_\nu\lambda c_\nu+\sum_{n=1}^{\infty} c_n\left[(\alpha_{\nu-n}+\alpha_{n-\nu}-\alpha_{n+\nu})+\right.$$
$$\left.a_\nu(-\beta_{\nu-n}+\beta_{n-\nu}+\beta_{n+\nu}+k_\nu(-1)^{\nu+1}\beta_n\right] = 0. \tag{14}$$

This is an infinite system of homogeneous linear algebraic equations with an infinite number of uncnowns which possesses the nontrivial solutions only if the determinant of the system vanishes. This determinant can easily be written:

$$\begin{vmatrix} k_1\lambda+(2\alpha_0-\alpha_2)+a_1(\beta_2+k_1\beta_1) & (\alpha_1-\alpha_3)+a_1(\beta_1+\beta_3+k_1\beta_2) & (\alpha_2-\alpha_4)+a_1(\beta_2+\beta_4+k_1\beta_3) \\ (\alpha_1-\alpha_3)+a_2(-\beta_1+\beta_3-k_2\beta_1) & k_2\lambda+(2\alpha_0-\alpha_4)+a_2(\beta_4-k_2\beta_2) & (\alpha_1-\alpha_5)+a_2(\beta_1+\beta_5-k_2\beta_3) \\ (\alpha_2-\alpha_4)+a_3(-\beta_2+\beta_4+k_3\beta_1) & (\alpha_1-\alpha_5)+a_3(-\beta_1+\beta_5+k_3\beta_2) & k_3\lambda+(2\alpha_0-\alpha_6)+a_3(\beta_6+k_3\beta_3) \end{vmatrix}=0 \quad (15)$$

In the developed form this is an infinite polynomial in λ, the roots of which are the sought eigenvalues of the equation (1). The smallest root of this equation can easily and quickly be found by means of successive approximations, as will be shown in an example.

In a number of cases interesting from an engineer's point of view the functions $\frac{\overline{B}}{\overline{\rho}_0^3 X}$ and $\frac{\overline{B}}{\overline{\rho}_0^4 X}\frac{d\overline{\rho}_0}{d\varphi}$ are given in the form:

$$\frac{\overline{B}}{\overline{\rho}_0^3 X}=\cos^{\varepsilon}\varphi \quad \text{and} \quad \frac{\overline{B}}{\overline{\rho}_0^4 X}\frac{d\overline{\rho}_0}{d\varphi}=b\cos^{\varepsilon-1}\varphi\sin\varphi\ . \tag{16}$$

For example for a parabolic arch with constant cross section there is $\varepsilon=7$, $b=3$; for a parabolic arch with variable rectangular cross section the height of which is varying like sec φ there is $\varepsilon=4$, $b=3$; if the breadth of the cross section is varying like sec φ there is $\varepsilon=6$, $b=3$; for a cathenoid with constant cross section there is $\varepsilon=5$, $b=2$; if the height of a rectangular cross section of a cathenoid is varying like sec φ then there is $\varepsilon=2$, $b=2$, etc.

In this case the coefficients β_μ can be expressed by means of the coefficients α_m.

From (5), (6) and (16) it is seen that

$$\alpha_m=\frac{1}{\varphi_0}\int_{-\varphi_0}^{+\varphi_0}\cos^{\varepsilon}\varphi\cos\frac{m\pi\varphi}{\varphi_0}d\varphi\ ,$$

$$\beta_\mu=\frac{1}{\varphi_0}\int_{-\varphi_0}^{+\varphi_0}b\cos^{\varepsilon-1}\varphi\sin\varphi\sin\frac{\mu\pi\varphi}{\varphi_0}d\varphi\ . \tag{17}$$

By means of partial integration of the second of these expressions we get readily:

$$\beta_\mu = \frac{b}{\varepsilon} \frac{\mu\pi}{\varphi_o} \alpha_\mu \tag{18}$$

and integrating directly the first of the expressions (17) we obtain:

$$\alpha_m = \frac{1}{2^{\varepsilon-2}\varphi_o} \sum_{p=0}^{p_1} \binom{\varepsilon}{p} \frac{\varepsilon - 2p}{(\varepsilon-2p)^2 - (m\pi/\varphi_o)^2} \sin(\varepsilon - 2p)\varphi_o \;, \tag{19}$$

where $p_1 = \frac{\varepsilon}{2} - 1$ for an even ε, and $p = \frac{\varepsilon - 1}{2}$ for odd ε . For α_o there must be taken half of the value obtained by (19).

From (19) and (18) it is easy to find α_m and β_μ for a given arch and then from the equation (15) by means of successive approximations the eigenvalues λ .

For example in the case of a parabolic arch with constant cross section ($\varepsilon = 7$ and $b = 3$) and the ratio $f/l = 0{,}2$ resp. $\varphi_o = 38^0 40'$ the determinant (15) takes the form

$$\begin{vmatrix} k_1\lambda + 1{,}2710 & 0{,}5498 & -0{,}0653 \\ 0{,}2991 & k_2\lambda + 1{,}2833 & 0{,}4621 \\ -0{,}0190 & 0{,}3243 & k_3\lambda + 1{,}2828 \end{vmatrix} = 0 \, . \tag{20}$$

Considering only the determinant of the first order we get $\lambda = 13{,}15$. The determinant of the second order gives 11,47 and from the determinant of the third order there is 11,11. The last of these values agrees completely with the value obtained by Lockschin by means of the numerical integration of the equation (1).

At last it should be mentionned that the solution is applicable also in cases where the form of the arch (resp. the corresponding loading) and the variation of the cross section are not given in analytical terms of φ. Then the necessary number of coefficients can easily be found by means of a numerical procedure (f. e. Runge's) or by means of an analysator, and then the equation (15) can be written.

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